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GRÜNWALD–LETNIKOV FRACTIONAL OPERATORS: FROM PAST TO PRESENT

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Abstract. In this paper, we establish a connection between the well-known Grünwald–Letnikov fractional operators which were defined in the mid–1800s and the recently defined fractional h-discrete operators. We prove that the Mittag-Leffler function of the Riemann-Liouville fractional calculus in continuous time is the limit of the Mittag-Leffler type function in fractional h-discrete calculus in discrete time when h approaches zero. In our study, we only focus on the backward difference operators which are also known as discrete nabla operators.

1. Introduction

Approximations of the Grünwald–Letnikov fractional operators, defined by Grünwald and Letnikov in [12, 15], to the Riemann-Liouville fractional operators have been extensively analyzed by Podlubny in his book [26]. These approximations of the Grünwald–Letnikov fractional operators for numerical calculations in several applications have appeared in much literature [1, 14, 22, 24, 25, 28, 27, 30, 31] over the years. While this is one direction of the study of fractional calculus in continuous time, mathematicians and scientists have started to focus more on fractional operators in discrete time [3, 8, 11], recently.

Nowadays, fractional calculus in discrete time is a highly popular area of research that involves contribution from many researchers to build theory and highlight applications. In the development of this theory, two discrete operators known as delta (Δ) and nabla (∇) take the foremost attention. In addition, calculus on time scales [6, 7] becomes an important tool and gives the momentum to form and establish building blocks of the theory by scientists. Discrete fractional calculus is called fractional h- discrete calculus if its domain is $h\mathbb{N}_a = \{a, a+h, a+2h, \ldots\}$ for any positive real number h and any real number a.

In this short paper, we aim to give a connection between what has been done in the 1800s and the current stage of development in the theory of discrete fractional calculus by considering the Grünwald–Letnikov fractional operators. By providing this connection for researchers who are interested in working in this area, we hope that the

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work on Grünwald–Letnikov fractional operators [1, 5, 8, 9, 17, 18, 19, 20, 22, 23, 24, 25, 28, 29, 30, 31, 32, 33] and the work on recent developments in discrete fractional calculus [2, 3, 4, 13, 16] will evolve in the same direction. This will eliminate the need to publish papers which have identical results but with different notation. In addition, we prove that the Mittag-Leffler function of Riemann-Liouville fractional calculus in continuous time is the limit of the Mittag-Leffler–type function of fractional h-discrete calculus when h approaches zero. Hence the finite sum which corresponds to the Mittag-Leffler–type function in h discrete calculus will be used to approximate the infinite sum which represents the Mittag-Leffler function in continuous time. This will provide a straightforward approximation for numerical calculations in this area.

2. Preliminaries

DEFINITION 1. Let $a \in \mathbb{R}$ and $h \in \mathbb{R}^+$. The backward h-difference operator for a function $f : h\mathbb{N}_a \longrightarrow \mathbb{R}$ is defined by

$$\nabla_h f(t) = \frac{f(t) - f(t-h)}{h}, \quad t = a+h,\, a+2h, \ldots,$$

where $h\mathbb{N}_a = \{a,\, a+h,\, a+2h, \ldots\}$.

REMARK 1. We note that

i) if $h = 1$, we have the backward difference operator, or nabla operator (∇)

$$(\nabla f)(t) = f(t) - f(t-1), \quad t \in \mathbb{N}_{a+1},$$

ii) if $\lim\limits_{h\to 0} \dfrac{f(t) - f(t-h)}{h}$ exists, then we have $f'(t) = \lim_{h\to 0} \nabla_h f(t)$.

DEFINITION 2. For any t, $r \in \mathbb{R}$ and $h > 0$, the h-rising factorial function is defined by

$$t_h^{\overline{r}} = h^r \frac{\Gamma(\frac{t}{h} + r)}{\Gamma(\frac{t}{h})},$$

where the quotient is well-defined. Here $\Gamma(\cdot)$ denotes the Euler gamma function.

In [7], Theorem 103 has the following special form if the time scale is considered as $h\mathbb{N}_a$.

THEOREM 1. *Let $t \in h\mathbb{N}_a$. If f is a real valued function defined on $h\mathbb{N}_a$, then the solution of the initial value problem*

$$\nabla_h^n y(t) = f(t), \quad \nabla_h^i y(a-h) = 0, \quad 0 \leqslant i \leqslant n-1$$

is given by

$$y(t) = \frac{1}{\Gamma(n)} \sum_{s=a/h}^{t/h} \big(t - \rho(sh)\big)_h^{\overline{n-1}} f(sh)h.$$

Here $y(t)$ represents n-fold summation of $f(t)$, i.e.

$$y(t) = \underbrace{\sum_{s=a/h}^{t/h} \sum_{s=a/h}^{t/h} \cdots \sum_{s=a/h}^{t/h}}_{n \text{ term}} f(t).$$

In other words, $y(t) = \nabla_h^{-n} f(t)$. Replacing n by any positive real number yields the following.

DEFINITION 3. [2] Let $\alpha > 0$ and a be two real numbers. For a function $f : h\mathbb{N}_a \longrightarrow \mathbb{R}$, the nabla h-fractional sum with order α is defined by

$$\nabla_{h,a}^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a/h}^{t/h} \big(t - \rho(sh)\big)_h^{\overline{\alpha-1}} f(sh)h, \quad t \in h\mathbb{N}_a,$$

where $h > 0$ and $\rho(t) = t - h$.

DEFINITION 4. The nabla h-fractional difference of order α in the sense of Riemann-Liouville is defined by

$$\nabla_{h,a}^{\alpha} f(t) := \nabla_h^n \nabla_{h,a}^{-(n-\alpha)} f(t), \quad t \in h\mathbb{N}_{a+nh},$$

where $a, \alpha \in \mathbb{R}$, $n - 1 < \alpha < n$, and n is a positive integer.

THEOREM 2. [7] *Let a be any real number, $h > 0$, and $t \in h\mathbb{N}_a$. Then for a function $f(t,\cdot) : \mathbb{N}_{\frac{a}{h}} \to \mathbb{R}$ the following identity is true.*

$$\nabla_h \sum_{s=a/h}^{t/h} f(t,s) = \sum_{s=a/h}^{t/h} \nabla_h f(t,s) + \frac{f(t-h, \frac{t}{h})}{h}.$$

Theorem 2 will be crucial in the proof of the following result.

THEOREM 3. *Assume $f : h\mathbb{N}_a \to \mathbb{R}$; $\alpha > 0$, $\alpha \notin \mathbb{N}_1$, and choose $n \in \mathbb{N}_1$ such that $n - 1 < \alpha < n$. Then,*

$$\nabla_{h,a}^{\alpha} f(t) := \frac{1}{\Gamma(-\alpha)} \sum_{s=a/h}^{t/h} (t - \rho(sh))_h^{\overline{-\alpha-1}} f(sh)h, \quad t \in h\mathbb{N}_a. \tag{1}$$

Proof. We prove the statement (1) using mathematical induction on n. Let $n = 1$. Then, $0 < \alpha < 1$. Consider

$$\begin{aligned}\nabla_{h,a}^{\alpha} f(t) &= \nabla_h \nabla_{h,a}^{-(1-\alpha)} f(t) \\ &= \nabla_h \left[\frac{1}{\Gamma(1-\alpha)} \sum_{s=a/h}^{t/h} (t - \rho(sh))_h^{\overline{-\alpha}} f(sh)h \right]\end{aligned}$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{s=a/h}^{t/h} \nabla_h \left[(t-\rho(sh))_h^{\overline{-\alpha}} \right] f(sh)h$$
$$+ \frac{1}{h} \left[(t-\rho(sh))_h^{\overline{-\alpha}} f(sh)h \right]_{t \to t-h,\ s \to \frac{t}{h}}$$
$$= \frac{1}{\Gamma(1-\alpha)} \sum_{s=a/h}^{t/h} \left[\frac{\Gamma(-\alpha+1)}{\Gamma(-\alpha-1+1)} (t-\rho(sh))_h^{\overline{-\alpha-1}} \right] f(sh)h + 0$$
$$= \frac{1}{\Gamma(-\alpha)} \sum_{s=a/h}^{t/h} (t-\rho(sh))_h^{\overline{-\alpha-1}} f(sh)h.$$

Assume the statement (1) is true for $n = k$. Then, we have

$$\nabla_{h,a}^{\alpha} f(t) = \nabla_h^k \nabla_{h,a}^{-(k-\alpha)} f(t) = \nabla_h^k \left[\frac{1}{\Gamma(k-\alpha)} \sum_{s=a/h}^{t/h} (t-\rho(sh))_h^{\overline{k-\alpha-1}} f(sh)h \right] \tag{2}$$
$$= \frac{1}{\Gamma(-\alpha)} \sum_{s=a/h}^{t/h} (t-\rho(sh))_h^{\overline{-\alpha-1}} f(sh)h. \tag{3}$$

Next, we prove the statement (1) is true for $n = k+1$. Here $k < \alpha < k+1$. Consider

$$\nabla_{h,a}^{\alpha} f(t) = \nabla_h^{k+1} \nabla_{h,a}^{-(k+1-\alpha)} f(t)$$
$$= \nabla_h^{k+1} \left[\frac{1}{\Gamma(k+1-\alpha)} \sum_{s=a/h}^{t/h} (t-\rho(sh))_h^{\overline{k-\alpha}} f(sh)h \right]$$
$$= \nabla_h^k \left(\nabla_h \left[\frac{1}{\Gamma(k+1-\alpha)} \sum_{s=a/h}^{t/h} (t-\rho(sh))_h^{\overline{k-\alpha}} f(sh)h \right] \right)$$
$$= \nabla_h^k \Bigg(\frac{1}{\Gamma(k+1-\alpha)} \sum_{s=a/h}^{t/h} \nabla_h \left[(t-\rho(sh))_h^{\overline{k-\alpha}} \right] f(sh)h$$
$$+ \frac{1}{h} \left[(t-\rho(sh))_h^{\overline{k-\alpha}} f(sh)h \right]_{t \to t-h,\ s \to \frac{t}{h}} \Bigg)$$
$$= \nabla_h^k \left(\frac{1}{\Gamma(k+1-\alpha)} \sum_{s=a/h}^{t/h} \left[\frac{\Gamma(k-\alpha+1)}{\Gamma(k-\alpha-1+1)} (t-\rho(sh))_h^{\overline{k-\alpha-1}} \right] f(sh)h + 0 \right)$$
$$= \nabla_h^k \left(\frac{1}{\Gamma(k-\alpha)} \sum_{s=a/h}^{t/h} (t-\rho(sh))_h^{\overline{k-\alpha-1}} f(sh)h \right)$$
$$= \nabla_h^k \nabla_{h,a}^{-(k-\alpha)} f(t) \quad \text{(From Definition 3)}$$
$$= \frac{1}{\Gamma(-\alpha)} \sum_{s=a/h}^{t/h} (t-\rho(sh))_h^{\overline{-\alpha-1}} f(sh)h. \quad \text{(From (2))}$$

Thus, the statement (1) is true for $n = k+1$. Hence, by principle of mathematical induction, the statement (1) is true for all $n \in \mathbb{N}_1$. $\square$

In view of Definition 3 and Theorem 3, the unified definition for α-th order nabla h-fractional sum and differences is as follows:

DEFINITION 5. Let α and a be two real numbers. The α-th order nabla h-fractional difference in the sense of Riemann–Liouville of a function $f : h\mathbb{N}_a \to \mathbb{R}$ is defined by

$$\nabla^{\alpha}_{h,a} f(t) := \frac{1}{\Gamma(-\alpha)} \sum_{s=a/h}^{t/h} (t-\rho(sh))_h^{\overline{-\alpha-1}} f(sh)h, \quad t \in h\mathbb{N}_a, \tag{4}$$

where $h > 0$.

3. Approximations

Next we recall the definition of the fractional order Grünwald–Letnikov like h-difference operator given in [17].

DEFINITION 6. Let α and a be two real numbers. The α-th order Grünwald–Letnikov like h-difference of a function $f : h\mathbb{N}_a \to \mathbb{R}$ is defined by

$$^{GL}\nabla^{\alpha}_{h,a} f(t) := \sum_{r=0}^{\frac{t-a}{h}} h^{-\alpha} \left[(-1)^r \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!} \right] f(t-rh), \quad t \in h\mathbb{N}_a,$$

where $h > 0$.

We assume that f in an integrable function on the set of positive real numbers.

LEMMA 1. *The following are valid:*

i) $\nabla^{\alpha}_{h,a} f(t) = ^{GL}\nabla^{\alpha}_{h,a} f(t), \quad t \in h\mathbb{N}_a.$

ii) If $\lim_{h\to 0} \left[\nabla^{\alpha}_{h,a} f(t)\right]$ *exists, then* $\lim_{h\to 0} \left[\nabla^{\alpha}_{h,a} f(t)\right] = \begin{cases} D^{\alpha}_a f(t), & \alpha \geqslant 0 \\ I^{\alpha}_a f(t), & \alpha < 0, \end{cases}$

for $t \in \mathbb{R}^+$, *where* $D^{\alpha}_a, I^{\alpha}_a$ *are Riemann-Liouville fractional derivative and integral operators, respectively.*

Proof. The proof of $i)$ follows from the lines below:

$$\begin{aligned} \nabla^{\alpha}_{h,a} f(t) &= \frac{1}{\Gamma(-\alpha)} \sum_{s=a/h}^{t/h} (t-\rho(sh))_h^{\overline{-\alpha-1}} f(sh)h \\ &= \frac{1}{\Gamma(-\alpha)} \sum_{r=0}^{\frac{t-a}{h}} (rh+h)_h^{\overline{-\alpha-1}} f(t-rh)h \quad (\text{Take } t-rh = sh) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(-\alpha)} \sum_{r=0}^{\frac{t-a}{h}} h^{-\alpha-1} \frac{\Gamma(\frac{rh+h}{h} - \alpha - 1)}{\Gamma(\frac{rh+h}{h})} f(t-rh)h \\
&= \sum_{r=0}^{\frac{t-a}{h}} h^{-\alpha} \frac{\Gamma(r-\alpha)}{\Gamma(r+1)\Gamma(-\alpha)} f(t-rh) \\
&= \sum_{r=0}^{\frac{t-a}{h}} h^{-\alpha} \left[\frac{-\alpha(-\alpha+1)\cdots(-\alpha+r-1)}{r!} \right] f(t-rh) \\
&= \sum_{r=0}^{\frac{t-a}{h}} h^{-\alpha} \left[(-1)^r \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!} \right] f(t-rh) \\
&=^{GL} \nabla_{h,a}^{\alpha} f(t).
\end{aligned}
$$

Thus, we have

$$\nabla_{h,a}^{\alpha} f(t) =^{GL} \nabla_{h,a}^{\alpha} f(t), \quad t \in h\mathbb{N}_a.$$

The proof of $ii)$ may be found in [26]. □

Next, we recall the well-known Mittag–Leffler function in fractional calculus and the Mittag-Leffler type function in fractional h-discrete calculus respectively.

DEFINITION 7. A two-parameter function of the Mittag–Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta > 0, \quad x \in \mathbb{R}. \tag{5}$$

DEFINITION 8. [2] Let μ, λ, $t_0 \in \mathbb{R}$ and $\nu > 0$. A two-parameter function of the Mittag–Leffler type in discrete time is defined by

$$\tilde{E}^h_{\lambda,\nu,\mu}(t,t_0) = \frac{1}{h^{\mu}} \frac{(t-t_0+h)_h^{\overline{\mu}}}{\Gamma(\mu+1)} + \frac{1}{h^{\mu}} \sum_{n=\frac{t_0}{h}+1}^{\frac{t}{h}} \lambda^{n-\frac{t_0}{h}} \frac{(t-\rho(nh))_h^{\overline{\nu(n-\frac{t_0}{h})+\mu}}}{\Gamma(\nu(n-\frac{t_0}{h})+\mu+1)}, \quad t \in h\mathbb{N}_{t_0}. \tag{6}$$

DEFINITION 9. [21] Let λ, $\beta \in \mathbb{R}$ and $\alpha > 0$. A discrete-time Mittag–Leffler function is defined by

$$E_{(\alpha,\beta)}(\lambda,n) = \sum_{k=0}^{\infty} \lambda^k \binom{n-k+k\alpha+\beta-1}{n-k}, \quad n \in \mathbb{Z}. \tag{7}$$

Our plan is to prove that the Mittag–Leffler type function in fractional h-discrete calculus approaches to the Mittag-Leffler function of fractional calculus when h is getting closer to zero. The following identity plays an important role in the proof.

(Wendel's Limit) For any real numbers x, a and b,

$$\lim_{x\to\infty}\left[x^{a-b}\frac{\Gamma(x+a)}{\Gamma(x+b)}\right]=1. \tag{8}$$

Next, we recall Tannery's Theorem since it will be a main tool in proof of following lemma.

THEOREM 4. [10] *Suppose we are given sequences $(a_n(p))\subset\mathbb{R}$ depending on $p\in\mathbb{N}_0$. Assume*

$$\lim_{p\to\infty} a_n(p)=a_n,$$

for all $n\in\mathbb{N}_0$,

$$|a_n(p)|\leqslant M_n, \quad \text{for all } n,p\in\mathbb{N}_0, \text{ where } \sum_{n=0}^{\infty}M_n<\infty.$$

Then

$$\lim_{p\to\infty}\sum_{n=0}^{p}a_n(p)=\lim_{p\to\infty}\sum_{n=0}^{\infty}a_n(p)=\sum_{n=0}^{\infty}a_n.$$

LEMMA 2. *The following are valid:*

i) $E_{(\nu,\mu+1)}(\lambda h^{\nu},\frac{t}{h})=\tilde{E}^h_{\lambda,\nu,\mu}(t,0)$ *for* $t\in h\mathbb{N}_0$.

ii) $\lim_{h\to0}\left[h^{\mu}\tilde{E}^h_{\lambda,\nu,\mu}(t,0)\right]=t^{\mu}E_{\nu,\mu+1}(\lambda t^{\nu})$.

Proof. The proof of $i)$ is straightforward. Hence we omit the proof. In Definition 8, take $t_0=0$. Then, we obtain

$$\begin{aligned}
\tilde{E}^h_{\lambda,\nu,\mu}(t,0)&=\frac{1}{h^{\mu}}\frac{(t+h)_h^{\overline{\mu}}}{\Gamma(\mu+1)}+\frac{1}{h^{\mu}}\sum_{n=1}^{\frac{t}{h}}\lambda^n\frac{(t-\rho(nh))_h^{\overline{\nu n+\mu}}}{\Gamma(\nu n+\mu+1)}\\
&=\frac{1}{h^{\mu}}\sum_{n=0}^{\frac{t}{h}}\lambda^n\frac{(t-\rho(nh))_h^{\overline{\nu n+\mu}}}{\Gamma(\nu n+\mu+1)}\\
&=\frac{1}{h^{\mu}}\sum_{n=0}^{\frac{t}{h}}\lambda^n\frac{(t-nh+h)_h^{\overline{\nu n+\mu}}}{\Gamma(\nu n+\mu+1)}\\
&=\frac{1}{h^{\mu}}\sum_{n=0}^{\frac{t}{h}}\lambda^n h^{\nu n+\mu}\frac{\Gamma(\frac{t-nh+h}{h}+\nu n+\mu)}{\Gamma(\frac{t-nh+h}{h})}\frac{1}{\Gamma(\nu n+\mu+1)}\\
&=\sum_{n=0}^{\frac{t}{h}}\lambda^n h^{\nu n}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)\Gamma(\nu n+\mu+1)}.
\end{aligned}\tag{9}$$

Consider

$$\begin{aligned}
&\lim_{h\to 0}\left[h^{\mu}\tilde{E}^{h}_{\lambda,\nu,\mu}(t,0)\right]\\
&=\lim_{h\to 0}\left[\sum_{n=0}^{\frac{t}{h}}\lambda^{n}h^{\nu n+\mu}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)\Gamma(\nu n+\mu+1)}\right]\\
&=\lim_{\frac{t}{h}\to\infty}\left[\sum_{n=0}^{\frac{t}{h}}\lambda^{n}h^{\nu n+\mu}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)\Gamma(\nu n+\mu+1)}\right]\\
&=\lim_{\frac{t}{h}\to\infty}\left[\sum_{n=0}^{\frac{t}{h}}\lambda^{n}t^{\nu n+\mu}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)\Gamma(\nu n+\mu+1)}\right]. \qquad (10)
\end{aligned}$$

To prove $ii)$, we first use Wendel's Limit. Then we have

$$\lim_{\frac{t}{h}\to\infty}\left[\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)}\right]=1. \qquad (11)$$

That is, for any given any $\varepsilon>0$ there exists $K=K(\varepsilon)>0$ such that for any $\frac{t}{h}>K$,

$$\left|\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)}-1\right|<\varepsilon, \qquad (12)$$

implying that

$$1-\varepsilon<\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)}<1+\varepsilon. \qquad (13)$$

Consider

$$\begin{aligned}
&\sum_{n=0}^{\infty}\lambda^{n}t^{\nu n+\mu}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)\Gamma(\nu n+\mu+1)}\\
&=\sum_{n=0}^{\infty}\frac{\lambda^{n}t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)}\\
&=\sum_{n=0}^{\infty}\frac{\lambda^{n}t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\left(\frac{t}{h}-n+1\right)^{\overline{\nu n+\mu}}\\
&=\sum_{n=0}^{\frac{t}{h}}\frac{\lambda^{n}t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\left(\frac{t}{h}-n+1\right)^{\overline{\nu n+\mu}}\\
&\quad+\sum_{n=\frac{t}{h}+1}^{\infty}\frac{\lambda^{n}t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\left(\frac{t}{h}-n+1\right)^{\overline{\nu n+\mu}}\\
&=L_1+L_2. \qquad (14)
\end{aligned}$$

We know that $\left(\frac{t}{h}-n+1\right)^{\overline{\nu n+\mu}}=0$ for all $n\in\mathbb{N}_{\frac{t}{h}+1}$. Thus,

$$L_2=\sum_{n=\frac{t}{h}+1}^{\infty}\frac{\lambda^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\left(\frac{t}{h}-n+1\right)^{\overline{\nu n+\mu}}=0. \tag{15}$$

Consequently, from (14) and (15), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\lambda^n t^{\nu n+\mu}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)\Gamma(\nu n+\mu+1)}\\
&=\sum_{n=0}^{\frac{t}{h}}\frac{\lambda^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\left(\frac{t}{h}-n+1\right)^{\overline{\nu n+\mu}}\\
&=\sum_{n=0}^{\frac{t}{h}}\frac{\lambda^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)}.
\end{aligned} \tag{16}$$

Now, consider

$$\begin{aligned}
&\lim_{h\to 0}\left[h^{\mu}\tilde{E}^{h}_{\lambda,\nu,\mu}(t,0)\right]\\
&=\lim_{\frac{t}{h}\to\infty}\left[\sum_{n=0}^{\frac{t}{h}}\lambda^n t^{\nu n+\mu}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)\Gamma(\nu n+\mu+1)}\right]\ \text{(By (10))}\\
&=\lim_{\frac{t}{h}\to\infty}\left[\sum_{n=0}^{\infty}\frac{\lambda^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)}\right]\ \text{(By (16))}.
\end{aligned} \tag{17}$$

We apply Tannery's theorem (Theorem 4) to (17). Here

$$a_n\left(\frac{t}{h}\right)=\frac{\lambda^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)},$$

and denote by

$$S_{\frac{t}{h}}=\sum_{n=0}^{\infty}a_n\left(\frac{t}{h}\right).$$

First, we show that

$$\lim_{\frac{t}{h}\to\infty}a_n\left(\frac{t}{h}\right)=b_n.$$

We have

$$\begin{aligned}
b_n&=\lim_{\frac{t}{h}\to\infty}a_n\left(\frac{t}{h}\right)\\
&=\lim_{\frac{t}{h}\to\infty}\left[\frac{\lambda^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)}\right]
\end{aligned}$$

$$= \frac{\lambda^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)} \lim_{\frac{t}{h}\to\infty} \left[\left(\frac{t}{h}\right)^{-(\nu n+\mu)} \frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)} \right]$$
$$= \frac{\lambda^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}. \quad \text{[By (11)]} \tag{18}$$

Next, we show that

$$\left| a_n\left(\frac{t}{h}\right) \right| \leqslant M_n, \quad \text{and} \quad \sum_{n=0}^{\infty} M_n < \infty.$$

For all $\frac{t}{h} \in \mathbb{N}_0^K$,

$$\left(\frac{t}{h}\right)^{-(\nu n+\mu)} \frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)}$$

is a non-negative continuous function. So, it is bounded for all $\frac{t}{h} \in \mathbb{N}_0^K$. Then, there exists a constant $A \geqslant 0$ such that

$$\left(\frac{t}{h}\right)^{-(\nu n+\mu)} \frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)} \leqslant A, \quad \frac{t}{h} \in \mathbb{N}_0^K. \tag{19}$$

Then,

$$\left| a_n\left(\frac{t}{h}\right) \right| = \left| \lambda^n t^{\nu n+\mu} \left(\frac{t}{h}\right)^{-(\nu n+\mu)} \frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)\Gamma(\nu n+\mu+1)} \right|$$
$$\leqslant \frac{|\lambda|^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)} \left| \left(\frac{t}{h}\right)^{-(\nu n+\mu)} \frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)} \right|$$
$$= \frac{|\lambda|^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)} \left[\left(\frac{t}{h}\right)^{-(\nu n+\mu)} \frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)} \right]$$
$$\leqslant A \frac{|\lambda|^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}. \quad \text{[By (19)]} \tag{20}$$

For $\frac{t}{h} > K$, consider

$$\left| \lambda^n t^{\nu n+\mu} \left(\frac{t}{h}\right)^{-(\nu n+\mu)} \frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)\Gamma(\nu n+\mu+1)} \right|$$
$$\leqslant \frac{|\lambda|^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)} \left| \left(\frac{t}{h}\right)^{-(\nu n+\mu)} \frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)} \right|$$
$$= \frac{|\lambda|^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)} \left[\left(\frac{t}{h}\right)^{-(\nu n+\mu)} \frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)} \right]$$
$$< (1+\varepsilon) \frac{|\lambda|^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}. \quad \text{[By (13)]} \tag{21}$$

Choose $M = \max\{A, (1+\varepsilon)\}$ and denote by

$$M_n = M \frac{|\lambda|^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}.$$

It follows from (20) and (21) that

$$\left|a_n\left(\frac{t}{h}\right)\right| \leqslant M_n.$$

Also,

$$\begin{aligned}\sum_{n=0}^{\infty} M_n &= M\sum_{n=0}^{\infty}\frac{|\lambda|^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}\\ &= Mt^{\mu}\sum_{n=0}^{\infty}\frac{(|\lambda|t^{\nu})^n}{\Gamma(\nu n+\mu+1)}\\ &= Mt^{\mu}E_{\nu,\mu+1}(|\lambda|t^{\nu}).\end{aligned}$$

Since

$$\sum_{n=0}^{\infty}\frac{|\lambda|^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}$$

is convergent for all $t \in \mathbb{R}$, we have

$$\sum_{n=0}^{\infty} M_n < \infty.$$

Thus, all the conditions of Tannery's theorem hypothesis are satisfied. Hence, by Tannery's theorem, we have

$$\begin{aligned}\lim_{\frac{t}{h}\to\infty} S_{\frac{t}{h}} &= \lim_{\frac{t}{h}\to\infty}\sum_{n=0}^{\infty} a_n\left(\frac{t}{h}\right)\\ &= \sum_{n=0}^{\infty}\left[\lim_{\frac{t}{h}\to\infty} a_n\left(\frac{t}{h}\right)\right]\\ &= \sum_{n=0}^{\infty} b_n\\ &= \sum_{n=0}^{\infty}\frac{\lambda^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}\\ &= t^{\mu}\sum_{n=0}^{\infty}\frac{(\lambda t^{\nu})^n}{\Gamma(\nu n+\mu+1)}\\ &= t^{\mu}E_{\nu,\mu+1}(\lambda t^{\nu}).\end{aligned}$$

Finally, from (17) and the above equality, we have

$$\begin{aligned}&\lim_{h\to 0}\left[h^{\mu}\tilde{E}^{h}_{\lambda,\nu,\mu}(t,0)\right]\\ &= \lim_{\frac{t}{h}\to\infty}\left[\sum_{n=0}^{\infty}\frac{\lambda^n t^{\nu n+\mu}}{\Gamma(\nu n+\mu+1)}\left(\frac{t}{h}\right)^{-(\nu n+\mu)}\frac{\Gamma(\frac{t}{h}-n+\nu n+\mu+1)}{\Gamma(\frac{t}{h}-n+1)}\right]\\ &= \lim_{\frac{t}{h}\to\infty} S_{\frac{t}{h}}\\ &= t^{\mu}E_{\nu,\mu+1}(\lambda t^{\nu}). \quad \square\end{aligned}$$

REMARK 2. While discrete Mittag-Leffler type function $\tilde{E}^{1}_{\lambda,\nu,\nu-1}(t,0)$ for $\nu \in (0,1)$, solves the equation

$$\nabla^{\nu}_{h,0}y(t) = \lambda y(t-h), \quad t \in h\mathbb{N}_h,$$

another Mittag-Leffler type function given in [4]

$$_h\hat{e}\left(\lambda, t_h^{\overline{\nu}}\right) = (1+\lambda h^{\nu})\frac{1}{h^{\nu-1}}\sum_{n=0}^{\infty}\frac{\lambda^n (t+h)_h^{\overline{\nu n+\nu-1}}}{\Gamma(\nu n+\nu)}, \quad t \in h\mathbb{N}_0$$

solves the equation

$$\nabla^{\nu}_{h,0}y(t) = \lambda y(t), \quad t \in h\mathbb{N}_h.$$

The following approximation can be proven using Tannery's theorem:

$$\lim_{h\to 0}\left[\frac{h^{\nu-1}}{(1+\lambda h^{\nu})}{}_h\hat{e}\left(\lambda, t_h^{\overline{\nu}}\right)\right] = t^{\nu-1}E_{\nu,\nu}(\lambda t^{\nu}).$$

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