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GENERALIZED WEIGHTED FRACTIONAL OSTROWSKI TYPE INEQUALITY WITH APPLICATIONS

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Abstract. We use Riemann-Liouville fractional integral to provide generalization of Weighted Ostrowski type inequality with bounded derivatives. Our results improved the inequalities of [14], and gave some applications.

1. Introduction

Fractional calculus is the field of mathematical analysis that deals with the investigation and applications of integrals and derivatives of random order. Fractional calculus is an old subject, starting from some assumptions of G. W. Leibniz $(1695, 1697)$ and L. Euler (1730), it has been studied extensively up to these days. Nowadays a lot of researchers and mathematicians are continuously working day and night in this attractive and interesting field. Due to the high importance and wide range of applications of fractional integral inequalities, several researchers have obtained various generalizations of fractional integral inequalities. Some generalizations can be found in these articles [2, 13, 14, 16].

A. M. Ostrowski has given a famous inequality in his paper [11] in 1938. This inequality could be used to estimate the deviation of functional value from its mean value. This famous inequality known as Ostrowski inequality can be found using the Montgomery identity.

In the following proposition, we state an inequality from [3].

PROPOSITION 1. *Let $f : I \to \mathbb{R}$ be a differentiable mappings on I^o such that $f \in L[a,b]$, where $a,b \in I$ and $a < b$. If $|f'(x)| \leqslant M \ \forall \ x \in (a,b)$ where M is positive real constant, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right]. \tag{1}$$

The constant $\frac{1}{4}$ is the best possible constant that it can not be replaced by the smaller one.

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We state the well known Montgomery identity from “Inequalities for Functions and Their Integrals and Derivatives” by D. S. Mitrinović et al. in [8].

PROPOSITION 2. *Let $f \in AC[a,b]$. Then*

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\int_a^b P(x,t)f'(t)dt, \tag{2}$$

where Peano kernel $P(x,t)$ is given as

$$P(x,t) = \begin{cases} t-a, \ \text{if} \quad t \in [a,x], \\ t-b, \ \text{if} \quad t \in (x,b]. \end{cases} \tag{3}$$

For our next result we need here definition of Riemann-Liouville fractional integral from [6].

DEFINITION 1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined as

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt$$
$$J_a^0 f(x) = f(x),$$

where gamma function $\Gamma(\alpha)$ is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\alpha^{-x}dx.$$

In [2] by using Riemann-Liouville fractional integrals, the authors obtained inequalities for differentiable functions that are linked with Ostrowski inequality, and discussed following proposition to prove their results.

PROPOSITION 3. *Let $f : I \to \mathbb{R}$ be differentiable mapping on I^0 with $a,b \in I$ $a < b$, $f' \in L_1[a,b]$ and for $\alpha \geqslant 1$. Then Montgomery fractional identity holds*

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^\alpha f(b) - J_a^{\alpha-1}(P_1(x,b)f(b)) + J_a^\alpha(P_1(x,b)f'(b)), \tag{4}$$

where $P_1(x,t)$ is the fractional Peano kernel defined by

$$P_1(x,t) = \begin{cases} \dfrac{t-a}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), \ \text{if} \quad t \in [a,x), \\ \dfrac{t-b}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), \ \text{if} \quad t \in [x,b]. \end{cases} \tag{5}$$

In [2] by using Riemann-Liouville fractional integrals, the authors also obtained inequalities for differentiable functions that are linked with weighted Ostrowski type inequality, that is given below.

PROPOSITION 4. *Let $w:[a,b]\to[0,\infty)$ be a probability density function, i.e., $\int_a^b w(t)dt=1$, and set $w(t)=\int_a^t w(x)dx$ for $a\leqslant t\leqslant b$, $w(t)=0$ for $t<a$ and $w(t)dt=1$ for $t>b$, $\alpha\geqslant 1$. Then the generalization of the weighted Montgomery identity for fractional integrals is in the following form:*

$$f(x)=(b-x)^{1-\alpha}\Gamma(\alpha)J_a^{\alpha}(w(b)f(b))-J_a^{\alpha-1}(K_w(x,b)f(b))+J_a^{\alpha}(K_w(x,b)f'(b)),$$

where the weighted fractional Peano kernal

$$K_w(x,t)=\begin{cases}\left[(b-x)^{1-\alpha}\Gamma(\alpha)\int_a^t w(t)dt\right], & \text{if} \quad a\leqslant t<x,\\ \left[(b-x)^{1-\alpha}\Gamma(\alpha)\int_b^t w(t)dt\right], & \text{if} \quad x\leqslant t\leqslant b.\end{cases}$$

In the next main section, we obtained some new results of generalized weighted fractional integral inequalities of Ostrowski type with bounded derivatives.

2. Generalized fractional Ostrowski type inequalities

We need to proof the following lemma for our main result.

LEMMA 1. *Let all suppositions of Proposition 4 be valid. Then the following identity holds*

$$\begin{aligned}f(x)&=2J_a^{\alpha}\left(K_{1w}(x,b)f'(b)\right)+\Gamma(\alpha)(b-x)^{1-\alpha}J_a^{\alpha}(f(b)w(b))-2J_a^{\alpha-1}\left(K_w(x,b)f(b)\right)\\&\quad-\frac{(b-x)^{1-\alpha}}{(b-a)^{1-\alpha}}\left(\int_a^x w(t)dt\right)J_a^0 f(a)-(b-x)^{1-\alpha}J_a^{\alpha-1}\left(\left(\int_x^b w(t)dt\right)f(b)\right)\end{aligned}$$

where $K_w(x,t)$ is as defined in Proposition 4 and $K_{1w}(x,t)$ is the fractional Peano kernel defined by

$$K_{1w}(x,t)=\begin{cases}\left[\int_a^t w(t)dt+\int_x^t w(t)dt\right]\frac{(b-x)^{1-\alpha}}{2}\Gamma(\alpha), & \text{if} \quad t\in[a,x),\\ \left[\int_b^t w(t)dt+\int_x^t w(t)dt\right]\frac{(b-x)^{1-\alpha}}{2}\Gamma(\alpha), & \text{if} \quad t\in[x,b].\end{cases}$$

Proof. Using Riemann-Liouville fractional integral operator on $K_{1w}(x,t)$, we get

$$\begin{aligned}&J_a^{\alpha}(K_{1w}(x,b)f'(b))\\&=\frac{1}{\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}K_{1w}(x,t)f'(t)dt\\&=\frac{(b-x)^{1-\alpha}}{2}\left[\int_a^x(b-t)^{\alpha-1}\left(\int_a^t w(t)dt+\int_x^t w(t)dt\right)\right.\\&\quad\left.\times f'(t)dt+\int_x^b(b-t)^{\alpha-1}\left(\int_b^t w(t)dt+\int_x^t w(t)dt\right)f'(t)dt\right]\\&=\frac{1}{2}\left[J_a^{\alpha}(K_w(x,b)f'(b))+(b-x)^{1-\alpha}\int_a^b(b-t)^{\alpha-1}\left(\int_x^t w(t)dt\right)f'(t)dt\right].\end{aligned} \tag{6}$$

We also have

$$\begin{aligned}
&\int_a^b (b-t)^{\alpha-1}\left(\int_x^t w(t)dt\right) f'(t)dt \\
&= (b-a)^{\alpha-1}\left(\int_a^x w(t)dt\right) f(a) \\
&\quad +(\alpha-1)\int_a^b f(t)(b-t)dt - \int_a^b f(t)(b-t)^{\alpha-1}w(t)dt. \qquad (7)
\end{aligned}$$

Now we have

$$\begin{aligned}
&J_a^\alpha \left(K_w(x,b)f'(b)\right) \\
&= f(x) - (b-x)^{1-\alpha}\Gamma(\alpha)J_a^\alpha(w(b)f(b)) + J_a^{\alpha-1}(K_w(x,b)f(b))]\,. \qquad (8)
\end{aligned}$$

Using (7) and (8) in (6), we get the result

$$\begin{aligned}
&J_a^\alpha(K_{1w}(x,b)f'(b)) \\
&= \frac{1}{2}\left[f(x)\int_a^b w(t)dt + J_a^{\alpha-1}(K_w(x,b)f(b)) + (b-x)^{1-\alpha}\Gamma(\alpha)J_a^\alpha f(b)\left(\int_x^t w(b)db\right)\right. \\
&\quad \left. -(b-x)^{1-\alpha}\Gamma(\alpha)J_a^\alpha(f(b)w(b)) + \frac{(b-x)^{1-\alpha}}{(b-a)^{1-\alpha}}J_a^0 f(a)\left(\int_x^a w(t)dt\right)\right]. \quad \square
\end{aligned}$$

REMARK 1. If we replace $w(t) = \dfrac{1}{b-a}$ in (6), then we get Corollary 2.4 of [10].

REMARK 2. If we replace $\alpha = 1$ and $w(t) = \dfrac{1}{b-a}$ in (6), then we get Corollary 2.5 of [10, 17].

By using Lemma 1, we obtain generalized Ostrowski–Grüss fractional integral inequality in the next theorem.

THEOREM 1. *Let f be a differentiable mapping on $[a,b]$ and $|f'(x)| \leqslant M$ for any $x \in [a,b]$. Then the following integral inequality holds*

$$\begin{aligned}
&\left|\frac{1}{2}f(x) - \Gamma(\alpha)\frac{(b-x)^{1-\alpha}}{2}J_a^\alpha(f(b)w(b)) + \frac{1}{2}J_a^{\alpha-1}(K_w(x,b)f(b))\right. \\
&\quad \left. + \frac{1}{2}(b-x)^{1-\alpha}J_a^{\alpha-1}f(b)\left(\int_x^t w(t)dt\right) + \frac{(b-x)^{1-\alpha}}{2(b-a)^{1-\alpha}}J_a^0 f(a)\left(\int_a^x w(t)dt\right)\right| \\
&\leqslant \frac{M}{\Gamma(\alpha)}A_{w,\alpha}(x,t) \qquad (9)
\end{aligned}$$

where

$$A_{w,\alpha}(x,t) = \int_a^b (b-t)^{\alpha-1}|K_1(x,t)|dt, \quad \alpha \geqslant 1.$$

Proof. From Lemma 1, consider

$$\begin{aligned}
&\left|J_a^{\alpha} K_1(x,b) f'(b)\right| \\
&= \Big|\frac{1}{2} f(x) - \Gamma(\alpha) \frac{(b-x)^{1-\alpha}}{2(b-a)} J_a^{\alpha}(f(b)w(b)) + \frac{1}{2} J_a^{\alpha-1}(K_w(x,b)f(b)) \\
&\quad + \frac{1}{2}(b-x)^{1-\alpha} J_a^{\alpha-1} f(b) \left(\int_x^t w(t)dt\right) + \frac{(b-x)^{1-\alpha}}{2(b-a)^{1-\alpha}} J_a^0 f(a) \left(\int_a^x w(t)dt\right)\Big| \\
&= \left|\frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} K_1(x,t) f'(t) dt\right| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} |K_1(x,t)||f'(t)| dt \\
&\leqslant \frac{M}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} |K_1(x,t)| dt \\
&\leqslant \frac{M}{\Gamma(\alpha)} A_{w,\alpha}(x,t). \quad \square
\end{aligned}$$

REMARK 3. If we replace $w(t) = \frac{1}{b-a}$ in (9), then we get Corollary 2.8 of [10].

REMARK 4. If we replace $w(t) = \frac{1}{b-a}$ and $\alpha = 1$ in (9), then we get Corollary 2.9 of [10].

REMARK 5. If we replace $w(t) = \frac{1}{b-a}$ with $\alpha = 1$ and $x = \frac{a+b}{2}$ in (9), then we get Corollary 2.11 of [10].

REMARK 6. If we put $x = a$ or $x = b$, $w(t) = \frac{1}{b-a}$ and $\alpha = 1$ in (9), then we get the Corollary 2.15 of [10] in this way we get a bound for trapezoidal inequality (Hermite-Hadamard right inequality).

3. Application in numerical integration

Let $I_n : a = z_0 < z_1 < \ldots < z_{n-1} < z_n = b$ be a division of the interval $[a,b]$ and let $h_i = z_{i+1} - z_i$, $z_i \leqslant \xi_i \leqslant z_{i+1}$ and $A_i = \frac{z_i + z_{i+1}}{2}$ for $i \in \{0, \ldots, n-1\}$.

Consider the general quadrature formula where

$$\begin{aligned}
Q_n(f,w,\xi) = \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{n-1} \Bigg[& \frac{f(\xi_i) + J_{z_i}^{\alpha-1}(K_w(\xi_i, z_{i+1}) f(z_{i+1}))}{(z_{i+1} - \xi_i)^{1-\alpha}} + J_{z_i}^{\alpha-1} f(z_{i+1}) \\
& \times \left(\int_{\xi_i}^t w(t)dt\right) + \frac{J_{z_i}^0 f(z_i)}{(h_i)^{1-\alpha}} \left(\int_{z_i}^{\xi_i} w(t)dt\right)\Bigg].
\end{aligned} \tag{10}$$

THEOREM 2. *Under the assumptions of Theorem 1, we have*

$$J_a^\alpha(f(b)w(b)) = Q_n(f,w,\xi) + R_n(f,w,\xi)$$

where $Q_n(f,w,\lambda)$ *is defined in* (10) *and the remainder satisfies the estimates*

$$|R_n(f,w,\xi)| \leqslant \sum_{i=0}^{n-1}\left[\frac{2M}{\Gamma(\alpha)}A_{(w,\alpha)}(\xi_i,t)\right] \tag{11}$$

where

$$A_{(w,\alpha)}(\xi_i,t) = \int_{z_i}^{z_{i+1}}(z_{i+1}-t)^{\alpha-1}|Z_{i+1}(\xi_i,t)|dt, \quad \alpha \geqslant 1.$$

Proof. We may apply inequality (9) on $[z_i, z_{i+1}]$,

$$|R(f,w,\xi)| = J_{z_i}^\alpha f((z_{i+1})(w_{i+1})) - \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n-1}\left[\frac{f(\xi_i)+J_{z_i}^{\alpha-1}(K_w(\xi_i,z_{i+1})f(z_{i+1}))}{(z_{i+1}-\xi_i)^{1-\alpha}}\right.$$
$$\left. + J_{z_i}^{\alpha-1}f(z_{i+1})\left(\int_{\xi_i}^t w(t)dt\right) + \frac{J_{z_i}^0 f(z_i)}{(h_i)^{1-\alpha}}\left(\int_{z_i}^{\xi_i}w(t)dt\right)\right]. \tag{12}$$

Summing the above inequality over i from 0 to $n-1$, we get

$$|R(f,w,\xi)|$$
$$= \sum_{i=0}^{n-1}J_{z_i}^\alpha f((z_{i+1})(w_{i+1})) - \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n-1}\left[\frac{f(\xi_i)+J_{z_i}^{\alpha-1}(K_w(\xi_i,z_{i+1})f(z_{i+1}))}{(z_{i+1}-\xi_i)^{1-\alpha}}\right.$$
$$\left. + J_{z_i}^{\alpha-1}f(z_{i+1})\left(\int_{\xi_i}^t w(t)dt\right) + \frac{J_{z_i}^0 f(z_i)}{(h_i)^{1-\alpha}}\left(\int_{z_i}^{\xi_i}w(t)dt\right)\right]. \tag{13}$$

According to (9), we have

$$|R_n(f,w,\lambda)| \leqslant \sum_{i=0}^{n-1}\left[\frac{2M}{\Gamma(\alpha)}A_{w,\alpha}(\xi_i,t)\right] \tag{14}$$

where

$$A_{(w,\alpha)}(\xi_i,t) = \int_{z_i}^{z_{i+1}}(z_{i+1}-t)^{\alpha-1}|Z_{i+1}(\xi_i,t)|dt, \quad \alpha \geqslant 1. \quad \square$$

REMARK 7. If we replace $w(t) = \frac{1}{h_i}$, in (11), $i \in \{0,\dots,n-1\}$, then we obtain the following corollary.

COROLLARY 1. *Let all the assumptions of Theorem 1 be valid. Then*

$$Q(f,w,\xi) = \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{n-1} \left[\frac{f(\xi_i)}{(z_{i+1}-\xi_i)^{1-\alpha}} + \frac{J_{z_i}^{\alpha-1}(P_1(\xi_i,z_{i+1})f(z_{i+1}))}{(z_{i+1}-\xi_i)^{1-\alpha}} \right.$$
$$\left. + J_{z_i}^{\alpha-1} f(z_{i+1}) \frac{(t-\xi_i)}{h_i} + \frac{(\xi_i - z_i) J_{z_i}^0 f(z_i)}{h_i^{2-\alpha}} \right]. \tag{15}$$

and

$$|R(f,w,\xi)|$$
$$\leqslant \frac{2M(z_{i+1}-\xi_i)^{\alpha-1}}{h_i}$$
$$\times \left[\frac{(\xi_i - z_i)}{2\alpha} \left\{ {h_i}^{\alpha} - (z_{i+1}-\xi_i)^{\alpha} + (z_{i+1}-\xi_i)^{\alpha+1} \right\} + \frac{2}{\alpha(\alpha+1)} \right.$$
$$\left. \times \left\{ \left(z_{i+1} - \frac{z_i+\xi_i}{2} \right)^{\alpha+1} - (z_{i+1}-\xi_i)^{\alpha+1} - \frac{{h_i}^{\alpha+1}}{2} + \left(\frac{z_{i+1}-\xi_i}{2} \right)^{\alpha+1} \right\} \right]. \tag{16}$$

REMARK 8. Put $\alpha = 1$ in (15) and (16), $i \in \{0,\dots,n-1\}$. Then we obtain the following corollary.

COROLLARY 2. *Let all suppositions of Theorem 1 be valid. Then*

$$Q(f,w,\xi) = \sum_{i=0}^{n-1} \left[f(\xi_i) + f(z_{i+1}) \left(P_1(\xi_i,z_{i+1}) + \frac{(t-\xi_i)}{h_i} \right) + \frac{(\xi_i - z_i) f(z_i)}{h_i} \right] \tag{17}$$

and

$$|R(f,w,\lambda)| \leqslant \sum_{i=0}^{n-1} \frac{M}{(z_{i+1}-z_i)} \left[(\xi_i - z_i)^2 + (z_{i+1}-\xi_i)^2 \right]. \tag{18}$$

REMARK 9. Let $\xi_i = \dfrac{z_i + z_{i+1}}{2} = A_i$, in (17) and (18), $i \in \{0,\dots,n-1\}$. Then we obtain the following corollary

COROLLARY 3. *Let all the assumptions of Theorem 1 be valid. Then*

$$\left| f(A_i) + \frac{f(z_i) + f(z_{i+1})}{2} \right| \leqslant \sum_{i=0}^{n-1} \frac{M}{2} h_i. \tag{19}$$

REMARK 10. Let $\xi_i = z_i$ or $\xi_i = z_{i+1}$ in (17) and (18), $i \in \{0,\dots,n-1\}$. Then we get the bound for trapezoidal inequality in the following corollary.

COROLLARY 4. *Let all suppositions of Theorem 1 be valid. Then*

$$Q(f,w,\xi) = \sum_{i=0}^{n-1}\left[\frac{f(z_i)+f(z_{i+1})}{2}\right]$$

and

$$|R(f,w,\xi)| \leqslant \sum_{i=0}^{n-1}\frac{M}{2}h_i.$$

4. Application for probability density function

Let X be a continuous random variable, the probability density function and the cumulative distribution function, respectively such that $f: [a,b] \to \mathbb{R}_+$ and $\Phi : [a,b] \to [0,1]$, defined as,

$$\Phi(x) = \int_a^x f(t)dt, \quad x \in [a,b],$$

and the expectations of the random variable X on $[a,b]$ are defined as,

$$E_w(X) = \Gamma(\alpha)J_a^\alpha(bw(b)f(b)) = \int_a^b t(b-t)^{\alpha-1}w(t)f(t)dt,$$
$$\Gamma(\alpha)J_a^\alpha(bw(b)f'(b)) = \int_a^b t(b-t)^{\alpha-1}w(t)\, f'(t)\, dt,$$
$$\Gamma(\alpha)J_a^{\alpha-1}(bw(b)f(b)) = (\alpha-1)\int_a^b t(b-t)^{\alpha-2}w(t)\, f(t)\, dt,$$
$$\Gamma(\alpha)J_a^\alpha(bw'(b)f(b)) = \int_a^b t(b-t)^{\alpha-1}w'(t)\, f(t)\, dt.$$

Then we have,

THEOREM 3. *Let f be a differentiable mapping on $[a,b]$ and $|\Phi'(x)| \leqslant M$ for any $x \in [a,b]$. Then the following integral inequality holds*

$$\Bigg|\Phi(x) + (b-x)^{1-\alpha}\left(J_a^{\alpha-1}(bw(b)f(b)) + J_a^\alpha(bw'(b)f(b)) + E_w(X)\right)$$
$$+J_a^{\alpha-1}(K_w(x,b)\Phi(b)) + (b-x)^{1-\alpha}J_a^{\alpha-1}\Phi(b)\left(\int_x^t w(t)dt\right)\Bigg|$$
$$\leqslant \frac{2M}{\Gamma(\alpha)}A_{(w,\alpha)}(x,t) \tag{20}$$

where

$$A_{(w,\alpha)}(x,t) = \int_a^b (b-t)^{\alpha-1}|K_1(x,t)|dt, \quad \alpha \geqslant 1.$$

Proof. By putting $f=\Phi$ in (2.5), we obtain (20). □

REMARK 11. If we replace $w(t)=\dfrac{1}{b-a}$ in (20).

COROLLARY 5. *Let all suppositions of Theorem 1 be valid. Then*

$$\begin{aligned}&\Bigg|f(x)-\frac{(b-x)^{1-\alpha}}{(b-a)}\left(J_a^{\alpha-1}(bf(b))+E(X)\right)+J_a^{\alpha-1}(P_1(x,b)f(b))\\&+\frac{(b-x)^{2-\alpha}}{(b-a)}\Gamma(\alpha)J_a^{\alpha-1}f(b)+\frac{(b-x)^{1-\alpha}}{(b-a)^{2-\alpha}}J_a^0f(a)(x-a)\Bigg|\\&\leqslant\frac{2M(b-x)^{1-\alpha}}{b-a}\left[\frac{(x-a)}{2\alpha}\left\{(b-a)^{\alpha}-(b-x)^{\alpha}+(b-x)^{\alpha+1}\right\}+\frac{1}{\alpha(\alpha+1)}\right.\\&\left.\times\left\{2\left(b-\frac{a+x}{2}\right)^{\alpha+1}-2(b-x)^{\alpha+1}-(b-a)^{\alpha+1}+2\left(\frac{b-x}{2}\right)^{\alpha+1}\right\}\right].\end{aligned}\tag{21}$$

5. Conclusion

In this article our target was to generalize the results of [14]. We have obtained Weighted fractional Ostrowski inequality with bounded derivatives by using the Riemann-Liouville integral. By using appropriate substitution we got different previous results as well as some better bounds stated in [14].

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