Fractional
Differential
Calculus
Volume 11, Number 2 (2021), 193–201
doi:10.7153/fdc-2021-11-13

ON ASYMPTOTIC PROPERTIES OF SOME NEUTRAL DIFFERENTIAL EQUATIONS INVOLVING RIEMANN–LIOUVILLE FRACTIONAL DERIVATIVE

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(*Communicated by W. Deng*)

Abstract. The main purpose of the present note is to investigate asymptotic properties of some neutral delay differential equations involving Reimann-Liouville fractional derivative by means of Lyapunov functions. Integer order derivatives are used to overcome the difficulties of calculating the derivatives of Lyapunov functions. Two examples are given to illustrate the results.

1. Introduction

As an extension of integer-order differentiation and integration, the subject of fractional calculus has drawn huge attention from many researchers. Fractional-order models own better description of memory and hereditary properties of various processes than integer-order ones. In recent decades, fractional differential equations [1, 2] have been proved to be an excellent tool in the modelling of many phenomena in various fields of electro-chemistry, diffusion, viscoelastic materials, control systems, biological systems and so on [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. There is no doubt that Lyapunov functional method provides a very effective approach to analyse stability of integer-order non-linear systems. Compared with integer-order systems, it is more difficult to choose Lyapunov functionals for fractional order cases, which results in many difficulties in investigating the asymptotic behaviour of such systems. For example, the well-known Leibniz rule does not hold for fractional derivative, this is one of the main reasons that there are very few practical algebraic criteria on stability of fractional-order systems. In order to overcome this difficulty, we construct suitable functionals, and calculate its first-order derivatives to derive stability conditions.

In the first stage of this paper, we give asymptotic stability results for solutions of the equation

$$ {}_{t_0}D_t^{\alpha}[x(t)+cx(t-\sigma)] = -a(t)g(x(t)) - b(t)f(x(t-r)), \quad (0<\alpha<1) \tag{1} $$

for all $t \geqslant t_1 = t_0 + \rho$, where $\rho = \max\{r, \sigma\}$ and the functions $a(t)$, $b(t)$, $f(x(t))$ and $g(x(t))$ are continuous in their respective arguments, with $c < 1$ and $f(0) = g(0) = 0$.

Mathematics subject classification (2020): 34K20, 34K40.

Keywords and phrases: Asymptotic properties, Lyapunov function, Riemann-Liouville fractional derivative, neutral differential equations.

It is also supposed that the derivatives $f'(x(t))$ and $g'(x(t))$ exist and are continuous. For each solution of (1), we assume the following initial condition :

$$ {}_{t_0}D_t^{-(1-\alpha)}x(t) = \phi(t), \quad t \in [-\rho, t_0], \quad \phi \in C([-\rho, 0], \mathbb{R}) $$

REMARK 1. The reader is referred to [1, 2, 14, 15] for general references related to neutral differential equations and fractional differential equations.

Also, define the operator ψ by : $\psi(x_t) = x(t) - cx(t-\sigma)$.

LEMMA 1. ([14]) *The operator ψ is said to be stable if the zero solution of the homogeneous difference equation $\psi(x_t) = 0, t \geqslant 0$ is uniformly asymptotically stable.*

LEMMA 2. ([14]) *The operator ψ is stable if $|c| < 1$.*

We will in the next stage, give results on stability of solutions for the more general equation

$$ {}_{t_0}D_t^{\alpha}[x(t) + cx(t-\sigma)] = -a(t)g(x(t)) - b(t)f(x(t-r)) - e(t)\int_{t-\delta}^{t} x(s)ds, \quad (0 < \alpha < 1) \tag{2} $$

for all $t \geqslant t_1 = t_0 + \rho$, where $\rho = \max\{r, \sigma, \delta\}$ and the functions are as previously defined, moreover, $e(t)$ is continuous with respect to its argument.

2. Preliminaries

This section is devoted to the introduction of some definitions and useful tools of fractional calculus.

DEFINITION 1. ([1]) The Riemann-Liouville fractional integral of order α for a function $x(t)$ is defined as

$$ {}_{t_0}D_t^{-\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}x(s)ds, \tag{3} $$

where $\alpha > 0, t > t_0$. The Gamma function $\Gamma(\alpha)$ is defined by the integral

$$ \Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1}e^{-s}ds. $$

DEFINITION 2. ([1]) The Riemann-Liouville fractional derivative of order α for a function $x(t)$ is defined as

$$ {}_{t_0}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-s)^{n-\alpha-1}x(s)ds, \tag{4} $$

where $0 \leqslant n-1 \leqslant \alpha < n$, $n \in \mathbb{Z}^+$.

LEMMA 3. ([1]) *If $\beta > \alpha > 0$, then the following equality*

$$_{t_0}D_t^{\alpha}\left({}_{t_0}D_t^{-\beta}x(t)\right) = {}_{t_0}D_t^{\alpha-\beta}x(t)$$

holds for sufficiently good functions $x(t)$. In particular, this relation holds if $x(t)$ is integrable.

LEMMA 4. ([3]) *Let $x(t) \in \mathbb{R}$ be a continuous and differentiable function. If the derivative of $x(t)$ is integrable, then the following inequality holds*

$$_{t_0}D_t^{\alpha}x^2(t) \leqslant 2x(t)_{t_0}D_t^{\alpha}x(t), \quad \forall\alpha \in (0,1). \tag{5}$$

3. Main results

In this section, by constructing suitable Lyapunov functionals, sufficient conditions are presented to ensure that the origin of equation (1) (respectively equation (2)) is asymptotically stable.

In the sequel we assume what follows hold. Suppose that there exist positive constants a_i, b_i, f_i, and g_i for $i = 0,1$, such that

i) $a_0 \leqslant a(t) \leqslant a_1, b_0 \leqslant b(t) \leqslant b_1$.

ii) $\left|f'(x)\right| \leqslant f_1$, and $\dfrac{f(x)}{x} \geqslant f_0 \;\; (x \neq 0)$, for all x.

iii) $\left|g'(x)\right| \leqslant g_1$, and $\dfrac{g(x)}{x} \geqslant g_0 \;\; (x \neq 0)$, for all x.

The first theorem concerns the asymptotic properties of solutions of equation (1) and is stated as follows

THEOREM 1. *Assume (i)–(iii) satisfied. Suppose also that*

$$2a_0g_0 > b_1 + c\left(a_1 + b_1 + a_1g_1^2\right) + b_1f_1^2(1+c)$$

then, the trivial solution of equation (1) is asymptotically stable.

Proof. We make use of the Lyapunov function

$$V(t) = \frac{1}{2}{}_{t_0}D_t^{\alpha-1}\psi^2(t) + \varepsilon_0\int_{t-\sigma}^{t} x^2(s)ds + \varepsilon_1\int_{t-r}^{t} x^2(s)ds, \tag{6}$$

where ε_0 and ε_1 are positive constants to be determined later with respect to the appropriate case. From the estimate

$$V(t) \geqslant \frac{1}{2}{}_{t_0}D_t^{\alpha-1}\psi^2(t), \tag{7}$$

it can be easily seen that the function (6) is positive definite in view of Lemma 3 since $\varepsilon_0 \int_{t-\sigma}^{t} x^2(s)ds + \varepsilon_1 \int_{t-r}^{t} x^2(s)ds \geqslant 0$.

Rewrite equation (1) as the equivalent system

$$\begin{cases} \psi(t) = x(t) + cx(t-\sigma), \\ {}_{t_0}D_t^{\alpha-1}\psi(t) = -a(t)g(x(t)) - b(t)f(x(t-r)). \end{cases} \tag{8}$$

The time derivative of (6) along trajectories of equation (8), by the use of Lemma 3, is given by

$$\begin{aligned} V'_{(8)}(t) &\leqslant \psi(t)_{t_0}D_t^{\alpha}\psi(t) + (\varepsilon_0+\varepsilon_1)x^2(t) - \varepsilon_0 x^2(t-\sigma) - \varepsilon_1 x^2(t-r) \\ &\leqslant (\varepsilon_0+\varepsilon_1)x^2(t) - \varepsilon_0 x^2(t-\sigma) - \varepsilon_1 x^2(t-r) \\ &\quad - a(t)g(x(t))x(t) - b(t)f(x(t-r))x(t) \\ &\quad - ca(t)g(x(t))x(t-\sigma) - cb(t)f(x(t-r))x(t-\sigma) \end{aligned}$$

With the help of the conditions (i)–(iii) and the inequality $2|uv| \leqslant u^2 + v^2$, one obtains

$$\begin{aligned} V'_{(8)}(t) \leqslant &\frac{1}{2}\left(-2a_0g_0 + b_1 + ca_1g_1^2 + 2\varepsilon_0 + 2\varepsilon_1\right)x^2(t) \\ &+ \frac{1}{2}\left(-2\varepsilon_0 + c\left(a_1+b_1\right)\right)x^2(t-\sigma) \\ &+ \frac{1}{2}\left(-2\varepsilon_1 + b_1f_1^2(1+c)\right)x^2(t-r). \end{aligned}$$

Take

$$2\varepsilon_0 = c\left(a_1+b_1\right),$$

and

$$2\varepsilon_1 = b_1f_1^2(1+c),$$

to get

$$V'_{(8)}(t) \leqslant \frac{1}{2}\left(-2a_0g_0 + b_1 + c\left(a_1+b_1+a_1g_1^2\right) + b_1f_1^2(1+c)\right)x^2(t).$$

Hence

$$V'_{(8)} \leqslant -k_0x^2, \tag{9}$$

where $k_0 = 2a_0g_0 - b_1 - c\left(a_1+b_1+a_1g_1^2\right) - b_1f_1^2(1+c) > 0$. We conclude that the trivial solution of equation (8) is asymptotically stable according to [14]. The proof is complete. □

We are now concerned with the asymptotic stability of the solutions for equation (2). First, for brevity, we make the notation

$$\gamma = b_1 + e_1 + ca_1g_1^2 + b_1f_1^2(1+c) + c\left(a_1+b_1+e_1\right), \tag{10}$$

and we rewrite equation (2) in the descriptor form

$$\begin{cases} \psi(x_t) = x(t) + cx(t-\sigma), \\ {}_{t_0}D_t^{\alpha}\psi(t) = -a(t)g(x(t)) - b(t)f(x(t-r)) - e(t)\int_{t-\delta}^{t} x(s)ds. \end{cases} \tag{11}$$

Our next main result is stated as follows

THEOREM 2. *Assume conditions* (i)–(iii) *satisfied. Suppose there exist positive constants* e_0, e_1 *and* ε *such that*

iv) $0 < e_0 \leqslant e(t) \leqslant e_1$.

v) $-2a_0g_0 + \gamma = -\varepsilon < 0$.

Then, the trivial solution of equation (11) *is asymptotically stable provided that*

$$\delta < \frac{2a_0g_0 - \gamma}{e_1(1+c)}.$$

Proof. Define a Lyapunov function by

$$V(t,x) = \frac{1}{2}{}_{t_0}D_t^{\alpha-1}\psi^2(t) + \mu\int_{t-\sigma}^{t} x^2(s)ds + \lambda\int_{t-r}^{t} x^2(s)ds + \eta\int_{-\delta}^{0}\int_{t+s}^{t} x^2(\theta)d\theta ds, \tag{12}$$

where μ, λ and η are positive constants to be determined later.

The time derivative of the Lyapunov function (12) along trajectories of equation (11) using Lemmas 3 and 4, is

$$\begin{aligned} V'_{(11)}(t) \leqslant & \psi(x_t)\, {}_{t_0}D_t^{\alpha}\psi(x_t) \\ & + {}_{t_0}D_t^{1}\left(\mu\int_{t-\sigma}^{t} x^2(s)ds + \lambda\int_{t-r}^{t} x^2(s)ds + \eta\int_{-\delta}^{0}\int_{t+s}^{t} x^2(\theta)d\theta ds\right) \\ \leqslant & x(t)\left(-a(t)g(x(t)) - b(t)f(x(t-r)) - e(t)\int_{t-\delta}^{t} x(s)ds\right) \\ & + cx(t-\sigma)\left(-a(t)g(x(t)) - b(t)f(x(t-r)) - e(t)\int_{t-\delta}^{t} x(s)ds\right) \\ & + (\mu+\lambda+\eta\delta)x^2 - \mu x^2(t-\sigma) - \lambda x^2(t-r) - \eta\int_{t-\delta}^{t} x^2(s)ds. \end{aligned}$$

With the help of conditions (i)–(iv) and the inequality $2|uv| \leqslant u^2 + v^2$, one obtains

$$\begin{aligned} V'_{(11)}(t) \leqslant & \frac{1}{2}\left(-2a_0g_0 + b_1 + e_1 + ca_1g_1^2 + 2\mu + 2\lambda + 2\eta\delta\right)x^2(t) \\ & + \frac{1}{2}\left(b_1f_1^2(1+c) - 2\lambda\right)x^2(t-r) \\ & + \frac{1}{2}\left(c(a_1+b_1+e_1) - 2\mu\right)x^2(t-\sigma) \\ & + \frac{1}{2}\left(e_1(1+c) - 2\eta\right)\int_{t-\delta}^{t} x^2(s)ds. \end{aligned}$$

Choose

$$2\lambda = b_1 f_1^2 (1+c),$$
$$2\mu = c\,(a_1+b_1+e_1)\,,$$

and

$$2\eta = e_1(1+c).$$

With the previous choice of constants λ,μ and η, using (10), we get

$$V'_{(11)}(t) \leqslant \frac{1}{2}\left(-2a_0g_0+\gamma+\delta c_1\left(1+\beta_1\right)\right)x^2.$$

From condition (v), there exists a positive constant k such that

$$V'_{(11)}(t) \leqslant -kx^2, \tag{13}$$

Noting that the operator $\psi(x_t)$ is stable, therefore, trivial solution of equation (2) is asymptotically stable according to [14], provided that

$$\delta < \frac{2a_0g_0-\gamma}{e_1(1+c)}.$$

This fact ends the proof. □

REMARK 2. If $e(t)=0$ in equation (2), then, the latter one reduces to the simpler case equation (1), and we can see that the result here still holds.

4. Examples

In this section, we give examples illustrating the obtained results.

4.1. Example 1

As a special case of equation (1), consider the following equation

$$\begin{aligned} {}_{t_0}D_t^{\alpha}\left[x(t)+\frac{1}{10}x(t-\sigma)\right] &= -\left(4+\frac{1}{2+t^2}\right)\times\left(x(t)+\frac{2x(t)}{10+|x(t)|}\right) \\ &\quad -\left(0.2+\frac{2}{10+t^2}\right)\times\left(0.5x(t-r)+\frac{x(t-r)}{10+|x(t-r)|}\right). \end{aligned} \tag{14}$$

Observing the functions over the equation (14), one can deduce the following

$$a_0 = 4 \leqslant a(t) = 4+\frac{1}{2+t^2} \leqslant 4.5 = a_1$$
$$b_0 = 0.2 \leqslant b(t) = 0.2+\frac{2}{10+t^2} \leqslant 0.4 = b_1,$$
$$c = \frac{1}{10} < 1, \quad \alpha \in (0,1),$$
$$g(x) = x+\frac{2x}{10+|x|},$$

and

$$f(x) = 0.5x + \frac{x}{10+|x|}.$$

It is clear, from the relation of $f(x)$ and $g(x)$, that $f(0) = g(0) = 0$, besides, since $0 \leqslant \frac{1}{10+|x|} \leqslant 1$, for all x, we have

$$\frac{g(x)}{x} \geqslant 1 = g_0, \quad \text{for all } x \neq 0,$$

and

$$\frac{f(x)}{x} \geqslant 0.5 = f_0, \quad \text{for all } x \neq 0.$$

Moreover

$$\left|g'(x)\right| = \left|1 + \frac{20}{(10+|x|)^2}\right| \leqslant 1.2 = g_1,$$

$$\left|f'(x)\right| = \left|0.5 + \frac{10}{(10+|x|)^2}\right| \leqslant 0.6 = f_1.$$

A simple calculation give

$$-2a_0g_0 + b_1 + c\left(a_1 + b_1 + a_1g_1^2\right) + b_1f_1^2(1+c) = -6.1 = -k_0 < 0.$$

Hence, the trivial solution of (14) is asymptotically stable by assumptions of Theorem 1.

4.2. Example 2

As a special case of equation (2), consider the following equation

$$\begin{aligned} {}_{t_0}D_t^{\alpha}\left[x(t) + \frac{1}{10}x(t-\sigma)\right] = &-\left(2 + \frac{1}{2+t^2}\right) \times \left(x(t) + \frac{2x(t)}{10+|x(t)|}\right) \\ &-\left(0.2 + \frac{2}{10+t^2}\right) \times \left(0.5x(t-r) + \frac{x(t-r)}{10+|x(t-r)|}\right) \\ &-\left(1.5 + \frac{1}{10+t^2}\right)\int_{t-\delta}^{t} x(s)ds \end{aligned} \tag{15}$$

$$a_0 = 2 \leqslant a(t) = 2 + \frac{1}{2+t^2} \leqslant 2.5 = a_1,$$

$$b_0 = 0.2 \leqslant b(t) = 0.2 + \frac{2}{10+t^2} \leqslant 0.4 = b_1,$$

$$e_0 = 1.5 \leqslant e(t) = 1.5 + \frac{1}{10+t^2} \leqslant 1.6 = e_1,$$

$$c = \frac{1}{10} < 1, \quad \alpha \in (0,1),$$

$$f(x) = 0.5x + \frac{x}{10+|x|},$$

and

$$g(x) = x + \frac{2x}{10+|x|}.$$

It is clear from the relation of $f(x)$ and $g(x)$, that $f(0) = g(0) = 0$, besides, since $0 \leqslant \frac{1}{10+|x|} \leqslant 1$, for all x, we have

$$\frac{g(x)}{x} \geqslant 1 = g_0, \qquad \text{for all } x \neq 0,$$

and

$$\frac{f(x)}{x} \geqslant 0.5 = f_0, \qquad \text{for all } x \neq 0.$$

Moreover

$$\left|g'(x)\right| = \left|1 + \frac{20}{(10+|x|)^2}\right| \leqslant 1.5 = g_1,$$
$$\left|f'(x)\right| = \left|0.5 + \frac{10}{(10+|x|)^2}\right| \leqslant 0.6 = f_1.$$

A simple calculation gives

$$-2a_0g_0 + b_1 + e_1 + ca_1g_1^2 + b_1f_1^2(1+c) + c\,(a_1+b_1+e_1) = -0.83 = -\gamma < 0.$$

Thus, the trivial solution of (15) is asymptotically stable by assumptions of Theorem 2.

Acknowledgement. The author would convey his deepest gratitude to the anonymous reviewers for their observations and valuable comments that led to the improvement of the quality of the paper.

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(Received April 23, 2021)

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