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INTEGRAL INEQUALITIES WITHIN THE FRAMEWORK OF GENERALIZED FRACTIONAL INTEGRALS

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Abstract. In this work, a new generalized fractional integral is defined and studied, and different relationships (equalities and inequalities) are obtained, which have as particular cases several of those reported in the literature. Hermite-Hadamard type inequalities are obtained for different kinds of functions such as symmetric, convex symmetric, Wright-quasi-convex and h-symmetrized convex.

1. Introduction

Integral Inequalities, has become one of the most dynamic areas of Mathematics, both pure and applied, which translates into the increase in researchers and results obtained, which has grown greatly in recent years. There is an inequality that is considered seminal, and that provides simple bounds for the integral mean value of a particular class of functions: convex functions, is the so-called Hermite-Hadamard inequality (see, e.g., [4, 14, 16]):

Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a convex function. Then, for $a, b \in I$ with $a < b$,

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1}$$

holds.

Here and in the following, let $\mathbb{R}, \mathbb{R}^+$, and $\mathbb{N}$ be the sets of real numbers, positive real numbers, and positive integers, respectively, and let $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Convex functions have their own importance, they have a much-required property, and that is that they are especially easy to minimize (the minimum of a function is convex in a global minimum), furthermore, as it is well said in [28], "the great milestone of optimization is not between linearity and nonlinearity, but between convexity and nonconvexity". For this reason, there is a very rich theory for solving convex optimization problems that has many practical applications (for example, circuit design, controller design, to solve some shape optimization, inverse and applied problems, etc.),

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see, for example, [10, 11, 12]. For readers interested in having a broader picture of the multitude of extensions and generalizations of the classical definition of convexity, we recommend the paper [23].

Inequality (1) has been the subject of much attention in recent years, and a multitude of results and extensions have appeared, not only referring to the Riemann Integral, but also to Fractional Integrals of the Riemann-Liouville type and Generalized Integrals (cf. [2, 7, 8, 9, 13, 17, 20, 23, 26, 31] and the references cited there.

The following definitions will be used in our work (see [4]).

DEFINITION 1. Let I be a nonempty interval on $\mathbb{R}$. Then a function $f : I \to \mathbb{R}$ is called *quasi-convex* on I (denoted by $f \in QC(I)$) if

$$f(tx+(1-t)y) \leqslant \max\{f(a), f(b)\} \quad (0 \leqslant t \leqslant 1; x, y \in I).$$

Clearly, any convex function is quasi-convex. Furthermore, there exists a quasi-convex function which is not convex.

DEFINITION 2. Let I be a nonempty interval on $\mathbb{R}$. Then a function $f : I \to \mathbb{R}$ is called *Wright-quasi-convex* on I (denoted by $f \in WQC(I)$) if

$$\frac{1}{2}\left[f(tx+(1-t)y)+f((1-t)x+ty)\right] \leqslant \max\{f(a), f(b)\} \quad (0 \leqslant t \leqslant 1; x, y \in I). \quad (2)$$

DEFINITION 3. Let I be a nonempty interval on $\mathbb{R}$. Then a function $f : I \to \mathbb{R}$ is called *Jensen-quasi-convex* on I (denoted by $f \in JQC(I)$)) if

$$f\left(\frac{x+y}{2}\right) \leqslant \max\{f(a), f(b)\} \quad (x, y \in I).$$

The relationship between these notions of convexity, are given in the following inclusion (see [4]).

$$QC(I) \subset WQC(I) \subset JQC(I)$$

The following result is known (see [4]).

THEOREM 1. *Let I be a nonempty interval on $\mathbb{R}$ and $a, b \in I$ with $a < b$. Also let $f \in WQC(I)$ be integrable on $[a, b]$. Then the following Hermite-Hadamard type inequality holds*

$$\frac{1}{b-a}\int_a^b f(t)dt \leqslant \max\{f(a), f(b)\}.$$

In [32] the following definition is presented.

DEFINITION 4. Let I and J be intervals on $\mathbb{R}$ with $(0,1) \subseteq J$. Also let $f : I \to \mathbb{R}_0^+$ be a function and $h : J \to \mathbb{R}_0^+$ a function with $h \neq 0$. Then f is called *h-convex* if

$$f(tx+(1-t)y) \leqslant h(t)f(x)+f(1-t)f(y) \quad (0 < t < 1; x, y \in I).$$

DEFINITION 5. Let $[a,b]\,(a<b)$ be an interval on $\mathbb{R}$ and $f:[a;b]\to\mathbb{C}$ a function. Then the *symmetrical transform* of f, denoted by $\check{f}$, is defined by

$$\check{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right]\quad (t\in[a,b]).$$

The *anti-symmetrical transform* of f on $[a,b]$, denoted by $\tilde{f}(t)$, is defined by

$$\tilde{f}(t):=\frac{1}{2}\left[f(t)-f(a+b-t)\right]\quad (t\in[a,b]).$$

Obviously, for any function f, $\check{f}+\tilde{f}=f$.

DEFINITION 6. Let h be the function in Definition 4. A function $f:[a,b]\to\mathbb{R}_0^+$ is called *h-symmetrized convex (concave)* on the interval $[a,b]$ if the symmetrical transform $\check{f}$ is h-convex (concave) on $[a,b]$.

THEOREM 2. *Let h be the function in Definition 4 and a function $f:[a,b]\to\mathbb{R}_0^+$ be h-symmetrized convex on the interval $[a,b]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leqslant\frac{f(x)+f(a+b-x)}{2}\leqslant\left[h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right]\frac{f(a)+f(b)}{2}.$$

For definitions 5 and 6, and Theorem 2, we refer to [3, 5].
For the following result and its consequences, see [3].

THEOREM 3. *Let $f:[a,b]\to\mathbb{R}(a<b)$ be a symmetrized convex function. Then, for any $x\in[a,b]$, we have*

$$f\left(\frac{a+b}{2}\right)\leqslant\check{f}(x)\leqslant\frac{f(a)+f(b)}{2}.$$

COROLLARY 1. *Let $f:[a,b]\to\mathbb{R}(a<b)$ be a symmetrized convex and integrable function. Then we have the following Hermite-Hadamard inequalities*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(t)dt\leqslant\frac{f(a)+f(b)}{2}.$$

In 1906 Fejér established an inequality of the type (1), adding a weight function (cf [6]).

THEOREM 4. *Let $f:[a,b]\to\mathbb{R}(a<b)$ be a convex function and $f\in L_1(a,b)$. Also let $g:[a,b]\to\mathbb{R}$ be nonnegative, integrable and symmetric to $\frac{a+b}{2}$. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx\leqslant\frac{1}{b-a}\int_a^b f(x)g(x)dx\leqslant\frac{f(a)+f(b)}{2}\int_a^b g(x)dx.$$

In this paper we will use the functions Γ (see [25, 27, 33, 34]) defined as $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t$, $\Re(z) > 0$.

Below we present the generalized integral operators that we will use from now on.

DEFINITION 7. The generalized fractional Riemann-Liouville integral of order α with $\alpha \in \mathbb{R}$, of an integrable function $f(x)$ on $[0,\infty)$, is given as follows:

$$\left({}^{\beta}J^{\alpha}_{F,a+}f\right)(x) = \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(t)dt}{F(\mathbb{F}(x,t),\beta)F(t,\alpha)}, \tag{3}$$

with $\mathbb{F}(x,t) = \int_t^x \frac{ds}{F(t,s)}$, and F an absolutly continuous positive function.

Under the conditions of the previous definition, we can enunciate the left and right integral operators in the following way.

DEFINITION 8. The left and right fractional generalized integrals of order $\beta \in \mathbb{C}$, $Re(\beta) > 0$, are defined by

$$\left({}^{\beta}J^{\alpha}_{F,a+}f\right)(x) = \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(t)dt}{F(\mathbb{F}_+(x,t),\beta)F(t-a,\alpha)} \tag{4}$$

$$\left({}^{\beta}J^{\alpha}_{F,b-}f\right)(x) = \frac{1}{\Gamma(\beta)}\int_x^b \frac{f(t)dt}{F(\mathbb{F}_-(t,x),\beta)F(b-t,\alpha)} \tag{5}$$

being

$$\mathbb{F}_+(x,t) = \int_t^x \frac{ds}{F(s-a,\alpha)},$$

$$\mathbb{F}_-(t,x) = \int_x^t \frac{ds}{F(b-s,\alpha)},$$

and $F(\mathbb{F}_+(t,x),1) = F(\mathbb{F}_-(t,x),1) = 1$.

REMARK 1. If in the previous definition we make $F(z,\alpha) = z^{(1-\alpha)}$, then we obtain the integral operators used in [30] and defined in [18], a generalization of classic Riemann-Liouville fractional integral, obtained from the operators of the kernel pointed at the beginning and $\beta = 1$. Obviously under the case of the previous kernel, if $\alpha = 1$, we get the classic Riemann Integral.

To promote the reading of our work, we must say that the structure of our work is as follows. First, we study the most important properties of the generalized operators of Definitions 7 and 8. In section 3, using these operators, we obtain different integral inequalities of the Hermite-Hadamard type for symmetric, symmetric convex, Wright-functions. quasi-convex and h-symmetrized convex. In the Conclusions, we highlight the strength of our results, showing how many well-known integral operators are obtained as a consequence of choosing certain kernels in our definitions.

2. First results

Now, we present some properties of integral operators generalized of Definition 8. One of the most required is the boundedness, presented below.

THEOREM 5. *Let* $f:[a,b]\to\mathbb{R}$ *be an integrable function,* $\alpha\in\mathbb{C}$, $\Re(\alpha)>0$, $k>0$. *Then both* ${}^{\beta}J^{\alpha}_{F,a+}f(t)$ *and* ${}^{\beta}J^{\alpha}_{F,b-}f(t)$ *exist for all* $x\in[a,b]$ *and* $\Re(\beta)>0$.

Proof. Let $C=[a,b]\times[a,b]$ and $K=F\left(\mathbb{F}_{+}(x,t),\beta\right)F(t-a,\alpha)$. From the fact that $K=K_{+}+K_{-}$, with

$$K_{+}(t,x)=\begin{cases}\frac{1}{F(\mathbb{F}_{+}(x,t),\beta)F(t-a,\alpha)} & a\leqslant t\leqslant x\leqslant b,\\ 0, & a\leqslant x\leqslant t\leqslant b\end{cases}\tag{6}$$

and

$$K_{-}(t,x)=\begin{cases}\frac{1}{F(\mathbb{F}_{-}(t,x),\beta)F(b-t,\alpha)}, & a\leqslant t\leqslant x\leqslant b,\\ 0, & a\leqslant x\leqslant t\leqslant b.\end{cases}\tag{7}$$

Since $K(t,x)$ is measurable on C, using Tonelli's theorem for iterated integrals (see [15]) we can obtain

$$\int_a^b\left[\int_a^b K(t,x)f(t)dt\right]dx\leqslant B\|f\|_1<\infty.\tag{8}$$

Hence, by Fubini's theorem, it follows that $\int_a^b K(t,x)f(t)dt$ is integrable over $[a,b]$ as a function of $x\in[a,b]$. This implies that ${}^{\beta}J^{\alpha}_{F,a+}f(t)$ exists. The existence of the right generalized integral ${}^{\beta}J^{\alpha}_{F,b-}f(t)$ can be proved in a similar manner. This completes the proof of the theorem. $\square$

Next, we study the continuity of the operator from Definition 8.

THEOREM 6. *Let* $\alpha\geqslant 1$, $k>0$ *and* $f\in L_1[a,b]$. *Then* ${}^{\beta}J^{\alpha}_{F,a+}f\in C[a,b]$.

Proof. Let x, $y\in[a,b]$, $x\leqslant y$ and $x\to y$. Then we have

$$\begin{aligned}&\left|{}^{\beta}J^{\alpha}_{F,a+}f(x)-{}^{\beta}J^{\alpha}_{F,a+}f(y)\right|\\ &=\frac{1}{\Gamma(\beta)}\left|\int_a^x\frac{f(t)\,dt}{F\left(\mathbb{F}(x,t),\beta\right)F(t,\alpha)}-\int_a^y\frac{f(t)\,dt}{F\left(\mathbb{F}(y,t),\beta\right)F(t,\alpha)}\right|\\ &=\frac{1}{\Gamma(\beta)}\left|\int_a^x\frac{f(t)\,dt}{F\left(\mathbb{F}(x,t),\beta\right)F(t,\alpha)}-\int_a^x\frac{f(t)\,dt}{F\left(\mathbb{F}(y,t),\beta\right)F(t,\alpha)}\right.\\ &\quad\left.-\int_x^y\frac{f(t)\,dt}{F\left(\mathbb{F}(y,t),\beta\right)F(t,\alpha)}\right|\end{aligned}$$

$$= \frac{1}{\Gamma(\beta)}\left|\int_a^x \frac{f(t)}{F(t,\alpha)}\cdot\left(\frac{1}{F(\mathbb{F}(x,t),\beta)}-\frac{1}{F(\mathbb{F}(y,t),\beta)}\right)dt - \int_x^y \frac{f(t)\,dt}{F(\mathbb{F}(y,t),\beta)F(t,\alpha)}\right|$$

$$\leqslant \frac{1}{\Gamma(\beta)}\left[\int_a^x \left|\frac{f(t)}{F(t,\alpha)}\right|\cdot\left|\frac{1}{F(\mathbb{F}(x,t),\beta)}-\frac{1}{F(\mathbb{F}(y,t),\beta)}\right|dt+\int_x^y \frac{|f(t)|\,dt}{|F(\mathbb{F}(y,t),\beta)F(t,\alpha)|}\right]$$

$$\leqslant \frac{1}{\Gamma(\beta)}\left[\int_a^x \left|\frac{f(t)}{F(t,\alpha)}\right|\cdot\left|\frac{1}{F(\mathbb{F}(x,t),\beta)}-\frac{1}{F(\mathbb{F}(y,t),\beta)}\right|dt+\frac{\|f(t)\|_{L_1[a,b]}}{|F(\mathbb{F}(y,t),\beta)F(t,\alpha)|}\right].$$

Since we have $\mathbb{F}(x,t)\to\mathbb{F}(y,t)$ as $x\to y$, then

$$\left|\frac{1}{F(\mathbb{F}(x,t),\beta)}-\frac{1}{F(\mathbb{F}(y,t),\beta)}\right|\to 0,$$

and also we get

$$\left|\frac{1}{F(\mathbb{F}(x,t),\beta)}-\frac{1}{F(\mathbb{F}(y,t),\beta)}\right|\leqslant \frac{2}{K_F(a,b)},$$

with $K_F(a,b)$ is a certain constant depending on the values of F, on the interval (a,b). Therefore, by dominated convergence theorem we obtain

$$\left|{}^{\beta}J^{\alpha}_{F,a+}f(x) - {}^{\beta}J^{\alpha}_{F,a+}f(y)\right|\to 0$$

as $x\to y$, i.e.,

$${}^{\beta}J^{\alpha}_{F,a+}f\in C[a,b]. \quad \square$$

Next, we discuss a desired property of the integral operator defined above: the commutativity and the semigroup property of the operator presented in Definition 8.

Hereinafter, the k-Γ Function will also be used, defined as follows.

$$\Gamma_k(z)=\int_0^{\infty}\tau^{z-1}e^{-\tau^k/k}\,\mathrm{d}\tau, k>0.$$

It is clear that if $k\to 1$ we have $\Gamma_k(z)\to\Gamma(z)$, $\Gamma_k(z)=(k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k)=z\Gamma_k(z)$. By other hand, we define the k-beta function as follows

$$B_k(u,v)=\frac{1}{k}\int_0^1\tau^{\frac{u}{k}-1}(1-\tau)^{\frac{v}{k}-1}d\tau,$$

notice that $B_k(u,v)=\frac{1}{k}B(\frac{u}{k},\frac{v}{k})$ and $B_k(u,v)=\frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}$.

REMARK 2. If F is additive, then the semigroup law is satisfied as shown below. For this we define, as a particular case of Definition (7) the following fractional integral operator

$$\left({}^{\frac{\beta}{k}}J^{\alpha}_{F,a+}f\right)(x)=\frac{1}{k\Gamma_k(\beta)}\int_a^x\frac{f(t)\,dt}{[\mathbb{F}(x,t)]^{1-\frac{\beta}{k}}F(t,\alpha)}$$

REMARK 3. This integral is not a minor case of the Definition 7, for example, if we consider $F(t,\alpha)=[h'(t)]^{-1}$ we obtain the (k,h)-operator fractional integral defined in [21] and if we put $F(t,\alpha)=t^{-\alpha}$ we obtain the (k,s)-operator of [29] (here the notation is changed).

THEOREM 7. *Let f be integrable function on $[a,b]$, $\beta\in\mathbb{R}$ and $\alpha\in[0,1]$. Then, we have*

$$ {}^{\frac{\beta}{k}}J^{\alpha}_{F,a+}\left({}^{\frac{\gamma}{k}}J^{\alpha}_{F,a+}f(x)\right)={}^{\frac{\beta+\gamma}{k}}J^{\alpha}_{F,a+}f(x)={}^{\frac{\gamma}{k}}J^{\alpha}_{F,a+}\left({}^{\frac{\beta}{k}}J^{\alpha}_{F,a+}f(x)\right), \tag{9}$$

for all $\alpha>0$, $\beta>0$, $\gamma>0$ $x\in[a,b]$.

Proof. Taking into account the remark 2 and Dirichlet's formula we have:

$$\begin{aligned}
&{}^{\frac{\beta}{k}}J^{\alpha}_{F,a+}\left({}^{\frac{\gamma}{k}}J^{\alpha}_{F,a+}f(x)\right)\\
&=\frac{1}{k\Gamma_k(\beta)}\int_a^x\frac{[\mathbb{F}(x,t)]^{\frac{\beta}{k}-1}}{F(t,\alpha)}\left(\frac{1}{k\Gamma_k(\gamma)}\int_a^t\frac{[\mathbb{F}(t,\tau)]^{\frac{\gamma}{k}-1}f(\tau)\,d\tau}{F(\tau,\alpha)}\right)dt\\
&=\frac{1}{k\Gamma_k(\beta)\,k\Gamma_k(\gamma)}\int_a^x\frac{f(\tau)}{F(\tau,\alpha)}\left(\int_\tau^x\frac{[\mathbb{F}(x,t)]^{\frac{\beta}{k}-1}[\mathbb{F}(t,\tau)]^{\frac{\gamma}{k}-1}}{F(t,\alpha)}dt\right)d\tau
\end{aligned}$$

Making $u=\frac{\mathbb{F}(t,\tau)}{\mathbb{F}(x,\tau)}$, we have

$$\begin{aligned}
\int_\tau^x\frac{[\mathbb{F}(x,t)]^{\frac{\beta}{k}-1}[\mathbb{F}(t,\tau)]^{\frac{\gamma}{k}-1}}{F(t,\alpha)}dt&=[\mathbb{F}(x,\tau)]^{\frac{\beta+\gamma}{k}-1}\int_0^1[u]^{\frac{\beta}{k}-1}[1-u]^{\frac{\gamma}{k}-1}du\\
&=[\mathbb{F}(x,\tau)]^{\frac{\beta+\gamma}{k}-1}B\left(\frac{\beta}{k},\frac{\gamma}{k}\right)
\end{aligned}$$

so

$$\begin{aligned}
&\frac{1}{k\Gamma_k(\beta)\,k\Gamma_k(\gamma)}\int_a^x\frac{f(\tau)}{F(\tau,\alpha)}[\mathbb{F}(x,\tau)]^{\frac{\beta}{k}-1}B\left(\frac{\beta}{k},\frac{\gamma}{k}\right)d\tau\\
&=\frac{B\left(\frac{\beta}{k},\frac{\gamma}{k}\right)}{k\Gamma_k(\beta)\,k\Gamma_k(\gamma)}\int_a^x\frac{f(\tau)}{F(\tau,\alpha)}[\mathbb{F}(x,\tau)]^{\frac{\beta+\gamma}{k}-1}d\tau\\
&=\frac{1}{k\Gamma_k(\beta+\gamma)}\int_a^x\frac{f(\tau)}{F(\tau,\alpha)}[L(x,\tau)]^{\frac{\beta+\gamma}{k}-1}d\tau\\
&=\left({}^{\frac{\beta+\gamma}{k}}J^{\alpha}_{F,a+}f\right)(x).
\end{aligned}$$

The second part of equality of (9) it is very easy to obtain. This completes the proof. □

Formally, we can define the fractional derivative, from Definition 8 as follows.

DEFINITION 9. Let α a real number satisfying $m-1<\alpha\leqslant m$ with $m\geqslant 1$ a positive integer. We call the left (and right) k-generalized fractional Riemann-Liouville derivative of order α to the defined by

$$N^{\alpha}_{F+}f(t)=\frac{d^m}{dt^m}\left[{}_{-}^{\beta}J^{m-\alpha}_{F,a+}f(t)\right], \tag{10}$$

$$N^{\alpha}_{F-}f(t)=\frac{d^m}{dt^m}\left[{}_{-}^{\beta}J^{m-\alpha}_{F,b-}f(t)\right]. \tag{11}$$

The above expressions are equivalent to

$$N^{\alpha}_{F+}f(t)=\begin{cases}\frac{d^m}{dt^m}\left[\frac{1}{\Gamma(\beta)}\int_a^t\frac{f(t)dt}{F(\mathbb{F}_{+}(x,t),\beta)F(t-a,\alpha)}\right], & m-1<\alpha\leqslant m,\\ \frac{d^m f(t)}{dt^m}, \quad \alpha=m,\end{cases} \tag{12}$$

and similarly to the right $N^{\alpha}_{F-}f(t)$.

REMARK 4. Another way to define these fractional derivatives is to consider them as the left (and right) inverse of the k-generalized fractional Riemann-Liouville integral with general kernel of order α, that is

$$N^{\alpha}_{F+}\left[{}_{-}^{\beta}J^{m-\alpha}_{F,a+}\right]=Id, \tag{13}$$

$$N^{\alpha}_{F-}\left[{}_{-}^{\beta}J^{m-\alpha}_{F,b-}\right]=Id. \tag{14}$$

REMARK 5. For convenience, we have written these last two results for $\alpha\in\mathbb{R}$, although they can be extended to $\alpha\in\mathbb{C}$, with $\Re(\alpha)>0$.

3. Hermite-Hadamard type inequalities

The following result will be used later in obtaining certain integral inequalities of the Hermite-Hadamard type.

LEMMA 1. *Let $\beta\in\mathbb{C}$ such that $Re(\beta)>0$, $I=[a,b]\subset\mathbb{R}$ and $f:I\to\mathbb{C}$ be an integrable function. Then*

$$\frac{1}{2}\left[\left({}^{\beta}J^{\alpha}_{F,a+}f\right)(x)+\left({}^{\beta}J^{\alpha}_{F,b-}f\right)(a+b-x)\right]=\left({}^{\beta}J^{\alpha}_{F,a+}\check{f}\right)(x) \tag{15}$$

and

$$\frac{1}{2}\left[\left({}^{\beta}J^{\alpha}_{F,a+}f\right)(a+b-x)+\left({}^{\beta}J^{\alpha}_{F,b-}f\right)(x)\right]=\left({}^{\beta}J^{\alpha}_{F,b-}\check{f}\right)(x) \tag{16}$$

Proof. We proof first (15). Starting from (5) that, for $a<x\leqslant b$,

$$\left({}^{\beta}J^{\alpha}_{F,b-}f\right)(x)=\frac{1}{\Gamma(\beta)}\int_x^b\frac{f(t)dt}{F(\mathbb{F}_{-}(t,x),\beta)F(b-t,\alpha)}$$

$$\left({}^{\beta}J^{\alpha}_{F,b-}f\right)(a+b-x)=\frac{1}{\Gamma(\beta)}\int_{a+b-x}^b\frac{f(t)dt}{F(\mathbb{F}_{-}(t,a+b-x),\beta)F(b-t,\alpha)} \tag{17}$$

Making a change of variable $t = a+b-u$ in the second member of (17), we obtain

$$\left({}^{\beta}J^{\alpha}_{F,b-}f\right)(a+b-x) = \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(a+b-u)du}{F(\mathbb{F}_-(a+b-u,a+b-x),\beta)F(u-a,\alpha)} \tag{18}$$

From (4), we have

$$\left({}^{\beta}J^{\alpha}_{F,a+}f\right)(x) = \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(t)dt}{F(\mathbb{F}_+(x,t),\beta)F(t-a,\alpha)} \tag{19}$$

Adding (18) and (19) member to member and using the definition 3, we get (15),

$$\begin{aligned}
&\left({}^{\beta}J^{\alpha}_{F,a+}f\right)(x) + \left({}^{\beta}J^{\alpha}_{F,b-}f\right)(a+b-x)\\
&= \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(t)dt}{F(\mathbb{F}_+(x,t),\beta)F(t-a,\alpha)}\\
&\quad + \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(a+b-u)du}{F(\mathbb{F}_-(a+b-u,a+b-x),\beta)F(u-a,\alpha)}\\
&= \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(t)}{F(\mathbb{F}_+(x,t),\beta)F(t-a,\alpha)}\\
&\quad + \frac{f(a+b-t)dt}{F(\mathbb{F}_-(a+b-t,a+b-x),\beta)F(t-a,\alpha)}dt\\
&= \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(t)+f(a+b-t)}{F(\mathbb{F}_+(x,t),\beta)F(t-a,\alpha)}dt
\end{aligned}$$

as

$$\begin{aligned}
\mathbb{F}_+(x,t) = \mathbb{F}_-(a+b-t,a+b-x) &= \int_{a+b-x}^{a+b-t} \frac{ds}{F(b-s,\alpha)}\\
&= \int_t^x \frac{ds}{F(b-(a+b-s),\alpha)} = \int_t^x \frac{ds}{F(s-a,\alpha)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{2}\left[\left({}^{\beta}J^{\alpha}_{F,a+}f\right)(x) + \left({}^{\beta}J^{\alpha}_{F,b-}f\right)(a+b-x)\right] &= \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(t)+f(a+b-t)}{F(\mathbb{F}_+(x,t),\beta)F(t-a,\alpha)}dt\\
&= \frac{1}{\Gamma(\beta)}\int_a^x \frac{\check{f}(t)}{F(\mathbb{F}_+(x,t),\beta)F(t-a,\alpha)}dt\\
&= \left({}^{\beta}J^{\alpha}_{F,a+}\check{f}\right)(x).
\end{aligned}$$

Analogously to (15) we can test equality (16). Details are left to the reader. □

REMARK 6. If we consider the kernel $F(t,\alpha) = t^{1-\alpha}$ in the previous result, we obtain Lemma 2.1 from [30].

Below we present Hermite-Hadamard inequalities involving the fractional integral operators (4) and (5).

THEOREM 8. *Let $\beta \in \mathbb{C}$ such that $Re(\beta) > 0$, $I = [a,b] \subset \mathbb{R}$ and $f : I \to \mathbb{C}$ be an integrable function. Then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\beta)}{\psi_{\alpha}^{\beta}(t-a)}\left[\left({}^{\beta}J_{F,a+}^{\alpha}\check{f}\right)(x)\right] \leqslant \frac{f(a)+f(b)}{2}. \tag{20}$$

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\beta)}{\psi_{\alpha}^{\beta}(b-t)}\left[\left({}^{\beta}J_{F,b-}^{\alpha}\check{f}\right)(x)\right] \leqslant \frac{f(a)+f(b)}{2}. \tag{21}$$

Proof. Following Theorem 3, since f is a convex symmetric function at $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right) \leqslant \check{f}(t) \leqslant \frac{f(a)+f(b)}{2} \tag{22}$$

If we multiply both sides of (22) by $\frac{1}{\Gamma(\beta)}\frac{1}{F(\mathbb{F}_{+}(x,t),\beta)F(t-a,\alpha)}$ and integrate from a to x $(a < t \leqslant x \leqslant b)$ with respect to the variable t, we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\frac{1}{\Gamma(\beta)}\int_{a}^{x}\frac{dt}{F(\mathbb{F}_{+}(x,t),\beta)F(t-a,\alpha)} \\ \leqslant & \frac{1}{\Gamma(\beta)}\int_{a}^{x}\frac{\check{f}(t)\,dt}{F(\mathbb{F}_{+}(x,t),\beta)F(t-a,\alpha)} \\ \leqslant & \frac{f(a)+f(b)}{2}\frac{1}{\Gamma(\beta)}\int_{a}^{x}\frac{dt}{F(\mathbb{F}_{+}(x,t),\beta)F(t-a,\alpha)} \end{aligned} \tag{23}$$

We obtain the desired inequality (20) by means of the relation of (15) with the second member of (23), and considering the following relation in the first and third terms:

$$\int_{a}^{x}\frac{dt}{F(\mathbb{F}_{+}(x,t),\beta)F(t-a,\alpha)} = \psi_{\alpha}^{\beta}(t-a)$$

Analogously to (20), we can prove (21). This completes the proof. □

REMARK 7. This result contains as a particular case Theorem 2.1 of [30], obtained from the previous putting $F(t,\alpha) = t^{1-\alpha}$.

In the following result we present some equalities referring to (4) and (5).

LEMMA 2. *Let $\beta \in \mathbb{C}$ such that $Re(\beta) > 0$, $I = [a,b] \subset \mathbb{R}$ and $f : I \to \mathbb{C}$ an integrable function. Then we have*

$$\frac{1}{2}\left[\left({}^{\beta}J_{F,x-}^{\alpha}f\right)(a) + \left({}^{\beta}J_{F,(a+b-x)+}^{\alpha}f\right)(b)\right] = \left({}^{\beta}J_{F,a-}^{\alpha}\check{f}\right)(x) \tag{24}$$

and

$$\frac{1}{2}\left[\left({}^{\beta}J_{F,(a+b-x)-}^{\alpha}f\right)(a) + \left({}^{\beta}J_{F,x+}^{\alpha}f\right)(b)\right] = \left({}^{\beta}J_{F,b+}^{\alpha}\check{f}\right)(x) \tag{25}$$

Proof. From (4), we have

$$\left({}^{\beta}J^{\alpha}_{F,(a+b-x)+}f\right)(b)=\frac{1}{\Gamma(\beta)}\int_{a+b-x}^{b}\frac{f(t)dt}{F(\mathbb{F}_{-}(t,a+b-x),\beta)F(b-t,\alpha)} \tag{26}$$

Making $u=a+b-t$ in (26), we obtain

$$\left({}^{\beta}J^{\alpha}_{F,(a+b-x)+}f\right)(b)=-\frac{1}{\Gamma(\beta)}\int_{x}^{a}\frac{f(a+b-u)du}{F(\mathbb{F}_{-}(a+b-u,a+b-x),\beta)F(u-a,\alpha)}.$$

Using (5), we get

$$\left({}^{\beta}J^{\alpha}_{F,x-}f\right)(a)=\frac{1}{\Gamma(\beta)}\int_{a}^{x}\frac{f(t)dt}{F(\mathbb{F}_{-}(t,x),\beta)F(b-t,\alpha)} \tag{27}$$

Finally, adding (26) and (27) member to member, and in view of the definition 3, we obtain the desired equality (24).

$$\begin{aligned}&\left({}^{\beta}J^{\alpha}_{F,x-}f\right)(a)+\left({}^{\beta}J^{\alpha}_{F,(a+b-x)+}f\right)(b)\\&=\frac{1}{\Gamma(\beta)}\int_{a}^{x}\frac{f(t)dt}{F(\mathbb{F}_{-}(t,x),\beta)F(b-t,\alpha)}\\&\quad-\frac{1}{\Gamma(\beta)}\int_{x}^{a}\frac{f(a+b-t)dt}{F(\mathbb{F}_{-}(a+b-u,a+b-x),\beta)F(u-a,\alpha)}\\&=\frac{1}{\Gamma(\beta)}\int_{a}^{x}\frac{f(t)}{F(\mathbb{F}_{-}(t,x),\beta)F(b-t,\alpha)}+\frac{f(a+b-t)}{F(\mathbb{F}_{-}(a+b-u,a+b-x),\beta)F(u-a,\alpha)}dt\end{aligned}$$

therefore

$$\begin{aligned}\frac{1}{2}\left[\left({}^{\beta}J^{\alpha}_{F,x-}f\right)(a)+\left({}^{\beta}J^{\alpha}_{F,(a+b-x)+}f\right)(b)\right]&=\frac{1}{\Gamma(\beta)}\int_{a}^{x}\frac{\check{f}(t)}{F(\mathbb{F}_{-}(t,x),\beta)F(b-t,\alpha)}\\&=\left({}^{\beta}J^{\alpha}_{F,a-}\check{f}\right)(x)\end{aligned}$$

where

$$\begin{aligned}\mathbb{F}_{-}(a+b-u,a+b-x)&=\int_{a+b-x}^{a+b-u}\frac{ds}{F(b-s,\alpha)}=-\int_{x}^{u}\frac{dz}{F(z-a,\alpha)}\\&=\int_{u}^{x}\frac{dz}{F(z-a,\alpha)}=\mathbb{F}_{+}(x,t).\end{aligned}$$

Obtaining of (25) is analogous to that of (24). In this way we end the proof. □

REMARK 8. As in the previous two remarks, this result covers Lemma 2.2 of [30].

THEOREM 9. *Let $\beta\in\mathbb{C}$ such that $Re(\beta)>0$, $I=[a,b]\subset\mathbb{R}$ and $f:I\to\mathbb{C}$ an integrable function. Then we have*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{\Gamma(\beta)}{\psi^{\beta}_{\alpha}(x-a)}\left[{}^{\beta}J^{\alpha}_{F,x-}f(a)+{}^{\beta}J^{\alpha}_{F,(a+b-x)+}f(b)\right]\leqslant\frac{f(a)+f(b)}{2}$$

and

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\beta)}{\psi_{\alpha}^{\beta}(b-x)}\left[{}^{\beta}J_{F,(a+b-x)-}^{\alpha}f(a) + {}^{\beta}J_{F,x+}^{\alpha}f(b)\right] \leqslant \frac{f(a)+f(b)}{2}.$$

Proof. The proof is analogous to theorem 8. □

Below we present several inequalities of the Hermite-Hadamard type for integral operators (4) and (5).

THEOREM 10. *Let $\beta \in \mathbb{C}$ such that $Re(\beta) > 0$, $I = [a,b]$ a real interval and $f : I \to \mathbb{C}$ be an integrable function. Also let $f : [a,b] \to \mathbb{R}$ be Wright-quasi-convex and integrable on $[a,b]$. Then*

$$\frac{\left({}^{\beta}J_{F,a+}^{\alpha}\check{f}\right)(x)}{\psi_{\alpha}^{\beta}(x-a)} = \frac{\left(\psi^{-1}\right)_{\alpha}^{\beta}(x-a)}{2}\left[\left({}^{\beta}J_{F,a+}^{\alpha}f\right)(x) + \left({}^{\beta}J_{F,b-}^{\alpha}f\right)(a+b-x)\right] \leqslant \max\{f(a), f(b)\} \tag{28}$$

$$\frac{\left({}^{\beta}J_{F,a+}^{\alpha}\check{f}\right)(b)}{\psi_{\alpha}^{\beta}(b-a)} = \frac{\left(\psi^{-1}\right)_{\alpha}^{\beta}(b-a)}{2}\left[\left({}^{\beta}J_{F,a+}^{\alpha}f\right)(b) + \left({}^{\beta}J_{F,b-}^{\alpha}f\right)(a)\right] \leqslant \max\{f(a), f(b)\} \tag{29}$$

$$\frac{\left({}^{\beta}J_{F,a+}^{\alpha}\check{f}\right)\left(\frac{a+b}{2}\right)}{\psi_{\alpha}^{\beta}\left(\frac{b-a}{2}\right)} = \frac{\left(\psi^{-1}\right)_{\alpha}^{\beta}\left(\frac{b-a}{2}\right)}{2}\left[\left({}^{\beta}J_{F,a+}^{\alpha}f\right)\left(\frac{a+b}{2}\right) + \left({}^{\beta}J_{F,b-}^{\alpha}f\right)\left(\frac{a+b}{2}\right)\right] \leqslant \max\{f(a), f(b)\} \tag{30}$$

Proof. Since $f : [a,b] \to \mathbb{R}$ is Wright-quasi-convex in $[a,b]$, making $x = a$, $y = b$ and $t = \frac{s-a}{b-a} \in [0,1]$ for $s \in [a,b]$ in (2), we obtain

$$\check{f}(s) = \frac{1}{2}\left[f(a+b-s) + f(s)\right] \leqslant \max\{f(a), f(b)\} \tag{31}$$

If by multiplying both members of (31) by $\frac{1}{\Gamma(\beta)}\frac{1}{F(\mathbb{F}_{+}(x,s),\beta)F(s-a,\alpha)}$ and integrating each term with respect to s from a to x $(a < s \leqslant x \leqslant b)$, we obtain

$$\frac{1}{\Gamma(\beta)}\int_{a}^{x}\frac{\check{f}(s)ds}{F(\mathbb{F}_{+}(x,s),\beta)F(s-a,\alpha)} \leqslant \max\{f(a), f(b)\}\frac{1}{\Gamma(\beta)}\int_{a}^{x}\frac{ds}{F(\mathbb{F}_{+}(x,s),\beta)F(s-a,\alpha)} \tag{32}$$

Using (15) to the first member of (32), we obtain

$$\frac{1}{2}\left[\left({}^{\beta}J_{F,a+}^{\alpha}f\right)(x) + \left({}^{\beta}J_{F,b-}^{\alpha}f\right)(a+b-x)\right] \leqslant \max\{f(a), f(b)\}\,\psi_{\alpha}^{\beta}(x-a),$$

which gives us the desired inequality (28).

Taking $x=b$ and $x=\frac{a+b}{2}$ in (28) give us the inequalities (29) and (30), respectively. This completes the proof. □

THEOREM 11. *Let $\beta\in\mathbb{C}$ such that $Re(\beta)>0$, $I=[a,b]$ a real interval and $f:I\to\mathbb{C}$ be an integrable function. Also let $f:[a,b]\to\mathbb{R}$ be Wright-quasi-convex and integrable on $[a,b]$. Then*

$$\frac{\Gamma(\beta)}{\psi_{\alpha}^{\beta}(t-a)}\left[{}^{\beta}J_{F,x-}^{\alpha}f(a)+{}^{\beta}J_{F,(a+b-x)+}^{\alpha}f(b)\right]\leqslant\max\{f(a),f(b)\}. \tag{33}$$

Proof. Analogous to the proof of Theorem 10, inequality 33 is established. □

THEOREM 12. *Suppose that the function $f:[a,b]\to[0,\infty)$ is h-symmetrized convex in the interval $[a,b]$, h and f are integrable in $[0,1]$ and $[a,b]$ respectively. Then*

$$\begin{aligned}&\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\frac{1}{\Gamma(\beta)}\psi_{\alpha}^{\beta}(t-a)\leqslant\left({}^{\beta}J_{F,a+}^{\alpha}\check{f}\right)(x)\\&\leqslant\frac{f(a)+f(b)}{2\Gamma(\beta)}\int_0^1\frac{1}{F(\mathbb{F}_+(x,(1-s)a+sx),\beta)F(s(x-a),\alpha)}\\&\quad\times\left[h\left(1-\frac{x-a}{b-a}s\right)+h\left(\frac{x-a}{b-a}s\right)\right](x-a)\,ds.\end{aligned}$$

Proof. Since h-symmetrized convex function we obtain

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leqslant\check{f}(t)\leqslant\left[h\left(\frac{b-t}{b-a}\right)+h\left(\frac{t-a}{b-a}\right)\right]\frac{f(a)+f(b)}{2}.$$

To prove the first inequality, we multiply the above by $\frac{1}{\Gamma(\beta)}\frac{1}{F(\mathbb{F}_+(x,t),\beta)F(t-a,\alpha)}$ and integrating with respect to t into $[a,x]$, we obtain

$$\begin{aligned}&\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\frac{1}{\Gamma(\beta)}\int_a^x\frac{1}{F(\mathbb{F}_+(x,t),\beta)F(t-a,\alpha)}\\&\leqslant\frac{1}{2}\left[\left({}^{\beta}J_{F,a+}^{\alpha}f\right)(x)+\left({}^{\beta}J_{F,b-}^{\alpha}f\right)(a+b-x)\right]\end{aligned}$$

from where we get the first inequality.

Similarly, if we multiply each term of the second inequality by $\frac{1}{\Gamma(\beta)}\frac{1}{F(\mathbb{F}_+(x,t),\beta)F(t-a,\alpha)}$ and integrating with respect to t into $[a,x]$, we obtain the other inequality

$$\begin{aligned}&\frac{1}{2}\left[\left({}^{\beta}J_{F,a+}^{\alpha}f\right)(x)+\left({}^{\beta}J_{F,b-}^{\alpha}f\right)(a+b-x)\right]\\&\leqslant\frac{f(a)+f(b)}{2\Gamma(\beta)}\int_a^x\frac{1}{F(\mathbb{F}_+(x,t),\beta)F(t-a,\alpha)}\left[h\left(\frac{b-t}{b-a}\right)+h\left(\frac{t-a}{b-a}\right)\right]dt\end{aligned}$$

for any $a < x \leqslant b$.

If we change the variable with $t = (1-s)a + sx$ for $s \in [0,1]$, i.e. $dt = (x-a)ds$, $\frac{b-t}{b-a} = 1 - \frac{b-t}{x-a}s$, $\frac{t-a}{b-a} = \frac{x-a}{b-a}s$ and $x - t = (1-s)(x-a)$, then we have

$$\begin{aligned}
&\frac{1}{2}\left[\left({}^{\beta}J^{\alpha}_{F,a+}f\right)(x) + \left({}^{\beta}J^{\alpha}_{F,b-}f\right)(a+b-x)\right] \\
&\leqslant \frac{f(a)+f(b)}{2\Gamma(\beta)} \int_0^1 \frac{1}{F(\mathbb{F}_+(x,(1-s)a+sx),\beta)F(s(x-a),\alpha)} \\
&\times \left[h\left(1 - \frac{x-a}{b-a}s\right) + h\left(\frac{x-a}{b-a}s\right)\right](x-a)ds \quad \square
\end{aligned}$$

REMARK 9. The theorems previously proved, contain as particular cases Theorem 2.2 and 2.3 of [30].

REMARK 10. Obviously our results generalize those obtained for Riemann-Liouville fractional integral operators (see [3]), which are obtained from operators (4) and (5) making $F(t,\alpha) = t^{1-\alpha}$ and $\beta = 1$.

4. Conclusions

In this work we have obtained various inequalities of the Hermite-Hadamard type, in the case of different notions of convexity, and using a generalized fractional operator, which allows obtaining as particular cases, several of those reported in the literature.

We want to point out, in addition to the observations made throughout the work, that with different choices of the F kernel we can obtain, as particular cases, several well-known integral operators. So, for example, if

1. $F(t,\alpha) = t^{-\alpha}$, $\alpha = 1$ and $\beta = 1$ we obtain the classic Riemann integral.
2. $F(t,\alpha) = t^{-\alpha}$ and $\beta = 1$ we have the fractional Riemann-Liouville integral.
3. Following the idea of the Remark 2, if we put $F(t,\alpha) = t^{-\alpha}$ with $\alpha = 1$, we can write the right sided operator as follows $\left({}^{\frac{\beta}{k}}J^{\alpha}_{F,a+}f\right)(x) = \frac{1}{k\Gamma_k(\beta)}\int_a^x \frac{f(t)dt}{(x-t)^{1-\frac{\beta}{k}}}$ and similarly the left sided integral. The k-Riemann-Liouville fractional integral of Mubeen and Habibullah (see [22]).
4. Taking $F(t,\alpha) = t^{-\alpha}$, we would obtain the Katugampola fractional integral of [19] (the notation is changed).
5. If we put $F = t^{\alpha}$ with $\alpha = 1$, then we get the right sided Hadamard fractional integral of [14].
6. An integral operator with non-singular kernel can also be obtained from our Definition 8. Thus, considering $F(t,\alpha) = exp\left[\frac{1-\alpha}{\alpha}t\right]$, if $\alpha = 1$ we have that $F = 1$. In this case $F(\mathbb{F}_+(x,t),\beta) = exp\left[\frac{1-\beta}{\beta}(x-t)\right]$, a slight modification of the operator defined by Kirane and Toberek in [1].

Considering the above, the following open question becomes clear. The integral inequalities obtained using the particular cases analyzed in I)-VI), can be generalized within the framework of our generalized operators of Definitions 7 and 8.

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