ON REFINEMENTS OF HERMITE–HADAMARD TYPE INEQUALITIES
WITH GENERALIZED FRACTIONAL INTEGRAL OPERATORS

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Abstract. In this paper we establish the refinements of Hermite-Hadamard type inequalities for generalized fractional integral operator through the instrument of convex functions.

1. Introduction

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions such as using convex mappings.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [15, p. 137], [6]). These inequalities state that if $F : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$, then

$$F\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leqslant \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\varkappa)d\varkappa \leqslant \frac{F(\kappa_1) + F(\kappa_2)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if F is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [2]–[5], [7]–[10], [12]–[14], [17]–[26]) and the references cited therein.

2. Preliminaries

Now, we reviewed some definitions and theorems which will be used in the proof of our main cumulative results.

In [16], Raina defined the following results connected with the general class of fractional integral operators.

$$\mathscr{F}_{\rho,\lambda}^{\sigma}(\varkappa) = \mathscr{F}_{\rho,\lambda}^{\sigma(0),\sigma(1),\cdots}(\varkappa) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} \varkappa^k \quad (\rho,\lambda > 0; |\varkappa| < \mathscr{R}), \quad (2)$$

where the coefficients $\sigma(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers and $\mathscr{R}$ is the set of real numbers. With the help of (2), in [16] and [1], Raina and Agarwal et. al defined the following left-sided and right-sided fractional integral operators, respectively, as follows:

$$\mathscr{J}_{\rho,\lambda,\kappa_1+;\omega}^{\sigma}F(\varkappa) = \int_{\kappa_1}^{\varkappa} (\varkappa - \xi)^{\lambda-1} \mathscr{F}_{\rho,\lambda}^{\sigma} \left[\omega(\varkappa - \xi)^{\rho} \right] F(\xi)d\xi, \quad \varkappa > \kappa_1, \quad (3)$$

$$\mathscr{J}_{\rho,\lambda,\kappa_2-;\omega}^{\sigma}F(\varkappa) = \int_{\varkappa}^{\kappa_2} (\xi - \varkappa)^{\lambda-1} \mathscr{F}_{\rho,\lambda}^{\sigma} \left[\omega(\xi - \varkappa)^{\rho} \right] F(\xi)d\xi, \quad \varkappa < \kappa_2, \quad (4)$$

where $\lambda, \rho > 0, \omega \in \mathbb{R}$, and $F(\xi)$ is such that the integrals on the right side exists.

It is easy to verify that $\mathscr{J}_{\rho,\lambda,\kappa_1+;\omega}^{\alpha}F(\varkappa)$ and $\mathscr{J}_{\rho,\lambda,\kappa_2-;\omega}^{\alpha}F(\varkappa)$ are bounded integral operators on $L(\kappa_1,\kappa_2)$, if

$$\mathfrak{M} := \mathscr{F}_{\rho,\lambda+1}^{\sigma} \left[\omega(\kappa_2 - \kappa_1)^{\rho} \right] < \infty. \quad (5)$$

In fact, for $F \in L(\kappa_1,\kappa_2)$, we have

$$\left\| \mathscr{J}_{\rho,\lambda,\kappa_1+;\omega}^{\alpha}F(\varkappa) \right\|_1 \leqslant \mathfrak{M}(\kappa_2 - \kappa_1)^{\lambda} \|F\|_1 \quad (6)$$

and

$$\left\| \mathscr{J}_{\rho,\lambda,\kappa_2-;\omega}^{\alpha}F(\varkappa) \right\|_1 \leqslant \mathfrak{M}(\kappa_2 - \kappa_1)^{\lambda} \|F\|_1, \quad (7)$$

where

$$\|F\|_p := \left(\int_{\kappa_1}^{\kappa_2} |F(\xi)|^p \, d\xi \right)^{\frac{1}{p}}.$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Here, we just point out that the classical Riemann-Liouville fractional integrals $I_{\kappa_1^+}^{\alpha}$ and $I_{\kappa_2^-}^{\alpha}$ of order α defined by (see, [11])

$$\left(I_{\kappa_1^+}^{\alpha}F \right)(\varkappa) := \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\varkappa} (\varkappa - \xi)^{\alpha-1} F(\xi)d\xi \quad (\varkappa > \kappa_1; \alpha > 0) \quad (8)$$

and

$$\left(I_{\kappa_2^-}^{\alpha} F\right)(\varkappa) := \frac{1}{\Gamma(\alpha)} \int_{\varkappa}^{\kappa_2} (\xi - \varkappa)^{\alpha-1} F(\xi) d\xi \quad (\varkappa < \kappa_2; \alpha > 0) \tag{9}$$

follow easily by setting

$$\lambda = \alpha, \ \sigma(0) = 1, \ \text{and } w = 0 \tag{10}$$

in (3) and (4), and the boundedness of (8) and (9) on $L(\kappa_1, \kappa_2)$ is also inherited from (6) and (7), (see, [1]).

The following Lemma will be very useful when we prove the main theorems.

LEMMA 1. [23, 25] *Let* $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ *be a convex function and* h *be defined by*

$$h(\xi) = \frac{1}{2} \left[F\left(\frac{\kappa_1 + \kappa_2}{2} - \frac{\xi}{2} \right) + F\left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\xi}{2} \right) \right].$$

Then h *is convex, increasing on* $[0, \kappa_2 - \kappa_1]$ *and for all* $\xi \in [0, \kappa_2 - \kappa_1]$,

$$F\left(\frac{\kappa_1 + \kappa_2}{2} \right) \leqslant h(\xi) \leqslant \frac{F(\kappa_1) + F(\kappa_2)}{2}.$$

In [23], Xiang obtained following important inequalities for the Riemann-Liouville fractional integrals utilizing the Lemma 1:

THEOREM 1. *Let* $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ *be a positive function with* $\kappa_1 < \kappa_2$ *and* $F \in L_1[\kappa_1, \kappa_2]$. *If* F *is a convex function on* $[\kappa_1, \kappa_2]$, *then* WH *is convex and monotonically increasing on* $[0, 1]$ *and*

$$F\left(\frac{\kappa_1 + \kappa_2}{2} \right) = WH(0) \leqslant WH(\xi) \leqslant WH(1) \tag{11}$$

$$= \frac{\Gamma(1 + \alpha)}{2(\kappa_2 - \kappa_1)^{\alpha}} \left[\left(I_{\kappa_1^+}^{\alpha} F \right)(\kappa_2) + \left(I_{\kappa_2^-}^{\alpha} F \right)(\kappa_1) \right]$$

with $\alpha > 0$ *where*

$$WH(\xi) = \frac{\alpha}{2(\kappa_2 - \kappa_1)^{\alpha}} \int_{\kappa_1}^{\kappa_2} F\left(\xi \varkappa + (1 - \xi) \frac{\kappa_1 + \kappa_2}{2} \right) \left((\kappa_2 - \varkappa)^{\alpha-1} + (\varkappa - \kappa_1)^{\alpha-1} \right) d\varkappa.$$

THEOREM 2. *Let* $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ *be a positive function with* $\kappa_1 < \kappa_2$ *and* $F \in L_1[\kappa_1, \kappa_2]$. *If* F *is a convex function on* $[\kappa_1, \kappa_2]$, *then* WP *is convex and monotonically increasing on* $[0, 1]$ *and*

$$\frac{\Gamma(1 + \alpha)}{2(\kappa_2 - \kappa_1)^{\alpha}} \left[\left(I_{\kappa_1^+}^{\alpha} F \right)(\kappa_2) + \left(I_{\kappa_2^-}^{\alpha} F \right)(\kappa_1) \right] = WP(0) \leqslant WP(\xi) \leqslant WP(1) \tag{12}$$

$$= \frac{F(\kappa_1) + F(\kappa_2)}{2}$$

with $\alpha > 0$ where

$$WP(\xi) = \frac{\alpha}{4\left(\kappa_2 - \kappa_1\right)^{\alpha}} \int\limits_{\kappa_1}^{\kappa_2} \left[F\left(\left(\frac{1+\xi}{2}\right)\kappa_1 + \left(\frac{1-\xi}{2}\right)\varkappa \right) \right.$$

$$\times \left(\left(\frac{2\kappa_2 - \kappa_1 - \varkappa}{2}\right)^{\alpha-1} + \left(\frac{\varkappa - \kappa_1}{2}\right)^{\alpha-1} \right)$$

$$+ F\left(\left(\frac{1+\xi}{2}\right)\kappa_2 + \left(\frac{1-\xi}{2}\right)\varkappa \right)$$

$$\left. \times \left(\left(\frac{\kappa_2 - \varkappa}{2}\right)^{\alpha-1} + \left(\frac{\varkappa + \kappa_2 - 2\kappa_1}{2}\right)^{\alpha-1} \right) \right] d\varkappa.$$

In [24], Yaldiz and Sarikaya gave the following Hermite-Hadamard inequality for the generalized fractional integral operators:

THEOREM 3. *Let $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex function on $[\kappa_1, \kappa_2]$ with $\kappa_1 < \kappa_2$, then the following inequalities for fractional integral operators hold*

$$F\left(\frac{\kappa_1 + \kappa_2}{2}\right)$$

$$\leqslant \frac{1}{2\left(\kappa_2 - \kappa_1\right)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma}\left[\omega\left(\kappa_2 - \kappa_1\right)^{\rho}\right]} \left[\mathcal{J}_{\rho,\lambda,\kappa_1+;\omega}^{\sigma} F\left(\kappa_2\right) + \mathcal{J}_{\rho,\lambda,\kappa_2-;\omega}^{\sigma} F\left(\kappa_1\right) \right]$$

$$\leqslant \frac{F\left(\kappa_1\right) + F\left(\kappa_2\right)}{2}$$

with $\lambda, \rho > 0, \omega \geqslant 0$.

The main purpose of this paper is to obtain the refinements of Hermite Hadamard type inequalities with the aid of generalized fractional integral operators which are the generalizations of the inequalities (11) and (12).

3. Refinements of Hermite-Hadamard type inequalities for generalized fractional integral operators

In this section, we will present two theorems for Hermite-Hadamard type inequalities with generalized fractional integral operators which is the refinements of previous work.

THEOREM 4. *Let $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a positive function with $\kappa_1 < \kappa_2$ and $F \in L_1[\kappa_1, \kappa_2]$. If F is a convex function on $[\kappa_1, \kappa_2]$, then BS is convex and monotonically increasing on $[0, 1]$ and for all $\rho, \lambda > 0$ and $w \geqslant 0$ we have the following inequalities*

for generalized fractional integral operators:

$$
F\left(\frac{\kappa_1+\kappa_2}{2}\right)
$$

$$
= BS(0) \leqslant BS(\xi) \leqslant BS(1) \tag{13}
$$

$$
= \frac{1}{2\left(\kappa_2-\kappa_1\right)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma\left[\omega\left(\kappa_2-\kappa_1\right)^\rho\right]}\left[\mathscr{J}_{\rho,\lambda,\kappa_1+;\omega}^\sigma F\left(\kappa_2\right)+\mathscr{J}_{\rho,\lambda,\kappa_2-;\omega}^\sigma F\left(\kappa_1\right)\right]
$$

where

$$
BS(\xi) = \frac{1}{2\left(\kappa_2-\kappa_1\right)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma\left[\omega\left(\kappa_2-\kappa_1\right)^\rho\right]} \int_{\kappa_1}^{\kappa_2} F\left(\xi\varkappa+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)
$$

$$
\times\left[\left(\kappa_2-\varkappa\right)^{\lambda-1}\mathscr{F}_{\rho,\lambda}^\sigma\left[\omega\left(\kappa_2-\varkappa\right)^\rho\right]+\left(\varkappa-\kappa_1\right)^{\lambda-1}\mathscr{F}_{\rho,\lambda+1}^\sigma\left[\omega\left(\varkappa-\kappa_1\right)^\rho\right]\right]d\varkappa
$$

and the coefficients $\sigma(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ are bounded sequences of positive real numbers.

    Proof. Firstly, $\xi_1,\ \xi_2,\ \beta\in[0,1]$, then

$$
BS((1-\beta)\xi_1+\beta\xi_2)
$$

$$
= \frac{1}{2\left(\kappa_2-\kappa_1\right)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma\left[\omega\left(\kappa_2-\kappa_1\right)^\rho\right]}
$$

$$
\times\int_{\kappa_1}^{\kappa_2} F\left(\left(\varkappa-\frac{\kappa_1+\kappa_2}{2}\right)((1-\beta)\xi_1+\beta\xi_2)+\frac{\kappa_1+\kappa_2}{2}\right)
$$

$$
\times\left[\left(\kappa_2-\varkappa\right)^{\lambda-1}\mathscr{F}_{\rho,\lambda}^\sigma\left[\omega\left(\kappa_2-\varkappa\right)^\rho\right]+\left(\varkappa-\kappa_1\right)^{\lambda-1}\mathscr{F}_{\rho,\lambda+1}^\sigma\left[\omega\left(\varkappa-\kappa_1\right)^\rho\right]\right]d\varkappa
$$

$$
= \frac{1}{2\left(\kappa_2-\kappa_1\right)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma\left[\omega\left(\kappa_2-\kappa_1\right)^\rho\right]}
$$

$$
\times\int_{\kappa_1}^{\kappa_2} F\left((1-\beta)\left[\left(\varkappa-\frac{\kappa_1+\kappa_2}{2}\right)\xi_1+\frac{\kappa_1+\kappa_2}{2}\right]+\beta\left[\left(\varkappa-\frac{\kappa_1+\kappa_2}{2}\right)\xi_2+\frac{\kappa_1+\kappa_2}{2}\right]\right)
$$

$$
\times\left[\left(\kappa_2-\varkappa\right)^{\lambda-1}\mathscr{F}_{\rho,\lambda}^\sigma\left[\omega\left(\kappa_2-\varkappa\right)^\rho\right]+\left(\varkappa-\kappa_1\right)^{\lambda-1}\mathscr{F}_{\rho,\lambda+1}^\sigma\left[\omega\left(\varkappa-\kappa_1\right)^\rho\right]\right]d\varkappa.
$$

As F is convex, we have

$$
F\left((1-\beta)\left[\left(\varkappa-\frac{\kappa_1+\kappa_2}{2}\right)\xi_1+\frac{\kappa_1+\kappa_2}{2}\right]+\beta\left[\left(\varkappa-\frac{\kappa_1+\kappa_2}{2}\right)\xi_2+\frac{\kappa_1+\kappa_2}{2}\right]\right)
$$

$$
\leqslant (1-\beta)F\left(\left(\varkappa-\frac{\kappa_1+\kappa_2}{2}\right)\xi_1+\frac{\kappa_1+\kappa_2}{2}\right)+\beta F\left(\left(\varkappa-\frac{\kappa_1+\kappa_2}{2}\right)\xi_2+\frac{\kappa_1+\kappa_2}{2}\right).
$$

Therefore, we have

$$
\begin{aligned}
& BS((1-\beta)\xi_1 + \beta\xi_2) \\
& \leqslant \frac{1}{2(\kappa_2 - \kappa_1)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega(\kappa_2 - \kappa_1)^\rho\right]} \\
& \times \int_{\kappa_1}^{\kappa_2} \left[(1-\beta)F\left(\left(\varkappa - \frac{\kappa_1 + \kappa_2}{2}\right)\xi_1 + \frac{\kappa_1 + \kappa_2}{2}\right) \right. \\
& \left. + \beta F\left(\left(\varkappa - \frac{\kappa_1 + \kappa_2}{2}\right)\xi_2 + \frac{\kappa_1 + \kappa_2}{2}\right) \right] \\
& \times \left[(\kappa_2 - \varkappa)^{\lambda-1}\mathscr{F}_{\rho,\lambda}^\sigma\left[\omega(\kappa_2 - \varkappa)^\rho\right] + (\varkappa - \kappa_1)^{\lambda-1}\mathscr{F}_{\rho,\lambda+1}^\sigma\left[\omega(\varkappa - \kappa_1)^\rho\right] \right] d\varkappa \\
& = \frac{1-\beta}{2(\kappa_2 - \kappa_1)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega(\kappa_2 - \kappa_1)^\rho\right]} \int_{\kappa_1}^{\kappa_2} F\left(\left(\varkappa - \frac{\kappa_1 + \kappa_2}{2}\right)\xi_1 + \frac{\kappa_1 + \kappa_2}{2}\right) \\
& \times \left[(\kappa_2 - \varkappa)^{\lambda-1}\mathscr{F}_{\rho,\lambda}^\sigma\left[\omega(\kappa_2 - \varkappa)^\rho\right] + (\varkappa - \kappa_1)^{\lambda-1}\mathscr{F}_{\rho,\lambda+1}^\sigma\left[\omega(\varkappa - \kappa_1)^\rho\right] \right] d\varkappa \\
& + \frac{\beta}{2(\kappa_2 - \kappa_1)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega(\kappa_2 - \kappa_1)^\rho\right]} \int_{\kappa_1}^{\kappa_2} F\left(\left(\varkappa - \frac{\kappa_1 + \kappa_2}{2}\right)\xi_2 + \frac{\kappa_1 + \kappa_2}{2}\right) \\
& \times \left[(\kappa_2 - \varkappa)^{\lambda-1}\mathscr{F}_{\rho,\lambda}^\sigma\left[\omega(\kappa_2 - \varkappa)^\rho\right] + (\varkappa - \kappa_1)^{\lambda-1}\mathscr{F}_{\rho,\lambda+1}^\sigma\left[\omega(\varkappa - \kappa_1)^\rho\right] \right] d\varkappa \\
& = (1-\beta)BS(\xi_1) + \beta BS(\xi_2).
\end{aligned}
$$

Thus, we deduce that BS is convex on $[0,1]$.

Then, using the change of variable, we obtain

$$
\begin{aligned}
BS(\xi) & = \frac{1}{2(\kappa_2 - \kappa_1)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega(\kappa_2 - \kappa_1)^\rho\right]} \int_{\kappa_1}^{\kappa_2} F\left(\xi\varkappa + (1-\xi)\frac{\kappa_1 + \kappa_2}{2}\right) \\
& \times \left[(\kappa_2 - \varkappa)^{\lambda-1}\mathscr{F}_{\rho,\lambda}^\sigma\left[\omega(\kappa_2 - \varkappa)^\rho\right] + (\varkappa - \kappa_1)^{\lambda-1}\mathscr{F}_{\rho,\lambda+1}^\sigma\left[\omega(\varkappa - \kappa_1)^\rho\right] \right] d\varkappa \\
& = \frac{1}{2(\kappa_2 - \kappa_1)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega(\kappa_2 - \kappa_1)^\rho\right]} \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} F\left(\xi\varkappa + (1-\xi)\frac{\kappa_1 + \kappa_2}{2}\right) \\
& \times \left[(\kappa_2 - \varkappa)^{\lambda-1}\mathscr{F}_{\rho,\lambda}^\sigma\left[\omega(\kappa_2 - \varkappa)^\rho\right] + (\varkappa - \kappa_1)^{\lambda-1}\mathscr{F}_{\rho,\lambda+1}^\sigma\left[\omega(\varkappa - \kappa_1)^\rho\right] \right] d\varkappa \\
& + \frac{1}{2(\kappa_2 - \kappa_1)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega(\kappa_2 - \kappa_1)^\rho\right]} \int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2} F\left(\xi\varkappa + (1-\xi)\frac{\kappa_1 + \kappa_2}{2}\right) \\
& \times \left[(\kappa_2 - \varkappa)^{\lambda-1}\mathscr{F}_{\rho,\lambda}^\sigma\left[\omega(\kappa_2 - \varkappa)^\rho\right] + (\varkappa - \kappa_1)^{\lambda-1}\mathscr{F}_{\rho,\lambda+1}^\sigma\left[\omega(\varkappa - \kappa_1)^\rho\right] \right] d\varkappa
\end{aligned}
$$

$$= \frac{1}{2\left(\kappa_2 - \kappa_1\right)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega\left(\kappa_2 - \kappa_1\right)^\rho\right]}$$

$$\times \int_0^{\kappa_2 - \kappa_1} F\left(\frac{\kappa_1 + \kappa_2}{2} - \frac{u\xi}{2}\right) \left[\left(\frac{\kappa_2 - \kappa_1}{2} + \frac{u}{2}\right)^{\lambda-1} \mathscr{F}_{\rho,\lambda}^\sigma \left[\omega\left(\frac{\kappa_2 - \kappa_1}{2} + \frac{u}{2}\right)^\rho\right]\right.$$

$$\left. + \left(\frac{\kappa_2 - \kappa_1}{2} - \frac{u}{2}\right)^{\lambda-1} \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega\left(\frac{\kappa_2 - \kappa_1}{2} - \frac{u}{2}\right)^\rho\right]\right] du$$

$$+ \frac{1}{2\left(\kappa_2 - \kappa_1\right)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega\left(\kappa_2 - \kappa_1\right)^\rho\right]}$$

$$\times \int_0^{\kappa_2 - \kappa_1} F\left(\frac{\kappa_1 + \kappa_2}{2} + \frac{u\xi}{2}\right) \left[\left(\frac{\kappa_2 - \kappa_1}{2} + \frac{u}{2}\right)^{\lambda-1} \mathscr{F}_{\rho,\lambda}^\sigma \left[\omega\left(\frac{\kappa_2 - \kappa_1}{2} + \frac{u}{2}\right)^\rho\right]\right.$$

$$\left. + \left(\frac{\kappa_2 - \kappa_1}{2} - \frac{u}{2}\right)^{\lambda-1} \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega\left(\frac{\kappa_2 - \kappa_1}{2} - \frac{u}{2}\right)^\rho\right]\right] du$$

$$= \frac{1}{2\left(\kappa_2 - \kappa_1\right)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega\left(\kappa_2 - \kappa_1\right)^\rho\right]}$$

$$\times \int_0^{\kappa_2 - \kappa_1} \left[F\left(\frac{\kappa_1 + \kappa_2}{2} - \frac{u\xi}{2}\right) + F\left(\frac{\kappa_1 + \kappa_2}{2} + \frac{u\xi}{2}\right)\right]$$

$$\times \left[\left(\frac{\kappa_2 - \kappa_1}{2} + \frac{u}{2}\right)^{\lambda-1} \mathscr{F}_{\rho,\lambda}^\sigma \left[\omega\left(\frac{\kappa_2 - \kappa_1}{2} + \frac{u}{2}\right)^\rho\right]\right.$$

$$\left. + \left(\frac{\kappa_2 - \kappa_1}{2} - \frac{u}{2}\right)^{\lambda-1} \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega\left(\frac{\kappa_2 - \kappa_1}{2} - \frac{u}{2}\right)^\rho\right]\right] du.$$

From Lemma 1, we have $h(\xi) = \frac{1}{2}\left[F\left(\frac{\kappa_1 + \kappa_2}{2} - \frac{\xi}{2}\right) + F\left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\xi}{2}\right)\right]$ is increasing on $[0, \kappa_2 - \kappa_1]$. Since

$$\left(\frac{\kappa_2 - \kappa_1}{2} + \frac{u}{2}\right)^{\lambda-1} \mathscr{F}_{\rho,\lambda}^\sigma \left[\omega\left(\frac{\kappa_2 - \kappa_1}{2} + \frac{u}{2}\right)^\rho\right]$$

$$+ \left(\frac{\kappa_2 - \kappa_1}{2} - \frac{u}{2}\right)^{\lambda-1} \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega\left(\frac{\kappa_2 - \kappa_1}{2} - \frac{u}{2}\right)^\rho\right]$$

is nonnegative, then $BS(\xi)$ is increasing on $[0, 1]$. Thus, using the facts that

$$BS(0) = F\left(\frac{\kappa_1 + \kappa_2}{2}\right)$$

and

$$BS(1) = \frac{1}{2\left(\kappa_2 - \kappa_1\right)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma \left[\omega\left(\kappa_2 - \kappa_1\right)^\rho\right]} \left[\mathscr{J}_{\rho,\lambda,\kappa_1+;\omega}^\sigma F(\kappa_2) + \mathscr{J}_{\rho,\lambda,\kappa_2-;\omega}^\sigma F(\kappa_1)\right],$$

we obtain the desired result. $\quad\square$

REMARK 1. If we choose $\lambda = \alpha$, $\sigma(0) = 1$, and $w = 0$ in Theorem 4, we the ineuality (13) reduces to ineuality (11).

THEOREM 5. *Let* $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ *be a positive function with* $\kappa_1 < \kappa_2$ *and* $F \in L_1[\kappa_1, \kappa_2]$. *If* F *is a convex function on* $[\kappa_1, \kappa_2]$, *then BY is convex and monotonically increasing on* $[0, 1]$ *and for all* $\rho, \lambda > 0$ *and* $w \geqslant 0$ *we have the following inequalities for generalized fractional integral operators:*

$$\frac{1}{2(\kappa_2 - \kappa_1)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma [\omega(\kappa_2 - \kappa_1)^\rho]} \left[\mathscr{J}_{\rho,\lambda,\kappa_1+;\omega}^\sigma F(\kappa_2) + \mathscr{J}_{\rho,\lambda,\kappa_2-;\omega}^\sigma F(\kappa_1) \right]$$
$$= BY(0) \leqslant BY(\xi) \leqslant BY(1)$$
$$= \frac{F(\kappa_1) + F(\kappa_2)}{2} \tag{14}$$

where

$$BY(\xi) = \frac{1}{4(\kappa_2 - \kappa_1)^\lambda \mathscr{F}_{\rho,\lambda+1}^\sigma [\omega(\kappa_2 - \kappa_1)^\rho]}$$

$$\times \int_{\kappa_1}^{\kappa_2} \left[F\left(\left(\frac{1+\xi}{2} \right) \kappa_1 + \left(\frac{1-\xi}{2} \right) \varkappa \right) \right.$$

$$\times \left(\left(\frac{2\kappa_2 - \kappa_1 - \varkappa}{2} \right)^{\lambda-1} \mathscr{F}_{\rho,\lambda}^\sigma \left[\omega \left(\frac{2\kappa_2 - \kappa_1 - \varkappa}{2} \right)^\rho \right] \right.$$

$$\left. + \left(\frac{\varkappa - \kappa_1}{2} \right)^{\lambda-1} \mathscr{F}_{\rho,\lambda}^\sigma \left[\omega \left(\frac{\varkappa - \kappa_1}{2} \right)^\rho \right] \right)$$

$$+ F\left(\left(\frac{1+\xi}{2} \right) \kappa_2 + \left(\frac{1-\xi}{2} \right) \varkappa \right)$$

$$\times \left(\left(\frac{\kappa_2 - \varkappa}{2} \right)^{\lambda-1} \mathscr{F}_{\rho,\lambda}^\sigma \left[\omega \left(\frac{\kappa_2 - \varkappa}{2} \right)^\rho \right] \right.$$

$$\left. \left. + \left(\frac{\varkappa + \kappa_2 - 2\kappa_1}{2} \right)^{\lambda-1} \mathscr{F}_{\rho,\lambda}^\sigma \left[\omega \left(\frac{\varkappa + \kappa_2 - 2\kappa_1}{2} \right)^\rho \right] \right) \right] d\varkappa$$

and the coefficients $\sigma(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ *are bounded sequences of positive real numbers.*

Proof. We note that if F is convex and g is linear, then the composition $F \circ g$ is convex. Moreover, we note that a positive constant multiple of a convex function and

sum of two convex functions are convex, thus

$$
F\left(\left(\frac{1+\xi}{2}\right)\kappa_1+\left(\frac{1-\xi}{2}\right)\varkappa\right)
$$
$$
\times\left(\left(\frac{2\kappa_2-\kappa_1-\varkappa}{2}\right)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{2\kappa_2-\kappa_1-\varkappa}{2}\right)^{\rho}\right]\right.
$$
$$
\left.+\left(\frac{\varkappa-\kappa_1}{2}\right)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{\varkappa-\kappa_1}{2}\right)^{\rho}\right]\right)
$$

and

$$
F\left(\left(\frac{1+\xi}{2}\right)\kappa_2+\left(\frac{1-\xi}{2}\right)\varkappa\right)
$$
$$
\times\left(\left(\frac{\kappa_2-\varkappa}{2}\right)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{\kappa_2-\varkappa}{2}\right)^{\rho}\right]\right.
$$
$$
\left.+\left(\frac{\varkappa+\kappa_2-2\kappa_1}{2}\right)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{\varkappa+\kappa_2-2\kappa_1}{2}\right)^{\rho}\right]\right)
$$

are convex, which implies that $BY(\xi)$ is convex. Then, by elementary calculus, we have

$$
BY(\xi)=\frac{1}{4\left(\kappa_2-\kappa_1\right)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\kappa_2-\kappa_1\right)^{\rho}\right]}
$$
$$
\times\int_{\kappa_1}^{\kappa_2}\left[F\left(\left(\frac{1+\xi}{2}\right)\kappa_1+\left(\frac{1-\xi}{2}\right)\varkappa\right)\right.
$$
$$
\times\left(\left(\frac{2\kappa_2-\kappa_1-\varkappa}{2}\right)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{2\kappa_2-\kappa_1-\varkappa}{2}\right)^{\rho}\right]\right.
$$
$$
\left.+\left(\frac{\varkappa-\kappa_1}{2}\right)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{\varkappa-\kappa_1}{2}\right)^{\rho}\right]\right)
$$
$$
+F\left(\left(\frac{1+\xi}{2}\right)\kappa_1+\left(\frac{1-\xi}{2}\right)\varkappa\right)
$$
$$
\times\left(\left(\frac{\kappa_2-\varkappa}{2}\right)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{\kappa_2-\varkappa}{2}\right)^{\rho}\right]\right.
$$
$$
\left.\left.+\left(\frac{\varkappa+\kappa_2-2\kappa_1}{2}\right)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{\varkappa+\kappa_2-2\kappa_1}{2}\right)^{\rho}\right]\right)\right]d\varkappa
$$

$$= \frac{1}{4 \left(\kappa_2 - \kappa_1 \right)^{\lambda} \mathscr{F}^{\sigma}_{\rho, \lambda+1} \left[\omega \left(\kappa_2 - \kappa_1 \right)^{\rho} \right]}$$

$$\times \int_0^{\kappa_2 - \kappa_1} \left[F \left(\kappa_1 + \left(\frac{1-\xi}{2} \right) u \right) \right.$$

$$\times \left(\left(\frac{2\kappa_2 - 2\kappa_1 - u}{2} \right)^{\lambda-1} \mathscr{F}^{\sigma}_{\rho, \lambda} \left[\omega \left(\frac{2\kappa_2 - 2\kappa_1 - u}{2} \right)^{\rho} \right] \right.$$

$$+ \left(\frac{u}{2} \right)^{\lambda-1} \mathscr{F}^{\sigma}_{\rho, \lambda} \left[\omega \left(\frac{u}{2} \right)^{\rho} \right] \right)$$

$$+ F \left(\kappa_2 - \left(\frac{1-\xi}{2} \right) u \right) \left(\left(\frac{u}{2} \right)^{\lambda-1} \mathscr{F}^{\sigma}_{\rho, \lambda} \left[\omega \left(\frac{u}{2} \right)^{\rho} \right] \right.$$

$$\left. \left. + \left(\frac{2\kappa_2 - 2\kappa_1 - u}{2} \right)^{\lambda-1} \mathscr{F}^{\sigma}_{\rho, \lambda} \left[\omega \left(\frac{2\kappa_2 - 2\kappa_1 - u}{2} \right)^{\rho} \right] \right) \right] du$$

$$= \frac{1}{4 \left(\kappa_2 - \kappa_1 \right)^{\lambda} \mathscr{F}^{\sigma}_{\rho, \lambda+1} \left[\omega \left(\kappa_2 - \kappa_1 \right)^{\rho} \right]}$$

$$\times \int_0^{\kappa_2 - \kappa_1} \left[F \left(\kappa_1 + \left(\frac{1-\xi}{2} \right) u \right) + F \left(\kappa_2 - \left(\frac{1-\xi}{2} \right) u \right) \right]$$

$$\times \left(\left(\frac{2\kappa_2 - 2\kappa_1 - u}{2} \right)^{\lambda-1} \mathscr{F}^{\sigma}_{\rho, \lambda} \left[\omega \left(\frac{2\kappa_2 - 2\kappa_1 - u}{2} \right)^{\rho} \right] \right.$$

$$+ \left(\frac{u}{2} \right)^{\lambda-1} \mathscr{F}^{\sigma}_{\rho, \lambda} \left[\omega \left(\frac{u}{2} \right)^{\rho} \right] \right) du$$

It follows that from Lemma 1 that $h(\xi) = \frac{1}{2} \left[F \left(\frac{\kappa_1 + \kappa_2}{2} - \frac{\xi}{2} \right) + F \left(\frac{\kappa_1 + \kappa_2}{2} + \frac{\xi}{2} \right) \right]$ and $k(\xi) = \kappa_2 - \kappa_1 - (1 - \xi) u$ are increasing on $[0, \kappa_2]$ and $[0, 1]$, respectively. Thus, $h(k(\xi)) = F \left(\kappa_1 + \left(\frac{1-\xi}{2} \right) u \right) + F \left(\kappa_2 - \left(\frac{1-\xi}{2} \right) u \right)$ is increasing on $[0, 1]$. Since

$$\left(\left(\frac{2\kappa_2 - 2\kappa_1 - u}{2} \right)^{\lambda-1} \mathscr{F}^{\sigma}_{\rho, \lambda} \left[\omega \left(\frac{2\kappa_2 - 2\kappa_1 - u}{2} \right)^{\rho} \right] + \left(\frac{u}{2} \right)^{\lambda-1} \mathscr{F}^{\sigma}_{\rho, \lambda} \left[\omega \left(\frac{u}{2} \right)^{\rho} \right] \right)$$

is non negative, then we deduce that BY is monotonically increasing on $[0, 1]$. Using the facts that

$$BY(0) = \frac{1}{2 \left(\kappa_2 - \kappa_1 \right)^{\lambda} \mathscr{F}^{\sigma}_{\rho, \lambda+1} \left[\omega \left(\kappa_2 - \kappa_1 \right)^{\rho} \right]} \left[\mathscr{J}^{\sigma}_{\rho, \lambda, \kappa_1+; \omega} F \left(\kappa_2 \right) + \mathscr{J}^{\sigma}_{\rho, \lambda, \kappa_2-; \omega} F \left(\kappa_1 \right) \right]$$

and

$$BY(1) = \frac{F \left(\kappa_1 \right) + F \left(\kappa_2 \right)}{2},$$

then one can obtain the required result. □

REMARK 2. If we choose $\lambda = \alpha$, $\sigma(0) = 1$, and $w = 0$ in Theorem 5, we the ineuality (14) reduces to ineuality (12).

4. Concluding remarks

In this study, we consider the refinements of Hermite-Hadamard type inequalities involving generalized fractional integral operators. The results presented in this study would provide generalizations of those given in earlier works.

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