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ON SOME COUPLED SYSTEMS OF FRACTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. We study certain coupled systems of fractional differential inclusions with several boundary conditions and we obtain the existence of solutions in the situation when the set-valued maps have non-convex values.

1. Introduction

In the last years one may find in the literature an increasing number of paper devoted to the study of coupled systems of fractional differential equations and inclusions [1, 2, 3, 4, 16, 18] etc. All these approaches provide the existence of solutions of such kind of problems by using standard fixed point techniques and contribute to the developement of the theory of differential equations and inclusions of fractional order ([6, 9, 14, 15, 17] etc.).

In the present paper we study three classes of coupled systems of fractional differential inclusions with certain boundary conditions and defined by set-valued maps that are Lipschitz in the state variables but without convex values. By proving the existence of solutions we extend or improve corresponding results in the literature [1, 16].

We consider first coupled systems of fractional differential inclusions with random parameters of the form

$$\begin{cases} (D^{\alpha_1,\beta_1}x)(t,w) \in F_1(t,x(t,w),y(t,w),w) & a.e.\ t \in I,\ w \in \Omega, \\ (D^{\alpha_2,\beta_2}y)(t,w) \in F_2(t,x(t,w),y(t,w),w) & a.e.\ t \in I,\ w \in \Omega, \end{cases} \tag{1.1}$$

$$\begin{cases} (I^{1-\gamma_1}x)(t,w)|_{t=0} = \varphi_1(w), & w \in \Omega, \\ (I^{1-\gamma_2}y)(t,w)|_{t=0} = \varphi_2(w), & w \in \Omega, \end{cases} \tag{1.2}$$

and

$$\begin{cases} (D_H^{\alpha_1,\beta_1}x)(t,w) \in G_1(t,x(t,w),y(t,w),w) & a.e.\ t \in J,\ w \in \Omega, \\ (D_H^{\alpha_2,\beta_2}y)(t,w) \in G_2(t,x(t,w),y(t,w),w) & a.e.\ t \in J,\ w \in \Omega, \end{cases} \tag{1.3}$$

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$$\begin{cases} (I_H^{1-\gamma_1}x)(t,w)|_{t=0} = \psi_1(w), & w \in \Omega, \\ (I_H^{1-\gamma_2}y)(t,w)|_{t=0} = \psi_2(w), & w \in \Omega, \end{cases} \tag{1.4}$$

where $\alpha_i \in (0,1)$, $\beta_i \in (0,1]$, $\gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i$, $I = [0,T]$, $J = [1,T]$, Ω is a measurable space, $w \in \Omega$ is a random parameter, $\varphi_i : \Omega \to \mathbf{R}$, $\psi_i : \Omega \to \mathbf{R}$ are measurable functions, $F_i : I \times \mathbf{R}^2 \times \Omega \to \mathscr{P}(\mathbf{R})$, $G_i : J \times \mathbf{R}^2 \times \Omega \to \mathscr{P}(\mathbf{R})$ are set-valued maps, $i = 1,2$, $D^{\alpha,\beta}$ is the Hilfer fractional derivative of order α and type β, I^γ is the left-side fractional integral of order $\gamma > 0$, $D_H^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order α and type β and I_H^γ is the left-sided Hadamard integral of order γ.

Finally, our study concerns coupled systems of mixed order fractional order differential inclusions with coupled integral fractional boundary conditions of the form

$$\begin{cases} D_C^\alpha x(t) \in H_1(t,x(t),y(t)) & a.e.\ t \in I, \\ D_{RL}^\beta y(t) \in H_2(t,x(t),y(t)) & a.e.\ t \in I, \end{cases} \tag{1.5}$$

$$\begin{cases} x(0) = \lambda D_C^p y(\eta) \\ y(0) = 0, \quad y(T) = \gamma I^q x(\xi), \end{cases} \tag{1.6}$$

with $\alpha \in (0,1], \beta \in (1,2]$, $p \in (0,1)$, $q > 0$, $\gamma, \lambda \in \mathbf{R}$, $\xi, \eta \in (0,T)$, D_C^α is the Caputo fractional derivative of order α, D_{RL}^β is the Riemann-Liouville fractional derivative of order β and $H_i : I \times \mathbf{R} \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ are given set-valued maps, $i = 1,2$.

Our aim is to use suitably the ideas of Filippov [10] to obtain the existence of solutions for problems (1.1)–(1.2), (1.3)–(1.4) and (1.5)–(1.6). For a differential inclusion without convexity in the right-hand side, Filippov's theorem [10] provides the existence of solutions starting from a given mapping which is usually called "quasi" solution. Moreover, the result contains an estimate between the "quasi" solution and the obtained solution.

The first paper in the literature that deals with coupled systems of fractional differential inclusions is [4]. In [4] several existence results for a class of coupled systems of fractional differential inclusions with coupled boundary conditions are obtained by using known fixed point theorems for set-valued maps. It is worth to mention that the first paper in the literature containing Filippov's approach applied to coupled systems of fractional differential inclusions is [8]. Moreover, Theorem 1 in [8] provides, in a particular case, a Filippov type existence result for the boundary problem studied in [4], a result which is similar to Theorem 3.8 in [4].

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel, while Section 3 is devoted to our existence theorems.

2. Preliminaries

Let (X,d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

As usual, we denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I,\mathbf{R})$ we denote the Banach space of all integrable functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_1 = \int_I |x(t)|\mathrm{d}t$. We introduce, also, the weighted space of continuous functions defined by $C_\gamma(I,\mathbf{R}) = \{x(.) : (0,T] \to \mathbf{R}; t \to t^{1-\gamma}x(t) \in C(I,\mathbf{R})\}$ with the norm $|x(.)|_{C_\gamma} = \sup_{t\in I}|t^{1-\gamma}x(t)|$ and $C^1_\gamma(I,\mathbf{R}) = \{x(.) : I \to \mathbf{R}; x'(.) \in C_\gamma(I,\mathbf{R})\}$ with the norm $|x(.)|_{C^1_\gamma} = |x(.)|_\infty + |x'(.)|_{C_\gamma}$.

DEFINITION 1. a) *The fractional integral of order* $r > 0$ of a Lebesgue integrable function $f : (0,T] \to \mathbf{R}$ is defined by

$$I^r f(t) = \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} f(s)ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(r) = \int_0^\infty t^{r-1}e^{-t}dt$.

b) *The Riemann-Liouville fractional derivative of order* $r > 0$ of a function Lebesgue integrable $f : (0,T] \to \mathbf{R}$ is defined by

$$D^r_{RL} f(t) = \frac{1}{\Gamma(n-r)} \frac{d^n}{dt^n} \int_0^t (t-s)^{-r+n-1} f(s)ds,$$

where $n = [r]+1$, provided the right-hand side is pointwise defined on $(0,\infty)$.

c) *The Caputo fractional derivative of order* $r > 0$ of an absolutely continuous function $f : I \to \mathbf{R}$ is defined by

$$D^r_C f(t) = \frac{1}{\Gamma(n-r)} \int_0^t (t-s)^{-r+n-1} f^{(n)}(s)\mathrm{d}s.$$

It is assumed implicitly that f is n times differentiable whose n-th derivative is absolutely continuous.

d) Let $\alpha \in (0,1)$, $\beta \in [0,1]$ and $f(.) \in L^1(I,\mathbf{R})$ with $I^{(1-\alpha)(1-\beta)} f \in AC(I,\mathbf{R})$. *The Hilfer fractional derivative of order* α *and type* β of f is defined by

$$(D^{\alpha,\beta} f)(t) = \left(I^{\beta(1-\alpha)}\left(\frac{d}{dt}(I^{(1-\alpha)(1-\beta)} f)\right)\right)(t) \quad a.e.\ (I).$$

We note that Hilfer fractional derivative, introduced in [11], has the Riemann-Liouville and Caputo fractional derivatives as particular cases; namely, $D^{\alpha,0} = D^\alpha_{RL}$ and $D^{\alpha,1} = D^\alpha_C$. For properties of such derivatives we refer, for example, to [12], but we shall use here only the above definitions.

Next $J = [1,T]$, $T > 1$, we consider the weighted space of continuous functions defined by $C_{\gamma,\ln}(J,\mathbf{R}) = \{x(.) : (1,T] \to \mathbf{R}; t \to (\ln t)^{1-\gamma}x(t) \in C(J,\mathbf{R})\}$ with the norm $|x(.)|_{C_{\gamma,\ln}} = \sup_{t\in J}|(\ln t)^{1-\gamma}x(t)|$.

Set $\delta = t\frac{d}{dt}$, $r > 0$, $n = [r]+1$ and $AC^n_\delta = \{u : J \to \mathbf{R}; \delta^{n-1}[u(.)] \in AC(J,\mathbf{R})\}$.

DEFINITION 2. a) *The Hadamard fractional integral of order* $r>0$ of a Lebesgue integrable function $g: J \to \mathbf{R}$ is defined by

$$I_H^r g(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\ln \frac{t}{s}\right)^{r-1} \frac{g(s)}{s} ds$$

provided the integral exists.

b) *The Hadamard fractional derivative of order* $r>0$ of a function $g: J \to \mathbf{R}$ is defined by

$$D_H^r g(t) = \frac{1}{\Gamma(n-r)} \delta^n \int_1^t \left(\ln \frac{t}{s}\right)^{n-r-1} \frac{g(s)}{s} ds.$$

c) *The Caputo-Hadamard fractional derivative of order* $r>0$ of function $g(.) \in AC_\delta^n$ is defined by

$$(D_{CH}^r g)(t) = (I_H^{n-r} \delta^n g)(t).$$

In particular, if $r \in (0,1]$ then $(D_{CH}^r g)(t) = (I_H^{1-r} \delta g)(t)$.

d) Let $\alpha \in (0,1)$, $\beta \in [0,1]$ and $g(.) \in L^1(J,\mathbf{R})$ with $I_H^{(1-\alpha)(1-\beta)} g \in AC(J,\mathbf{R})$. *The Hilfer-Hadamard fractional derivative of order* α *and type* β of g is defined by

$$(D_H^{\alpha,\beta} g)(t) = (I_H^{\beta(1-\alpha)}(D_H^{\gamma} g))(t) \quad a.e.\ (I).$$

We note that Hilfer-Hadammard fractional derivative has the Hadamard and Caputo-Hadamard fractional derivatives as particular cases; namely, $D_H^{\alpha,0} = D_H^{\alpha}$ and $D_H^{\alpha,1} = D_{CH}^{\alpha}$. For properties of such derivatives we refer, for example, to [11, 13], but we shall use here only the above definitions.

In what follows $(\Omega, \mathscr{A})$ is a measurable space, $F_i : I \times \mathbf{R}^2 \times \Omega \to \mathscr{P}(\mathbf{R})$ are given set-valued maps, $i=1,2$, $x(.,.) : I \times \Omega \to \mathbf{R}$, $y(.,.) : I \times \Omega \to \mathbf{R}$ are such that $x(.,w), y(.,w) \in C_\gamma(I,\mathbf{R})$ $\forall w \in \Omega$. We define the set of selections of F_1 and F_2 by

$$S_{F_1,x,y}(w) = \{f_1 : I \times \Omega \to \mathbf{R};\quad f_1(t,w) \in F_1(t,x(t,w),y(t,w),w) \quad a.e.\ (I)\},$$

$$S_{F_2,x,y}(w) = \{f_2 : I \times \Omega \to \mathbf{R};\quad f_2(t,w) \in F_2(t,x(t,w),y(t,w),w) \quad a.e.\ (I)\}.$$

The next technical result is proved in [1].

LEMMA 1. *a) Let* $h \in C_\gamma(I,\mathbf{R})$. *Then the unique solution of the Cauchy problem* $(D^{\alpha,\beta}u)(t) = h(t)$, $(I^{1-\gamma}u)(0) = \varphi$ *is* $u(t) = \frac{\varphi}{\Gamma(\gamma)} t^{\gamma-1} + I^\alpha h(t)$.

b) Let $h \in C_{\gamma,\ln}(I,\mathbf{R})$. *Then the unique solution of the Cauchy problem* $(D_H^{\alpha,\beta}u)(t) = h(t)$, $(I_H^{1-\gamma}u)(0) = \psi$ *is* $u(t) = \frac{\psi}{\Gamma(\gamma)} (\ln t)^{\gamma-1} + I_H^\alpha h(t)$.

DEFINITION 3. By a solution of problem (1.1)–(1.2) we understand two functions $x(.,.) : I \times \Omega \to \mathbf{R}$, $y(.,.) : I \times \Omega \to \mathbf{R}$ such that $x(t,.)$, $y(t,.)$ are measurable for any $t \in I$, $x(.,w)$, $y(.,w) \in C_\gamma(I,\mathbf{R})$ $\forall w \in \Omega$ and satisfies equations

$$x(t,w) = \frac{\varphi_1(w)}{\Gamma(\gamma)} t^{\gamma-1} + (I^\alpha f_1(.,w))(t),\ y(t,w) = \frac{\varphi_2(w)}{\Gamma(\gamma)} t^{\gamma-1} + (I^\alpha f_2(.,w))(t),$$

where $f_i \in S_{F_i,x,y}(w)$ with $f_i(.,w) \in L^1(I,\mathbf{R})$ $\forall w \in \Omega$, $i=1,2$.

DEFINITION 4. By a solution of problem (1.3)–(1.4) we understand two functions $x(.,.) : J \times \Omega \to \mathbf{R}$, $y(.,.) : J \times \Omega \to \mathbf{R}$ such that $x(t,.)$, $y(t,.)$ are measurable for any $t \in J$, $x(.,w), y(.,w) \in C_{\gamma,\ln}(I,\mathbf{R})$ $\forall w \in \Omega$ and satisfies equations

$$x(t,w) = \frac{\psi_1(w)}{\Gamma(\gamma)} t^{\gamma-1} + (I_H^\alpha g_1(.,w))(t),\ y(t,w) = \frac{\psi_2(w)}{\Gamma(\gamma)} t^{\gamma-1} + (I_H^\alpha g_2(.,w))(t),$$

where $g_i \in S_{G_i,x,y}(w)$ with $g_i(.,w) \in L^1(I,\mathbf{R})$ $\forall w \in \Omega$, $i = 1,2$.

LEMMA 2. *Consider* $h_1, h_2 \in C(I,\mathbf{R})$ *and* $\alpha \in (0,1]$, $\beta \in (1,2]$. *Then the solution of the linear system*

$$\begin{cases} D_C^\alpha x(t) = h_1(t) & a.e.\ t \in I, \\ D_{RL}^\beta y(t) = h_2(t) & a.e.\ t \in I, \end{cases}$$

with boundary conditions (1.6) is equivalent to the following system of integral equations

$$\begin{aligned} x(t) &= I^\alpha h_1(t) - \tfrac{\lambda}{\Lambda}[T^{\beta-1} I^{\beta-p} h_2(\eta) - \tfrac{\Gamma(\beta)}{\Gamma(\beta-p)} \eta^{\beta-p-1} (\gamma I^{q+\alpha} h_1(\xi) - I^\beta h_2(T))] \\ y(t) &= I^\beta h_2(t) + \tfrac{t^{\beta-1}}{\Lambda}[I^\beta h_2(T) - \gamma I^{q+\alpha} h_1(\xi) - \lambda\gamma \tfrac{\xi^q}{\Gamma(1+q)} I^{\beta-p} h_2(\eta)], \end{aligned} \tag{2.1}$$

where it is assumed that $\Lambda := T^{\beta-1} + \lambda\gamma \frac{\Gamma(\beta)\xi^q \eta^{\beta-p-1}}{\Gamma(1+q)\Gamma(\beta-p)} \neq 0$.

The proof may be found in [16], namely Lemma 2.5.

REMARK 1. If we denote

$$\mathscr{H}_1(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \chi_{[0,t]}(s) + \frac{\lambda}{\Lambda} \cdot \frac{\gamma\Gamma(\beta)\eta^{\beta-p-1}}{\Gamma(\beta-p)} \cdot \frac{(\xi-s)^{q+\alpha-1}}{\Gamma(q+\alpha)} \chi_{[0,\xi]}(s),$$

$$\mathscr{H}_2(t,s) = -\frac{\lambda}{\Lambda} T^{\beta-1} \cdot \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)} - \frac{\lambda}{\Lambda} \eta^{\beta-p-1} \cdot \frac{(T-s)^{\beta-1}}{\Gamma(\beta-p)},$$

$$\mathscr{H}_3(t,s) = -\frac{\gamma t^{\beta-1}}{\Lambda} \cdot \frac{(t-s)^{q+\alpha-1}}{\Gamma(q+\alpha)} \chi_{[0,\xi]}(s),$$

$$\mathscr{H}_4(t,s) = \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \chi_{[0,t]}(s) + \frac{t^{\beta-1}}{\Lambda} \left(\frac{(T-s)^{\beta-1}}{\Gamma(\beta)} - \frac{\lambda\gamma\xi^q}{\Gamma(1+q)} \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)} \chi_{[0,\eta]}(s) \right).$$

then the solutions in (2.1) may be rewritten as

$$\begin{aligned} x(t) &= \textstyle\int_0^T \mathscr{H}_1(t,s) h_1(s) ds + \int_0^T \mathscr{H}_2(t,s) h_2(s) ds, \quad t \in I, \\ y(t) &= \textstyle\int_0^T \mathscr{H}_3(t,s) h_1(s) ds + \int_0^T \mathscr{H}_4(t,s) h_(s) ds, \quad t \in I. \end{aligned}$$

Moreover, if $\beta > p+1$, for any $t,s \in I$, we have

$$|\mathscr{H}_1(t,s)| \leqslant \frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{|\lambda|}{|\Lambda|} \cdot \frac{|\gamma|\Gamma(\beta)\eta^{\beta-p-1}}{\Gamma(\beta-p)} \cdot \frac{\xi^{q+\alpha-1}}{\Gamma(q+\alpha)} =: M_1,$$

$$|\mathscr{H}_2(t,s)| \leqslant \frac{|\lambda|}{|\Lambda|}T^{\beta-1}\frac{\eta^{\beta-p-1}}{\Gamma(\beta-p)} + \frac{|\lambda|}{|\Lambda|}\eta^{\beta-p-1}\frac{T^{\beta-1}}{\Gamma(\beta-p)} =: M_2,$$

$$|\mathscr{H}_3(t,s)| \leqslant \frac{|\gamma|T^{\beta-1}}{|\Lambda|}\frac{\xi^{q+\alpha-1}}{\Gamma(q+\alpha)} =: M_3,$$

$$|\mathscr{H}_4(t,s)| \leqslant \frac{T^{\beta-1}}{\Gamma(\beta)} + \frac{T^{2\beta-2}}{|\Lambda|\Gamma(\beta)} + \frac{|\lambda|}{|\Lambda|}\frac{|\gamma|T^{\beta-1}\xi^q}{\Gamma(1+q)}\frac{\eta^{\beta-p-1}}{\Gamma(\beta-p)} =: M_4.$$

DEFINITION 5. By solution of problem (1.5)–(1.6) we mean the functions $x(.) \in C^1(I,\mathbf{R})$, $y(.) \in C^2(I,\mathbf{R})$ such that there exist functions $h_1(.), h_2(.) \in L^1(I,\mathbf{R})$ that satisfies $h_1(t) \in H_1(t,x(t),y(t))$ *a.e.* (I), $h_2(t) \in H_2(t,x(t),y(t))$ *a.e.* (I) and (2.1) is satisfied.

The next lemma [5] contains a selection result for set-valued maps and is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem.

LEMMA 3. *Consider X a separable Banach space, B is the closed unit ball in X, $W : I \to \mathscr{P}(X)$ is a set-valued map with nonempty closed values and $c : I \to X, r : I \to \mathbf{R}_+$ are measurable functions. If*

$$W(t) \cap (c(t) + r(t)B) \neq \emptyset \quad a.e.\ (I),$$

then the set-valued map $t \to W(t) \cap (c(t) + r(t)B)$ has a measurable selection.

3. Main results

We treat first problem (1.1)–(1.2). We need the following assumptions.

HYPOTHESIS H1. i) $F_i(.,.,.,.) : I \times \mathbf{R}^2 \times \Omega \to \mathscr{P}(\mathbf{R})$ *have nonempty closed values and the set-valued maps* $(t,w) \to F_i(t,u,v,w)$ *are measurable* $\forall u, v \in \mathbf{R}$, $i = 1,2$.

ii) *There exists measurable and bounded functions* $l_i(.,.) : I \times \Omega \to (0,\infty)$ *such that, for all* $w \in \Omega$, $F_i(t,.,.,w)$ *satisfy the following Lipschitz condition*

$$d_H(F_i(t,x_1,y_1,w), F_i(t,x_2,y_2,w)) \leqslant t^{1-\gamma_i} l_i(t,w)(|x_1 - x_2| + |y_1 - y_2|),$$

$\forall\, t \in I$, $x_1, x_2, y_1, y_2 \in \mathbf{R}$, $i = 1,2$.

Denote $l_i^* := \sup_{w\in\Omega} |l_i(.,w)|_\infty$, $i = 1,2$ and $L = \frac{l_1^* T^{1+\alpha_1-\gamma_1}}{\Gamma(1+\alpha_1)} + \frac{l_2^* T^{1+\alpha_2-\gamma_2}}{\Gamma(1+\alpha_2)}$.

THEOREM 1. *Assume that Hypothesis H1 is satisfied and $L < 1$. Let $u(.,.), z(.,.) : I \times \Omega \to \mathbf{R}$ be such that $u(t,.)$ and $z(t,.)$ are measurable for any $t \in I$, $u(.,w) \in C_{\gamma_1}(I,\mathbf{R})$, $z(.,w) \in C_{\gamma_2}(I,\mathbf{R})$ $\forall w \in \Omega$, $(I^{1-\gamma_1}u)(0+,w) = \Phi_1(w)$, $(I^{1-\gamma_2}z)(0+,w) = \Phi_2(w)$, $w \in \Omega$ with $\Phi_1, \Phi_2 : \Omega \to \mathbf{R}$ measurable functions and there exist $p(.,.), q(.,.) : I \times \Omega \to \mathbf{R}$, $p(t,.)$, $q(t,.)$ are measurable functions for any $t \in I$, $(I^{\alpha_1} p(.,w))(T) < +\infty$, $(I^{\alpha_2} q(.,w))(T) < +\infty$ $\forall w \in \Omega$ and such that $d((D^{\alpha_1,\beta_1}u)(t,w), F_1(t,u(t,w),z(t,w),w)) \leqslant p(t,w)$, $d((D^{\alpha_2,\beta_2}z)(t,w), F_2(t,u(t,w),z(t,w),w)) \leqslant q(t,w)$ a.e. $t \in I$, $\forall w \in \Omega$.*

Then there exists $(x(.,.),y(.,.))$ *a solution of problem (1.1)–(1.2) satisfying for all* $w\in\Omega$

$$\begin{aligned}&|x(.,w)-u(.,w)|_{C_{\gamma_1}}+|y(.,w)-z(.,w)|_{C_{\gamma_2}}\\ &\leqslant\frac{1}{1-L}\Big[\frac{1}{\Gamma(\gamma_1)}|\varphi_1(w)-\Phi_1(w)|\frac{1}{\Gamma(\gamma_2)}|\varphi_2(w)-\Phi_2(w)|\\ &\quad++T^{1-\gamma_1}(I^{\alpha_1}p(.,w))(T)+T^{1-\gamma_2}(I^{\alpha_2}q(.,w))(T)\Big].\end{aligned}\tag{3.1}$$

Proof. The multifunctions $t\to F_i(t,u(t,w),z(t,w),w)$ are measurable with closed values and for almost all $t\in I$

$$F_1(t,u(t,w),z(t,w),w)\cap\{(D^{\alpha_1,\beta_1}u)(t,w)+p(t,w)[-1,1]\}\neq\emptyset,$$
$$F_2(t,u(t,w),z(t,w),w)\cap\{(D^{\alpha_2,\beta_2}z)(t,w)+q(t,w)[-1,1]\}\neq\emptyset.$$

It follows from Lemma 3 that there exists $f_1(.,.),g_1(.,.):I\times\Omega\to\mathbf{R}$ such that $f_1(.,w)$, $q_1(.,w)$ are measurable for any $w\in\Omega$, $f_1(t,w)\in F_1(t,u(t,w),z(t,w),w)$, $g_1(t,w)\in F_2(t,u(t,w),z(t,w),w)$ *a.e.* $t\in I$ $\forall w\in\Omega$ veryfing

$$\begin{aligned}&|f_1(t,w)-(D^{\alpha_1,\beta_1}u)(t,w)|\leqslant p(t,w)\quad a.e.\,t\in I,\ \forall w\in\Omega,\\ &|g_1(t,w)-(D^{\alpha_2,\beta_2}z)(t,w)|\leqslant q(t,w)\quad a.e.\,t\in I,\ \forall w\in\Omega.\end{aligned}\tag{3.2}$$

Define $x_1(t,w)=\frac{\varphi_1(w)}{\Gamma(\gamma_1)}t^{\gamma_1-1}+(I^{\alpha_1}f_1(.,w))(t)$, $y_1(t,w)=\frac{\varphi_2(w)}{\Gamma(\gamma_2)}t^{\gamma_2-1}+(I^{\alpha_2}g_1(.,w))(t)$; one has

$$t^{1-\gamma_1}|x_1(t,w)-u(t,w)|\leqslant\tfrac{1}{\Gamma(\gamma_1)}|\varphi_1(w)-\Phi_1(w)|+t^{1-\gamma_1}(I^{\alpha_1}p(.,w))(t),$$
$$t^{1-\gamma_2}|y_1(t,w)-z(t,w)|\leqslant\tfrac{1}{\Gamma(\gamma_2)}|\varphi_2(w)-\Phi_2(w)|+t^{1-\gamma_2}(I^{\alpha_2}q(.,w))(t),$$

$$|x_1(.,w)-u(.,w)|_{C_{\gamma_1}}\leqslant\tfrac{1}{\Gamma(\gamma_1)}|\varphi_1(w)-\Phi_1(w)|+T^{1-\gamma_1}(I^{\alpha_1}p(.,w))(T),$$
$$|y_1(.,w)-z(.,w)|_{C_{\gamma_2}}\leqslant\tfrac{1}{\Gamma(\gamma_2)}|\varphi_2(w)-\Phi_2(w)|+T^{1-\gamma_2}(I^{\alpha_2}q(.,w))(T),$$

and therefore,

$$\begin{aligned}&|x_1(.,w)-u(.,w)|_{C_{\gamma_1}}+|y_1(.,w)-z(.,w)|_{C_{\gamma_2}}\\ &\leqslant\frac{1}{\Gamma(\gamma_1)}|\varphi_1(w)-\Phi_1(w)|+\frac{1}{\Gamma(\gamma_2)}|\varphi_2(w)-\Phi_2(w)|\\ &\quad+T^{1-\gamma_1}(I^{\alpha_1}p(.,w))(T)+T^{1-\gamma_2}(I^{\alpha_2}q(.,w))(T)=:K.\end{aligned}$$

Next we construct the sequences $x_n(.,.),y_n(.,.):I\times\Omega\to\mathbf{R}$, $x_n(.,w)$, $y_n(.,w)\in C(I,\mathbf{R})$, for any $w\in\Omega$, $f_n(.,.),g_n(.,.):I\times\Omega\to\mathbf{R}$, $f_n(.,w)$, $g_n(.,w)\in L^1(I,\mathbf{R})$ for any $w\in\Omega$, $n\geqslant1$ with the following properties

$$\begin{aligned}&x_n(t,w)=\tfrac{\varphi_1(w)}{\Gamma(\gamma_1)}t^{\gamma_1-1}+(I^{\alpha_1}f_n(.,w))(t),\quad t\in I,\ w\in\Omega,\\ &y_n(t,w)=\tfrac{\varphi_2(w)}{\Gamma(\gamma_2)}t^{\gamma_2-1}+(I^{\alpha_2}g_n(.,w))(t),\quad t\in I,\ w\in\Omega,\end{aligned}\tag{3.3}$$

$$f_n(t,w) \in F_1(t,x_{n-1}(t,w),y_{n-1}(t,w),w) \quad a.e. t \in I,\ w \in \Omega,$$
$$g_n(t,w) \in F_2(t,x_{n-1}(t,w),y_{n-1}(t,w),w) \quad a.e. t \in I,\ w \in \Omega, \tag{3.4}$$

$$|f_{n+1}(t,w) - f_n(t,w)| \leqslant l_1^*(|x_n(.,w) - x_{n-1}(.,w)|_{C_{\gamma_1}} + |y_n(.,w) - y_{n-1}(.,w)|_{C_{\gamma_2}}),$$
$$a.e. t \in I,\ w \in \Omega,$$
$$|g_{n+1}(t,w) - g_n(t,w)| \leqslant l_2^*(|x_n(.,w) - x_{n-1}(.,w)|_{C_{\gamma_1}} + |y_n(.,w) - y_{n-1}(.,w)|_{C_{\gamma_2}}),$$
$$a.e. t \in I,\ w \in \Omega. \tag{3.5}$$

If we done this construction, then from (3.2)–(3.5) we have for all $w \in \Omega$

$$|x_{n+1}(.,w) - x_n(.,w)|_{C_{\gamma_1}} + |y_{n+1}(.,w) - y_n(.,w)|_{C_{\gamma_2}} \leqslant KL^n.$$

Indeed, assume that the last inequality is true for $n-1$ and we prove it for n. One has

$$\begin{aligned} & t^{1-\gamma_1}|x_{n+1}(t,w) - x_n(t,w)| \\ & \leqslant t^{1-\gamma_1}\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}|f_{n+1}(s,w) - f_n(s,w)|ds \\ & \leqslant T^{1-\gamma_1} l_1^* \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}(|x_n(.,w) - x_{n-1}(.,w)|_{C_{\gamma_1}} + |y_n(.,w) - y_{n-1}(.,w)|_{C_{\gamma_2}})ds \\ & = T^{1-\gamma_1} l_1^* \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)}(|x_n(.,w) - x_{n-1}(.,w)|_{C_{\gamma_1}} + |y_n(.,w) - y_{n-1}(.,w)|_{C_{\gamma_2}}). \end{aligned}$$

Similarly,

$$\begin{aligned} & t^{1-\gamma_2}|y_{n+1}(t,w) - y_n(t,w)| \\ & \leqslant T^{1-\gamma_2} l_2^* \frac{T^{\alpha_2}}{\Gamma(\alpha_2+1)}(|x_n(.,w) - x_{n-1}(.,w)|_{C_{\gamma_1}} + |y_n(.,w) - y_{n-1}(.,w)|_{C_{\gamma_2}}). \end{aligned}$$

Therefore,

$$\begin{aligned} & |x_{n+1}(.,w) - x_n(.,w)|_{C_{\gamma_1}} + |y_{n+1}(.,w) - y_n(.,w)|_{C_{\gamma_2}} \\ & \leqslant L(|x_n(.,w) - x_{n-1}(.,w)|_{C_{\gamma_1}} + |y_n(.,w) - y_{n-1}(.,w)|_{C_{\gamma_2}}). \end{aligned}$$

and

$$|x_{n+1}(.,w) - x_n(.,w)|_{C_{\gamma_1}} + |y_{n+1}(.,w) - y_n(.,w)|_{C_{\gamma_2}} \leqslant L \cdot KL^{n-1} = KL^n.$$

Therefore, $\{x_n(.,w)\}$ and $\{y_n(.,w)\}$ are Cauchy sequences in the Banach space $C(I,\mathbf{R})$, hence converging uniformly to some $x(.,w), y(.,w) \in C(I,\mathbf{R})$, for any $w \in \Omega$,. Therefore, by (3.5), for almost all $t \in I$, the sequences $\{f_n(t,w)\}$, $\{g_n(t,w)\}$ are Cauchy in $\mathbf{R}$, for any $w \in \Omega$. Let $f(.,w)$ be the pointwise limit of $f_n(.,w)$ and $g(.,w)$ be the pointwise limit of $g_n(.,w)$.

For any $t \in I, w \in \Omega$ we have the estimate

$$\begin{aligned} &|x_n(t,w)-u(t,w)|+|y_n(t,w)-z(t,w)| \\ \leqslant &|x_1(t,w)-u(t,w)|+|y_1(t,w)-z(t,w)| \\ &+\sum_{i=1}^{n-1}(|x_{i+1}(t,w)-x_i(t,w)|+|y_{i+1}(t,w)-y_i(t,w)|) \\ \leqslant &K+\sum_{i=1}^{n-1}KL^i \leqslant K\cdot\frac{1}{1-L}. \end{aligned} \tag{3.6}$$

At the same time, from (3.2), (3.5) and (3.6) we obtain for almost all $t \in I$ and any $w \in \Omega$

$$\begin{aligned} &|f_n(t,w)-(D^{\alpha_1,\beta_1}u)(t,w)|+|g_n(t,w)-(D^{\alpha_2,\beta_2}z)(t,w)| \\ \leqslant &\sum_{i=1}^{n-1}(|f_{i+1}(t,w)-f_i(t,w)|+|g_{i+1}(t,w)-g_i(t,w)|) \\ &+|f_1(t,w)-(D^{\alpha_1,\beta_1}u)(t,w)|+|g_1(t,w)-(D^{\alpha_2,\beta_2}z)(t,w)| \\ \leqslant &\sum_{i=1}^{n-1}(t^{1-\gamma_1}l_1(t,w)+t^{1-\gamma_2}l_2(t,w))K\frac{1}{1-L}+p(t,w)+q(t,w). \end{aligned}$$

Hence the sequences $f_n(.,w), g_n(.,w)$ are integrably bounded and therefore, $f(.,w)$, $g(.,w) \in L^1(I,\mathbf{R})$ for any $w \in \Omega$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.3), (3.4) we deduce that $(x(.,.),y(.,.))$ is a solution of (1.1)–(1.2). Finally, passing to the limit in (3.6) we obtain the desired estimates on $x(.,.)$ and $y(.,.)$.

It remains the construction of the sequences $x_n(.,.), f_n(.,.)$ and $y_n(.,.)$, $g_n(.,.)$ with the properties in (3.3)–(3.5). This construction will be realized by induction.

We note that the first step is already realized. Next, assume that for some $N \geqslant 1$ we already constructed $x_n(.,.), y_n(.,.) : I \times \Omega \to \mathbf{R}$, $x_n(.,w), y_n(.,w) \in C(I,\mathbf{R})$, for any $w \in \Omega$ and $f_n(.,.), g_n(.,.) : I \times \Omega \to \mathbf{R}$, $f_n(.,w), g_n(.,w) \in L^1(I,\mathbf{R})$ for any $w \in \Omega$, $n = 1,2,\dots N$ satisfying (3.3), (3.5) for $n = 1,2,\dots,N$ and (3.4) for $n = 1,2,\dots,N-1$. The set-valued maps $t \to F_i(t,x_N(t,w),y_N(t,w),w)$, $i = 1,2$ are measurable. Moreover, the maps $t \to t^{1-\gamma_i}l_i(t,w)(|x_N(t,w)-x_{N-1}(t,w)|+|y_N(t,w)-y_{N-1}(t,w)|)$, $i = 1,2$ are measurable. By the lipschitzianity of $F_i(t,.,.,w)$, $i = 1,2$ we have that for almost all $t \in I$ and any $w \in \Omega$

$$\begin{aligned} &F_1(t,x_N(t,w),y_N(t,w),w)\cap\{f_N(t,w)+t^{1-\gamma_1}l_1(t,w)(|x_N(t,w)-x_{N-1}(t,w)| \\ &+|y_N(t,w)-y_{N-1}(t,w)|)[-1,1]\} \neq \emptyset \end{aligned}$$

$$\begin{aligned} &F_2(t,x_N(t,w),y_N(t,w),w)\cap\{g_N(t,w)+t^{1-\gamma_2}l_2(t,w)(|x_N(t,w)-x_{N-1}(t,w)| \\ &+|y_N(t,w)-y_{N-1}(t,w)|)[-1,1]\} \neq \emptyset. \end{aligned}$$

Using again Lemma 3 we find that there exists $f_{N+1}(.,.), g_{N+1}(.,.) : I \times \Omega \to \mathbf{R}$ such that $f_{N+1}(.,w)$, $g_{N+1}(.,w)$ are measurable for any $w \in \Omega$, $f_{N+1}(t,w) \in F_1(t,x_N(t,w),$

$y_N(t,w),w)$, $g_{N+1}(t,w) \in F_2(t,x_N(t,w),y_N(t,w),w)$ *a.e.* $t \in I$, $\forall w \in \Omega$ and

$$\begin{aligned} &|f_{N+1}(t,w) - f_N(t,w)| \\ &\leqslant t^{1-\gamma_1} l_1(t,w)(|x_N(t,w) - x_{N-1}(t,w)| + |y_N(t,w) - y_{N-1}(t,w)|) \\ &\quad a.e.\, t \in I, \forall w \in \Omega, \\ &|g_{N+1}(t,w) - g_N(t,w)| \\ &\leqslant t^{1-\gamma_2} l_2(t,w)(|x_N(t,w) - x_{N-1}(t,w)| + |y_N(t,w) - y_{N-1}(t,w)|) \\ &\quad a.e.\, t \in I, \forall w \in \Omega. \end{aligned}$$

We define $x_{N+1}(.,.)$ and $y_{N+1}(.,.)$ as in (3.3) with $n = N+1$. Therefore, $f_{N+1}(.,.)$ and $g_{N+1}(.,.)$ satisfies (3.4) and (3.5) and the proof is complete. □

The assumptions in Theorem 1 are satisfied, in particular, for $u = z = 0$, $\Phi_1 = \Phi_2 = 0$, $p = l_1$ and $q = l_2$. We obtain the following consequence of Theorem 1.

COROLLARY 1. *Hypothesis H1 is satisfied,* $d(0,F_i(t,0,0,w)) \leqslant l_i(t,w)$ *a.e.* $t \in I$, $\forall w \in \Omega$, $i = 1,2$ *and* $L < 1$.

Then there exists $(x(.,.),y(.,.))$ *a solution of problem (1.1)–(1.2) satisfying for all* $w \in \Omega$

$$\begin{aligned} &|x(.,w)|_{C_{\gamma_1}} + |y(.,w)|_{C_{\gamma_2}} \\ &\leqslant \frac{1}{1-L}\Big[\frac{1}{\Gamma(\gamma_1)}|\varphi_1(w)| + \frac{1}{\Gamma(\gamma_2)}|\varphi_2(w)| \\ &\quad + T^{1-\gamma_1}(I^{\alpha_1} l_1(.,w))(T) + T^{1-\gamma_2}(I^{\alpha_2} l_2(.,w))(T)\Big]. \end{aligned}$$

REMARK 2. Problem (1.1)–(1.2) with F_1 and F_2 single-valued maps was studied in [1, 2]. In [1] the functions that define the problem are assumed to be Carathéodory and in [2] these functions are Lipschitz. Corollary 1 above may be regarded as an extension to the set-valued framework of Theorem 3 in [2] whose proof uses a random version of the contraction principle.

Next, we are concerned with problem (1.3)–(1.4).

HYPOTHESIS H2. i) $G_i(.,.,.,.) : I \times \mathbf{R}^2 \times \Omega \to \mathscr{P}(\mathbf{R})$ *have nonempty closed values and the set-valued maps* $(t,w) \to G_i(t,u,v,w)$ *are measurable* $\forall u,v \in \mathbf{R}$, $i = 1,2$.

ii) *There exists measurable and bounded functions* $k_i(.,.) : I \times \Omega \to (0,\infty)$ *such that, for all* $w \in \Omega$, $G_i(t,.,.,w)$ *satisfy the following Lipschitz condition*

$$d_H(G_i(t,x_1,y_1,w), G_i(t,x_2,y_2,w)) \leqslant (\ln t)^{1-\gamma_i} k_i(t,w)(|x_1 - x_2| + |y_1 - y_2|),$$

$\forall t \in I$, $x_1,x_2,y_1,y_2 \in \mathbf{R}$, $i = 1,2$.

Denote $k_i^* := \sup_{w\in\Omega} |k_i(.,w)|_\infty$, $i = 1,2$ and $k = \frac{k_1^*(\ln T)^{1+\alpha_1-\gamma_1}}{\Gamma(1+\alpha_1)} + \frac{k_2^*(\ln T)^{1+\alpha_2-\gamma_2}}{\Gamma(1+\alpha_2)}$.

THEOREM 2. *Assume that Hypothesis H2 is satisfied and* $k < 1$. *Let* $u(.,.), z(.,.) : I \times \Omega \to \mathbf{R}$ *be such that* $u(t,.)$ *and* $z(t,.)$ *are measurable for any* $t \in I$, $u(.,w) \in$

$C_{\gamma_1,\ln}(I,\mathbf{R})$, $z(.,w)\in C_{\gamma_2,\ln}(I,\mathbf{R})$ $\forall w\in\Omega$, $(I^{1-\gamma_1}u)(0+,w)=\Psi_1(w)$, $(I^{1-\gamma_2}z)(0+,w)=\Psi_2(w)$, $w\in\Omega$ with $\Psi_1,\Psi_2:\Omega\to\mathbf{R}$ measurable maps and there exist $p(.,.),q(.,.):I\times\Omega\to\mathbf{R}$, $p(t,.)$, $q(t,.)$ are measurable functions for any $t\in I$, $(I_H^{\alpha_1}p(.,w))(T)<+\infty$, $(I_H^{\alpha_2}q(.,w))(T)<+\infty$ $\forall w\in\Omega$ and such that $d((D_H^{\alpha_1,\beta_1}u)(t,w),G_1(t,u(t,w),z(t,w),w))\leqslant p(t,w)$, $d((D_H^{\alpha_2,\beta_2}z)(t,w),G_2(t,u(t,w),z(t,w),w))\leqslant q(t,w)$ a.e. $t\in I$, $\forall w\in\Omega$.

Then there exists $(x(.,.),y(.,.))$ a solution of problem (1.3)–(1.4) satisfying for all $w\in\Omega$

$$\begin{aligned}&|x(.,w)-u(.,w)|_{C_{\gamma_1,\ln}}+|y(.,w)-z(.,w)|_{C_{\gamma_2,\ln}}\\ \leqslant\ &\frac{1}{1-k}\Big[\frac{1}{\Gamma(\gamma_1)}|\psi_1(w)-\Psi_1(w)|+\frac{1}{\Gamma(\gamma_2)}|\psi_2(w)-\Psi_2(w)|\\ &+(\ln T)^{1-\gamma_1}(I_H^{\alpha_1}p(.,w))(T)+(\ln T)^{1-\gamma_2}(I_H^{\alpha_2}q(.,w))(T)\Big].\end{aligned}$$

The proof of Theorem 2 is similar to the proof of Theorem 1.

Theorems 1 and 2 may be interpreted as extensions to coupled systems of fractional differential inclusions with random parameters of similar results in [7] obtained for "single" fractional differential inclusions with random parameters.

Finally, we consider problem (1.5)–(1.6).

HYPOTHESIS H3. i) *$H_i(.,.,.):I\times\mathbf{R}\times\mathbf{R}\to\mathscr{P}(\mathbf{R})$, $i=1,2$ have nonempty closed values and are $\mathscr{L}(I)\otimes\mathscr{B}(\mathbf{R}\times\mathbf{R})$ measurable.*

(ii) *There exist $m_i(.)\in L^1(I,(0,\infty))$ such that, for almost all $t\in I$, $H_i(t,.,.)$ are $m_i(t)$-Lipschitz in the sense that*

$$d_H(H_1(t,x_1,y_1),H_2(t,x_2,y_2))\leqslant m_1(t)(|x_1-x_2|+|y_1-y_2|)\ \forall\, x_1,x_2,y_1,y_2\in\mathbf{R}.$$

$$d_H(H_1(t,x_1,y_1),H_2(t,x_2,y_2))\leqslant m_2(t)(|x_1-x_2|+|y_1-y_2|)\ \forall\, x_1,x_2,y_1,y_2\in\mathbf{R}.$$

We use next the following notation: $M(t)=M_1m_1(t)+M_2m_2(t)+M_3m_1(t)+M_4m_2(t)$, $t\in I$.

THEOREM 3. *Assume that Hypothesis H3 is satisfied, $\Lambda\neq 0$, $\beta>p+1$ and $|M(.)|_1<1$.*

Consider $u(.)\in C^1(I,\mathbf{R})$, $z(.)\in C^2(I,\mathbf{R})$ with $u(0)=\lambda D_C^p z(\eta)$, $z(0)=0$, $z(T)=\gamma I^q u(\xi)$ and there exists $p(.),q(.)\in L^1(I,\mathbf{R}_+)$ verifying $d(D_C^\alpha u(t),H_1(t,u(t),z(t)))\leqslant p(t)$ a.e. (I) and $d(D_{RL}^\beta z(t),H_2(t,u(t),z(t)))\leqslant q(t)$ a.e. (I).

Then there exists $(x(.),y(.))$ a solution of problem (1.5)–(1.6) satisfying for all $t\in I$

$$|x(t)-u(t)|+|y(t)-z(t)|\leqslant\frac{(M_1+M_3)|p(.)|_1+(M_2+M_4)|q(.)|_1}{1-|M(.)|_1}.$$

The proof of Theorem 3 is similar to the proof of Theorem 1 (see also the proof of Theorem 1 in [8]).

If in Theorem 3 we take $u=z=0$, $p=m_1$ and $q=m_2$ we get the following consequence of Theorem 3.

COROLLARY 2. *Assume that Hypothesis H3 is satisfied,* $\Lambda \neq 0$*,* $\beta > p+1$*,* $d(0, H_1(t,0,0)) \leqslant m_1(t)$*,* $d(0, H_2(t,0,0)) \leqslant m_2(t)$ *a.e.* (I) *and* $|M(.)|_1 < 1$*.*

Then there exists $(x(.), y(.))$ *a solution of problem (1.5)–(1.6) satisfying for all* $t \in I$

$$|x(t)| + |y(t)| \leqslant \frac{(M_1+M_3)|m_1(.)|_1 + (M_2+M_4)|m_2(.)|_1}{1-|M(.)|_1}. \tag{3.7}$$

REMARK 3. A similar existence result to the one in Corollary 2 may be found in [16], namely Theorem 3.7. Its proof is performed by using the set-valued contraction principle. It is worth to mention that the approach in [1], apart from the requirement that the values of $F(.,.)$ are compact, does not provides a priori bounds for solutions as in (3.7).

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