Fractional Differential Calculus
Volume 12, Number 2 (2022), 101–114
doi:10.7153/fdc-2022-12-07

EXISTENCE AND UNIQUENESS RESULTS OF IMPULSIVE FRACTIONAL NEUTRAL PANTOGRAPH INTEGRO DIFFERENTIAL EQUATIONS WITH DELAY

Ravi Ilavarasi and Kandasamy Malar

(*Communicated by N. Nymoradi*)

Abstract. In this paper, we study the existence and uniqueness results of Impulsive fractional neutral pantograph integro-differential equations with delay. The results are obtained by using the Krasnoselskii fixed point theorem. Finally examples are given to illustrate the main result obtained in this article.

1. Introduction

Fractional calculus is the mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order [22, 29]. In recent years, it is extensively applied to various fields such as viscous elastic mechanics, power fractal networks, electronic circuits [4, 21]. In [23], one can find some recent developments in the fields of fractional dynamics. Fractional Differential Equations emerged as a new branch of applied mathematics which has been used for many mathematical fields such in science and engineering. Benson [8], presented some applications of fractional calculus in the study of convective and diffusion of solutes in natural porous or fractured media. In [25, 26], we can see applications of fractional differential equations in complex dynamics, biological tissues, viscoelastic materials, signal processing, thermal systems and heat conduction. Concerning the development of theory and applications of fractional calculus, we refer to the monographs of [22, 28, 29] and papers [31, 32, 5, 6, 30]. Some recent results on the existence of solutions for fractional integro-differential and fractional differential equations can be found in [1, 2, 11, 12] and the references therein. Fractional delay differential equations arise in many applications, such as automatic control, long transmission lines, economy and biology [26]. The fractional pantograph equations, as a kind of fractional delay differential equations, plays an important role.

Impulsive differential equations are suitable mathematical model to simulate the evolution of large classes of real process. The impulsive differential equations arising from the real world problems to describe the dynamics of process in which sudden,

Mathematics subject classification (2020): 34A08, 34K05, 34K45.
Keywords and phrases: Fractional integro-differential equations, pantograph equations, impulsive conditions, delay.

discontinuous jumps occur. Such processes are naturally seen in biology, physics, engineering [24, 17].

Neutral differential equation appear in many areas of physical mathematics and, for this reason, these equations have received much attention over the last few decades. Some good literature for ordinary neutral functional differential equations are the books of Benchohra et al. [7], Graef et al. [14], and the references therein. On the other hand, for partial neutral functional differential equations we refer the reader to Balachandran [9], Hale [16].

In this paper we consider the following impulsive fractional neutral pantograph integro-differential equation of the form

$$\begin{aligned} {}^{C}\mathscr{D}^{\alpha}\big[u(t)-\mathfrak{A}(t)u(t-1)\big] &= \mathscr{F}(t,u(t),u(\lambda t))+\int_0^{qt}\mathscr{G}_1(t,s,u(s))ds \\ &\quad +\int_0^{t}\mathscr{G}_2(t,s,u(s))ds, \quad t\in J:=[0,T],\ t\neq t_k \\ \Delta u|_{t=t_k} &= \mathscr{I}_k(u(t_k));\quad k=1,2,\ldots,m \\ u(0) &= u_0, \\ u(t) &= \phi(t), \quad t\in[-1,0] \end{aligned} \tag{1.1}$$

where $0<\alpha,\ \lambda,\ q<1$, ${}^{C}\mathscr{D}^{\alpha}$ is the Caputo fractional derivative and $\mathscr{F}: J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\mathscr{G}_i: J\times J\times\mathbb{R}\to\mathbb{R}$ are continuous for $i=1,2$. $\mathfrak{A}(t)$ and $\mathscr{I}_k: X\to X$ are continuous function for $k=1,2,\ldots,m$. Here $0=t_0<t_1<t_2<\ldots<t_m<t_{m+1}=T$, $\Delta u(t_k)=u(t_k^+)-u(t_k^-)$, $u(t_k^+)=\lim_{h\to 0}u(t_k+h)$, $u(t_k^-)=\lim_{h\to 0}u(t_k-h)$, $k=1,2,\ldots,m$ represent the right and left limits of $u(t)$ *at* $t=t_k$, respectively.

Pantograph type equations have been studied extensively owing to the numerous application in which these equation arise. The name pantograph originated from the work of Ockendon and Taylor [27] on the collection of current by the pantograph head of an electric locomotive. The pantograph type equations are appeared in modelling of various problems in engineering and sciences such as biology, economy, control and electrodynamics [3]. Balachandran and Kiruthika [10] studied the existence of solutions of abstract fractional pantograph equations by using the fractional calculus and fixed point theorems. Yüzbasi et al. [33] investigated the numerical solution of generalized pantograph equation with a linear functional argument by virtue of introducing a collocation method based on the Bessel polynomials for the approximate solution of the pantograph equations. For some applications of this equation we refer [18, 19, 20, 15]. Due to its importance in many applied fields, it is interesting to study the fractional model of the pantograph equations.

Motivated by the above mentioned works, the main aim of this paper is to establish the existence and uniqueness solutions for the neutral pantograph integro differential equation with impulsive condition and delay (1.1) by using contraction mapping theorem and the fixed point theorem of Krasnoselskii. To best of our knowledge there is some new results in this paper.

The organization of the paper is as follows. In Section 2, we recall some basic well known results and some notations. In Section 3, we discuss the existence and uniqueness results. In Section 4 two examples are given to illustrating our results is presented.

2. Preliminaries

In this section we need some basic definitions and properties of fractional calculus that are used in this article. Let X be a Banach space and $[a,b] \subset \mathbb{R}$ be a finite interval and assume that $\alpha, \beta, \gamma \in \mathbb{C}$ and $\Re(z) = Real(z)$ for $z \in \mathbb{C}$. $\mathfrak{C}(J,X)$ be the Banach space of continuous functions $u(t)$ with $u(t) \in J$ for $t \in J := [0,T]$.

For convenience of the reader, we need to introduce some notations and properties of fractional calculus which will be used in the proof of our results. Let $\mathfrak{N}$ be the set of natural numbers, and let $\Re$ be the set of real numbers and $\Re_+ = (0,\infty)$.

Use the notations

$$\begin{aligned}
&\prod_{l=1}^{t-1} f(l) = 1,\\
&\prod_{l=t-n}^{t} f(l) = f(t-n)f(t-n+1)\ldots f(t),\\
&\prod_{\tau=t}^{t-1} f(\tau) = 0, \qquad (2.1)\\
&\prod_{\tau=t-n}^{t} f(\tau) = f(t-n)+f(t-n+1)+\ldots+f(t)
\end{aligned}$$

for $n \in \mathbb{N}$, $t \in \Re_+$ and arbitrary real function f. The difference operator Δ is defined by $\Delta f(t) = f(t+1) - f(t)$, where the function $f(t)$ is defined for $t \in \Re_+$. The difference operator Δ_t is defined by $\Delta_t g(t,a) = g(t+1,a) - g(t,a)$, where the function $g(t,a)$ is defined for a, $t \in \Re_+$.

Let t_0 be a positive real number and set

$$\begin{aligned}
t_{-1} &= \min\{\inf\{\lambda(s) : s \geqslant t_0\}, t_0 - 1\},\\
t_n &= \inf\{s : \lambda(s) > t_{n-1}\} \qquad (2.2)
\end{aligned}$$

for $n = 1,2,\ldots$. Then $\{t_n\}$ is an increasing sequence such that

$$\begin{aligned}
&\lim_{n\to\infty} t_n = \infty,\\
&\bigcup_{n=1}^{\infty} [t_{n-1}, t_n) = [t_0, \infty), \qquad (2.3)\\
&\lambda(t) \in \bigcup_{i=0}^{n} [t_{i-1}, t_i), \qquad t_n \geqslant t < t_{n+1}, \ n = 0,1,2,\ldots
\end{aligned}$$

Fix a point t such that $t_n \leqslant t < t_{n+1}$, define the natural number $\mathfrak{K}(t)$ such that $t - \mathfrak{K}(t) - 1 < t_n$ and $t - \mathfrak{K}(t) \geqslant t_n$, and the set $T(t)$ is defined by

$$T(t) = \{t - \mathfrak{K}(t), t - \mathfrak{K}(t) + 1, \ldots, t-1, t\}. \qquad (2.4)$$

A solution of (1.1) is a function $u(t)$ which is defined for $t \geqslant t_{-1}$ and satisfies (1.1) for $t \geqslant t_0$. For a given a real bounded continuous function ϕ on $t_{-1} \leqslant t < t_0$, (1.1) has a unique solution $u(t)$ satisfying the initial condition $u(t) = \phi(t)$.

DEFINITION 2.1. [22] The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$ are defined by

$$(\mathscr{I}^{\alpha}_{a+}\mathscr{F})(u) = \frac{1}{\Gamma(\alpha)} \int_a^u \frac{\mathscr{F}(s)}{(u-s)^{1-\alpha}} ds, \quad u > a, \quad \Re(\alpha) > 0, \tag{2.5}$$

where $\Gamma(.)$ is the gamma function.

DEFINITION 2.2. [22] The Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{C}$ are defined by

$$\begin{aligned}(\mathscr{D}^{\alpha}_{a+}\mathscr{F})(u) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{du^n} \int_a^u \frac{\mathscr{F}(s)}{(u-s)^{\alpha-n+1}} ds \\ &= \frac{d^n}{du^n} (\mathscr{I}^{n-\alpha}_{a+}\mathscr{F})(u), \quad u > a, \ \Re(\alpha) \geqslant 0, \end{aligned}\tag{2.6}$$

respectively, where $n = [\Re(\alpha)] + 1$ when $\alpha \notin \mathbb{N}_0 = \{0, 1, \ldots\}$ ($[\alpha]$ denotes the integer part of α).

DEFINITION 2.3. [22] The Caputo fractional derivative of order α on $[a, b]$ is defined by

$$({}^C\mathscr{D}^{\alpha}_{a+}y)(u) = \left(\mathscr{D}^{\alpha}_{a+} \left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!} (t-a)^k \right] \right)(u). \tag{2.7}$$

When $a = 0$ we denote $\mathscr{I}^{\alpha}_{a+}y$ and ${}^C\mathscr{D}^{\alpha}_{a+}y$ by $\mathscr{I}^{\alpha}y$ and ${}^C\mathscr{D}^{\alpha}y$. The semigroup properties of the fractional integral operator $\mathscr{I}^{\alpha}_{a+}$ and the composition relation between the fractional integral operator $\mathscr{I}^{\alpha}_{a+}$ and the fractional differentiation operator $\mathscr{D}^{\beta}_{a+}$ are given by the following Lemma (see [22], Lemma 2.9).

LEMMA 2.1. [22] *Lemma 2.9 Let* $\Re(\alpha), \Re(\beta) > 0$ *and* $\mathscr{F}(u) \in \mathfrak{C}[a, b]$. *Then for any* $u \in [a, b]$ *the following assertions are true:*

(a)

$$(\mathscr{I}^{\alpha}_{a+}\mathscr{I}^{\beta}_{a+}\mathscr{F})(u) = (\mathscr{I}^{\alpha+\beta}_{a+}\mathscr{F})(u). \tag{2.8}$$

(b)

$$(\mathscr{D}^{\alpha}_{a+}\mathscr{I}^{\alpha}_{a+}\mathscr{F})(u) = \mathscr{F}(u). \tag{2.9}$$

(c) If $\Re(\alpha) > \Re(\beta)$, *then*

$$(\mathscr{D}^{\beta}_{a+}\mathscr{I}^{\alpha}_{a+}\mathscr{F})(u) = (\mathscr{I}^{\alpha-\beta}_{a+}\mathscr{F})(u). \tag{2.10}$$

(d) Let $n = [\Re(\alpha)] + 1$ *for* $\Re(\alpha) \notin \mathbb{N}$ *and* $\mathscr{F}_{n-\alpha}(u) = (\mathscr{I}_{a+}^{n-\alpha}\mathscr{F})(u) \in \mathfrak{C}^n[a,b]$*, then*

$$(\mathscr{I}_{a+}^{\alpha}\mathscr{D}_{a+}^{\alpha}\mathscr{F})(u) = \mathscr{F}(u) - \sum_{k=1}^{n} \frac{\mathscr{F}_{n-\alpha}^{(n-k)}(a)}{\Gamma(\alpha-k+1)}(u-a)^{\alpha-k}. \tag{2.11}$$

Let $\mathfrak{C}_\gamma[a,b]$ be the space of functions $\mathscr{F}$ given on $(a,b]$ such that $(u-a)^\gamma\mathscr{F}(u) \in \mathfrak{C}[a,b]$ with the norm

$$\|\mathscr{F}\|_{\mathfrak{C}_\gamma} = \|(u-a)^\gamma\mathscr{F}(u)\|_{\mathfrak{C}} := \sup_{u\in[a,b]} |(u-a)^\gamma\mathscr{F}(u)|.$$

Notice that for $\gamma = 0$, $\mathfrak{C}_\gamma[a,b] = \mathfrak{C}[a,b]$. The following Lemma ([22] Lemma 2.8 (a)) is concerning with the continuity of the fractional integral operator $\mathscr{I}_{a+}^{\alpha}$ from the space $\mathfrak{C}_\gamma[a,b]$ into $\mathfrak{C}[a,b]$.

LEMMA 2.2. [22] *Lemma* 2.8(*a*) *Let* $\Re(\alpha) > 0$ *and* $0 \leqslant \Re(\gamma) \leqslant 1$. *If* $\Re(\gamma) \leqslant \Re(\alpha)$, *then the fractional integration operator* $\mathscr{I}_{a+}^{\alpha}$ *is bounded from* $\mathfrak{C}_\gamma[a,b]$ *into* $\mathfrak{C}[a,b]$*:*

$$\|\mathscr{I}_{a+}^{\alpha}\mathscr{F}\|_{\mathfrak{C}} \leqslant \mathfrak{K}_0\|\mathscr{F}\|_{\mathfrak{C}_\gamma},$$
$$\mathfrak{K}_0 = (b-a)^{\Re(\alpha-\gamma)}\frac{\Gamma(\Re(\alpha)|\Gamma(1-\Re(\gamma))|}{|\Gamma(\alpha)|\Gamma(1+\Re(\alpha-\gamma))}.$$

The following result ([22] *Lemma* 2.21, *part* (*a*)) *mentions that the Caputo fractional differentiation operator* ${}^{C}\mathscr{D}_{a+}^{\alpha}$ *is the left inverse of the Riemann Liouville fractional integration operator* $\mathscr{I}_{a+}^{\alpha}$ *when* $\Re(\alpha) \notin \mathbb{N}_0$ *or* $\alpha \in \mathbb{N}$.

LEMMA 2.3. *Let* $\alpha \in \mathbb{C}$ *with* $\Re(\alpha) > 0$ *and* $y(u) \in \mathfrak{C}[a,b]$. *If* $\Re(\alpha) \notin \mathbb{N}$ *or* $\alpha \in \mathbb{N}$, *then*

$$({}^{C}\mathscr{D}_{a+}^{\alpha}\mathscr{I}_{a+}^{\alpha}y)(u) = y(u). \tag{2.12}$$

THEOREM 2.4. [13] *Let* E *be a closed convex and nonempty subset of a Banach space* X. *Let* A *and* B *be two operators such that*

(i) $Au + Bv \in E$ *whenever* $u, v \in E$,

(ii) A *is compact and continuous,*

(iii) B *is a contraction mapping.*

Then there exists $z \in E$ *such that* $z = Az + Bz$.

3. Existence and uniqueness results

Consider the Banach Space $\mathfrak{C}(J)$ with the norm $\left\|u\right\|_{\mathfrak{C}} = \sup_{t\in J}|u(t)|$. By defining

$$(\mathfrak{G}_1 u)(t) := \int_0^{qt} \mathscr{G}_1(t,s,u(s))ds, \tag{3.1}$$

$$(\mathfrak{G}_2 u)(t) := \int_0^{t} \mathscr{G}_2(t,s,u(s))ds, \tag{3.2}$$

we will introduce an integral equation corresponding to problem (1.1) in the next lemma.

LEMMA 3.1. *Let $\mathscr{F} : J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\mathscr{G}_i : J\times J\times\mathbb{R}\to\mathbb{R}$, $i=1,2$ be continuous function. Then the function $u\in\mathfrak{C}(J)$ satisfies problem (1.1) if and only if u is a solution of the fractional integral equation*

$$\begin{aligned} u(t) = u_0 &- (\mathscr{I}^{\alpha}_{0+}\mathfrak{A}(t)u(t-1)) + (\mathscr{I}^{\alpha}_{0+}\mathscr{F}(s,u(s),u(\lambda s)))(t) \\ &+ (\mathscr{I}^{\alpha}_{0+}((\mathfrak{G}_1 u)(s) + (\mathfrak{G}_2 u)(s)))(t) \\ &+ \sum_{k=1}^{m} \mathscr{I}_q(t-t_k)(\mathscr{I}^{\alpha}_{0+}\mathscr{I}_k(u(t_k))). \end{aligned} \tag{3.3}$$

Proof. Let $u\in\mathfrak{C}(J)$ satisfy the problem (1.1). Then using Lemma 2.1 (d), the definition of the Caputo fractional derivative and by applying the operator $\mathscr{I}^{\alpha}$ to both sides of equation (2.5) we have

$$\begin{aligned} [u(t) - u_0] = (\mathscr{I}^{\alpha}_{0+}\mathfrak{A}(t)u(t-1) &+ (\mathscr{I}^{\alpha}_{0+}\mathscr{F}(s,u(s),u(\lambda s)))(t) \\ &+ (\mathscr{I}^{\alpha}_{0+}((\mathfrak{G}_1 u)(s) + (\mathfrak{G}_2 u)(s)))(t) \\ &+ \sum_{k=1}^{m} \mathscr{I}_q(t-t_k)(\mathscr{I}^{\alpha}_{0+}\mathscr{I}_k(u(t_k))). \end{aligned}$$

Conversely, assume that $u\in\mathfrak{C}(J)$ satisfies (3.3). From (3.3) and Lemma 2.2, we have $u(0)=u_0$. Using Lemma 2.3 by applying ${}^{C}\mathscr{D}^{\alpha}$ to both sides of equation (3.3) and the fact that the Caputo fractional derivative of constant functions is zero, we deduce that u satisfies (1.1). □

Define

$$\Delta = \big\{(t,s) : 0 \leqslant s \leqslant t\big\}, \quad \Delta_q = \big\{(t,s) : 0 \leqslant s \leqslant qt\big\}.$$

We consider the following assumptions:

$(H1)$ $0 < \mathfrak{A}(t) < 1$ and $\mathfrak{A}(0) = 0$.

$(H2)$ There exists continuous real function $\lambda(t)$ which satisfies the following : when $t>0$, $0<\lambda(t)<t-\delta(t)$, where $0<\delta^{'}(t)<1$ and $lim_{t\to\infty}\lambda(t)=\infty$; when $t=0$, $\delta(0)=0$, which implies $\lambda(0)=0$.

$(H3)$ $\mathscr{F} : J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a continuous function $a : J\to[0,\infty)$ such that

$$\big|\mathscr{F}(t,x,y) - \mathscr{F}(t,z,w)\big| \leqslant a(t)\big(\big|x-z\big| + \big|y-w\big|\big),$$

for all $t \in J$ and all $x,y,z,w \in \mathbb{R}$.

$(H4)$ $\mathscr{G}_i : J \times J \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$ are continuous and there exist $a_1 : \Delta_q \to [0,\infty)$ and $a_2 : \Delta \to [0,\infty)$ such that $b_1(t) := \int_0^{qt} a_1(t,s)ds \in \mathfrak{C}(J)$, $b_2(t) := \int_0^t a_2(t,s)ds \in \mathfrak{C}(J)$ and

$$\left|\mathscr{G}_i(t,s,x)\right| \leqslant a_i(t,s)(1+|x|).$$

$(H5)$ The function $\mathscr{I}_k : X \to X$ are continuous and there exists a constant $\rho > 0$ such that

$$\left\|\mathscr{I}_k(u)\right\| \leqslant \rho\left\|u\right\| \quad \text{for all } u,v \in B_{r_0} \text{ and } k = 1,2,\ldots,m.$$

$(H6)$ There are positive constant $\delta_k > 0$ such that

$$\left\|\mathscr{I}_k(u) - \mathscr{I}_k(v)\right\| \leqslant \delta_k\left\|u - v\right\|.$$

$(H7)$

$$\|u_0\| + \frac{T^\alpha}{\Gamma(\alpha+1)}\left(\left\|b_1+b_2\right\|_{\mathfrak{C}} + 2\left\|a\right\|_{\mathfrak{C}}\right) + \sum_{k=1}^{m} \rho\left\|u\right\| < 1.$$

Let $\mathfrak{B}_r \subset \mathfrak{C}(J)$ be the closed ball centered at 0 with radius r and put

$$\tilde{\mathscr{F}} := \sup\left\{\left|\mathscr{F}(t,0,0)\right| : t \in J\right\}, \tag{3.4}$$

$$r_0 := \frac{1 + |u_0| + \frac{T^\alpha}{\Gamma(\alpha+1)}\left(\left\|b_1+b_2\right\|_{\mathfrak{C}} + \tilde{\mathscr{F}}\right)}{1 - \frac{T^\alpha}{\Gamma(\alpha+1)}\left(\left\|b_1+b_2\right\|_{\mathfrak{C}} + 2\left\|a\right\|_{\mathfrak{C}}\right)}, \tag{3.5}$$

and define

$$(\mathscr{A}u)(t) := \left(\mathscr{I}_{0+}^{\alpha}((\mathfrak{G}_1 u)(s) + (\mathfrak{G}_2 u)(s))\right)(t).$$

LEMMA 3.2. *Let the assumptions $(H3)$, $(H4)$ and $(H7)$ be satisfied. Then the operator $\mathscr{A}$ maps $\mathfrak{B}_{r_0}$ into itself and $\mathscr{A} : \mathfrak{B}_{r_0} \to \mathfrak{B}_{r_0}$ is continuous and compact.*

Proof. By $(H4)$ and Lemma 2.2, we have $\mathscr{A}u \in \mathfrak{C}(J)$. Now we proceed the proof by the following steps.

Step 1. $\mathscr{A}(\mathfrak{B}_{r_0}) \subset \mathfrak{B}_{r_0}$.

Let $u \in \mathfrak{B}_{r_0}$. Using assumption $(H4)$ and $(H7)$, for $t \in J$ we have

$$\begin{aligned}
\left|(\mathscr{A}u)(t)\right| &\leqslant \left(\mathscr{I}_{0+}^{\alpha}\left(\left|(\mathfrak{G}_1 u)(s)\right| + \left|(\mathfrak{G}_2 u)(s)\right|\right)\right)(t) \\
&\leqslant \left(\mathscr{I}_{0+}^{\alpha}\left(\int_0^{qs} a_1(s,\tau)d\tau + \int_0^{s} a_2(s,\tau)d\tau\right)|u(s)|\right)(t) \\
&\quad + \left(\mathscr{I}_{0+}^{\alpha}\left(\left[\int_0^{qs} a_1(s,\tau)d\tau + \int_0^{s} a_2(s,\tau)d\tau\right]|u(s)|\right)\right)(t) \\
&\leqslant \frac{T^\alpha}{\Gamma(\alpha+1)}\left(\left\|b_1+b_2\right\|_{\mathfrak{C}}\right) + r_0\frac{T^\alpha}{\Gamma(\alpha+1)}\left(\left\|b_1+b_2\right\|_{\mathfrak{C}}\right) \\
&\leqslant r_0.
\end{aligned} \tag{3.6}$$

Step 2. $\mathscr{A}:\mathfrak{B}_{r_0}\to\mathfrak{B}_{r_0}$ is continuous.

Fix $\varepsilon>0$ and take arbitrarily $u,v\,\varepsilon\,\mathfrak{B}_{r_0}$ such that $\|u-v\|\leqslant\varepsilon$. For $t\in J$ we have

$$\big|(\mathfrak{G}_i u)(t)-(\mathfrak{G}_i v))(t)\big|\leqslant\int_0^t\big|\mathscr{G}_i(t,s,u(s))-\mathscr{G}_i(t,s,v(s))\big|ds\leqslant\mathfrak{W}_{r_0}(\mathscr{G}_i,\varepsilon)T, \tag{3.7}$$

where

$$\mathfrak{W}_{r_0}(\mathscr{G}_i,\varepsilon)$$
$$=\sup\big\{\big|\mathscr{G}_i(t,s,u_1)-\mathscr{G}_i(t,s,u_2)\big|:t,s\in J\;u_1,u_2\in[-r_0,r_0],\;\big|u_1-u_2\big|\leqslant\varepsilon\big\},$$

for $i=1,2$. Then using (3.7) we have

$$\begin{aligned}\big|(\mathscr{A}u)(t)-(\mathscr{A}v)(t)\big|&\leqslant\Big(\mathscr{I}_{0+}^{\alpha}\big(\big|(\mathfrak{G}_1u)(s)-(\mathfrak{G}_1v)(s)\big|+\big|(\mathfrak{G}_2u)(s)-(\mathfrak{G}_2v)(s)\big|\big)\Big)(t)\\&\leqslant\frac{\Big(\mathfrak{W}_{r_0}(\mathscr{G}_1,\varepsilon)+\mathfrak{W}_{r_0}(\mathscr{G}_2,\varepsilon)\Big)T^{\alpha+1}}{\Gamma(\alpha+1)}.\end{aligned}\tag{3.8}$$

By the uniform continuity of $\mathscr{G}_i,\ i=1,2$ on bounded subsets of $J\times\mathbb{R}\times\mathbb{R}$, we conclude that $\mathfrak{W}_{r_0}(\mathscr{G}_i,\varepsilon)\to0$ as $\varepsilon\to0$. Thus, the inequality (3.8) implies that $\mathscr{A}:\mathfrak{B}_{r_0}\to\mathfrak{B}_{r_0}$ is continuous.

Step 3. $\mathscr{A}(\mathfrak{B}_{r_0})$ is an equicontinuous subset of $\mathfrak{C}(J)$.

Assumption $(H4)$ implies that, for any $u\in\mathfrak{B}_{r_0}$ and $s\in J$ we have

$$\begin{aligned}\big|(\mathfrak{G}_1u)(s)\big|&\leqslant\int_0^{qs}\big|\mathscr{G}_1(s,\tau,u(\tau))\big|d\tau\leqslant\int_0^{qs}a_1(s,\tau)\big(1+\big|u(\tau)\big|\big)d\tau\\&\leqslant\big(1+r_0\big)b_1(s),\end{aligned}\tag{3.9}$$

and similarly

$$\big|(\mathfrak{G}_2u)(s)\big|\leqslant\big(1+r_0\big)b_2(s).\tag{3.10}$$

Now let $\mathfrak{T}_1,\mathfrak{T}_2\in J$ and $\mathfrak{T}_1<\mathfrak{T}_2$. By (3.9) and (3.10), for any $u\in\mathfrak{B}_{r_0}$ we have

$$\begin{aligned}\big|(\mathscr{A}u)(\mathfrak{T}_1)-(\mathscr{A}u)(\mathfrak{T}_2)\big|&\leqslant\Big|\Big(\mathscr{I}_{0+}^{\alpha}\big(\mathfrak{G}_1u\big)(s)\Big)(\mathfrak{T}_1)-\Big(\mathscr{I}_{0+}^{\alpha}\big(\mathfrak{G}_1u\big)(s)\Big)(\mathfrak{T}_2)\Big|\\&\quad+\Big|\Big(\mathscr{I}_{0+}^{\alpha}\big(\mathfrak{G}_2u\big)(s)\Big)(\mathfrak{T}_2)-\Big(\mathscr{I}_{0+}^{\alpha}\big(\mathfrak{G}_2u\big)(s)\Big)(\mathfrak{T}_2)\Big|\\&\leqslant\frac{r_0+1}{\Gamma(\alpha)}\int_0^{\mathfrak{T}_1}\big(b_1+b_2\big)(s)\Big(\frac{1}{(\mathfrak{T}_1-s)^{1-\alpha}}-\frac{1}{(\mathfrak{T}_2-s)^{1-\alpha}}\Big)ds\\&\quad+\frac{r_0+1}{\Gamma(\alpha)}\int_{\mathfrak{T}_1}^{\mathfrak{T}_2}\frac{\big(b_1+b_2\big)(s)}{(\mathfrak{T}_2-s)^{1-\alpha}}ds\\&\leqslant\frac{\big\|b_1+b_2\big\|_{\mathfrak{C}}}{\Gamma(\alpha+1)}\Big(2\big(\mathfrak{T}_2-\mathfrak{T}_1\big)^{\alpha}+\mathfrak{T}_2^{\alpha}-\mathfrak{T}_1^{\alpha}\Big)\big(r_0+1\big).\end{aligned}\tag{3.11}$$

The right hand side of inequality (3.11) tends to zero as $\mathfrak{T}_1\to\mathfrak{T}_2$. So, by Step 1–Step 3 and Arzela-Ascoli theorem, we conclude that $\mathscr{A}:\mathfrak{B}_{r_0}\to\mathfrak{B}_{r_0}$ is continuous and compact. □

THEOREM 3.3. *Under assumptions* $(H1)$–$(H7)$, *problem* (1.1) *has at least one solution in the space* $\mathfrak{C}(J)$.

Proof. Define the operator $\mathscr{C},\mathscr{D}$ on $\mathfrak{C}(J)$ by

$$(\mathscr{C}u)(t) := \left[u_0 - \left(\mathscr{I}^{\alpha}_{0+}(\mathfrak{A}(t)u(t-1))\right)\right] + \left(\mathscr{I}^{\alpha}_{0+}\mathscr{F}\left(s,u(s),u(\lambda s)\right)\right)(t). \tag{3.12}$$

$$(\mathscr{D}w)(t) := \sum_{k=1}^{m}\mathscr{I}_q(t-t_k)\mathscr{I}^{\alpha}_{0+}\mathscr{I}_k(u(t_k)). \tag{3.13}$$

Due to the continuity of $\mathscr{F}$ and by Lemma 2.2, the operator $\mathscr{C},\mathscr{D}$ are well defined and $\mathscr{C}u \in \mathfrak{C}(J)$, $\mathscr{D}u \in \mathfrak{C}(J)$ for any $u \in \mathfrak{C}(J)$.

By assumption $(H1)$-$(H7)$ and inequality (3.6), for any $u,v \in \mathfrak{B}_{r_0}$ and $t \in J$ we have

$$\begin{aligned}
&\left|(\mathscr{A}u)(t)+(\mathscr{C}v)(t)+(\mathscr{D}w)(t)\right| \\
&\leqslant \left|(\mathscr{A}u)(t)\right| + \left|u_0\right| - \left(\mathscr{I}^{\alpha}_{0+}\left|\mathfrak{A}(0)v(-1)\right|\right) + \left(\mathscr{I}^{\alpha}_{0+}\left|(\mathscr{F}(s,v(s),v(\lambda s)) - \mathscr{F}(s,0,0)\right|\right)(t) \\
&\quad + \left(\mathscr{I}^{\alpha}_{0+}\left|\mathscr{F}(s,0,0)\right|\right)(t) + \left|\sum_{k=1}^{m}\mathscr{I}_k(v(t_k)\right| \\
&\leqslant \left|u_0\right| + \left(\mathscr{I}^{\alpha}_{0+}\left[b_1+b_2\right]\right)(t) + r_0\left(\mathscr{I}^{\alpha}_{0+}\left[b_1+b_2\right]\right)(t) + 2r_0\left(\mathscr{I}^{\alpha}_{0+}a\right)(t) \\
&\quad + \left(\mathscr{I}^{\alpha}_{0+}\left|\mathscr{F}(s,0,0)\right|\right)(t) + \left(\sum_{k=1}^{m}\rho\left[\left|v\right|\right]\right)(t) \\
&\leqslant 1 + \left|u_0\right| + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\left(\left\|b_1+b_2\right\|_{\mathfrak{C}} + 2\left\|a\right\|_{\mathfrak{C}}\right)r_0 \\
&\quad + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\left(\left\|b_1+b_2\right\|_{\mathfrak{C}} + \tilde{\mathscr{F}}\right) + \left(\sum_{k=1}^{m}\rho\left[\left|v\right|\right]\right)(t) \\
&\leqslant r_0.
\end{aligned}$$

Thus, $\mathscr{A}u+\mathscr{C}v+\mathscr{D}v \in \mathfrak{B}_{r_0}$ for any $u,v \in \mathfrak{B}_{r_0}$. Also using $(H1)$, $(H3)$ and $(H6)$, for $u,v \in \mathfrak{C}(J)$ we obtain

$$\begin{aligned}
\left|(\mathscr{C}u)(t)-(\mathscr{C}v)(t)\right| &\leqslant \left(\mathscr{I}^{\alpha}_{0+}a(s)\left[\left|u(s)-v(s)\right| - \left|u(\lambda s)-v(\lambda s)\right|\right]\right)(t) \\
&\leqslant 2\frac{T^{\alpha}}{\Gamma(\alpha+1)}\left\|a\right\|_{\mathfrak{C}}\left\|u-v\right\|_{\mathfrak{C}}, \ t \in J.
\end{aligned} \tag{3.14}$$

Assumption $(H7)$ and inequality (3.14) implies that $\mathscr{C},\mathscr{D}$ is a contraction mapping. Then by Lemma 3.2, assumptions of Theorem 2.4 are satisfied and there exists $z \in \mathfrak{C}(J)$ such that $\mathscr{A}z+\mathscr{C}z+\mathscr{D}z = z$. By the fact that the fixed points of $\mathscr{A}+\mathscr{C}+\mathscr{D}$ are the solutions of integral equation (3.3), we conclude that (3.3) has at least one solution in $\mathfrak{C}(J)$. Finally Lemma 3.1 implies that problem (1.1) has at least one solution in $\mathfrak{C}(J)$. □

In the next theorem we will prove the existence of a unique solution for problem (1.1).

THEOREM 3.4. *Let* $(H1)$ *and* $(H7)$ *be satisfied and the following assumption hold* $(H4)^{'}$*:* $\mathscr{G}_i : J \times J \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$ *are continuous and there exist* $a_i : J \times J \to [0,\infty)$, $i = 1,2$ *such that*

$$b_1(t) := \int_0^{qs} a_1(t,s)ds \in \mathfrak{C}(J),$$
$$b_2(t) := \int_0^{t} a_2(t,s)ds \in \mathfrak{C}(J),$$

and

$$\left|\mathscr{G}_i(t,s,x) - \mathscr{G}_i(t,s,y)\right| \leqslant a_i(t,s)\left|x-y\right|. \tag{3.15}$$

The problem (1.1) *has a unique solution on* J*.*

Proof. By Lemma 3.1, it is enough to prove that integral equation (3.3) has a unique solution. Define the operator $\mathfrak{F}$ on $\mathfrak{C}(J)$ as

$$\begin{aligned}(\mathfrak{F}u)(t) := u_0 &- \mathscr{I}_{0+}^{\alpha}\big(\mathfrak{A}(t)u(t-1)\big)\\ &+ \Big(\mathscr{I}_{0+}^{\alpha}\mathscr{F}(s,u(s),u(\lambda s))\Big)(t) + \Big(\mathscr{I}_{0+}^{\alpha}\Big(\big(\mathfrak{G}_1 u)(s) + \big(\mathfrak{G}_2 u\big)(s)\Big)\Big)(t)\\ &+ \sum_{k=1}^{m} \mathscr{I}_q(t-t_k)\mathscr{I}_{0+}^{\alpha}\mathscr{I}_k(u(t_k)).\end{aligned} \tag{3.16}$$

By the continuity of $\mathscr{F}, \mathscr{G}_1, \mathscr{G}_2$ and by Lemma 2.2, one can easily find that $\mathfrak{F}u \in \mathfrak{C}(J)$ for any $u \in \mathfrak{C}(J)$. Obviously, the fixed points of $\mathfrak{F}$ are the solutions of integral equation (3.3). In the sequel we prove that $\mathfrak{F}$ is a contraction mapping and then by the Banach contraction principal, $\mathfrak{F}$ has a unique fixed point. Let $u,v \in \mathfrak{C}(J)$. By $(H1)$, $(H4)^{'}$, for any $t \in J$ we have

$$\begin{aligned}\Big|(\mathfrak{F}u)(t) - (\mathfrak{F}v)(t)\Big| &\leqslant \big(\mathfrak{A}(t)\big)\big[u(t-1) - v(t-1)\big]\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\left|\mathscr{F}(s,u(s),u(\lambda s)) - \mathscr{F}(s,v(s),v(\lambda s))\right|}{(t-s)^{1-\alpha}}ds\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\left|(\mathfrak{G}_1 u)(s) - (\mathfrak{G}_1 v)(s)\right| + \left|(\mathfrak{G}_2 u)(s) - (\mathfrak{G}_2 v)(s)\right|}{(t-s)^{1-\alpha}}ds\\ &\quad + \sum_{k=1}^{m} \mathscr{I}_q(t-t_k)\left|(\mathscr{I}_k(u(t_k))) - (\mathscr{I}_k(v(t_k)))\right|.\\ &\leqslant \mathfrak{A}(t)\left\|u-v\right\|_{\mathfrak{C}}\\ &\quad + 2\left\|u-v\right\|_{\mathfrak{C}}\big(\mathscr{I}_{0+}^{\alpha}a\big)(t) + \left\|u-v\right\|_{\mathfrak{C}}\big(\mathscr{I}_{0+}^{\alpha}\big(b_1+b_2\big)\big)(t)\\ &\quad + \sum_{k=1}^{m} \delta_k\left\|u-v\right\|_{\mathfrak{C}}(\mathscr{I}_{0+}^{\alpha})(t).\end{aligned}$$

Therefore

$$\begin{aligned}
\|\mathfrak{F}u-\mathfrak{F}v\| &\leqslant \frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big[\Big(\mathfrak{A}(t)+2\|a\|_{\mathfrak{C}}+\|b_1+b_2\|_{\mathfrak{C}}\Big)+\big(\sum_{k=1}^{m}\delta_k\big)\Big]\|u-v\|_{\mathfrak{C}}.\\
\|\mathfrak{F}u-\mathfrak{F}v\| &\leqslant \wp\|u-v\|_{\mathfrak{C}},
\end{aligned}$$

where $\wp=\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big(\mathfrak{A}(t)+2\|a\|_{\mathfrak{C}}+\|b_1+b_2\|_{\mathfrak{C}}\Big)+\big(\sum_{k=1}^{m}\delta_k\big)\big]$, $\wp<1$. Assumption $(H7)$, shows that $\mathfrak{F}$ is a contraction mapping and its completes the proof. □

3.1. Examples

Consider the following fractional neutral impulsive integro-differential equation with initial condition

$$\begin{aligned}
{}^{C}\mathscr{D}^{0.7}u(t) &= \frac{\tan h(u(t)+u(\frac{1}{3})(t))}{16(1+t^2)}\\
&\quad+\int_0^{\frac{1}{5}}\frac{u(s)}{1+32\sqrt{(t-s)}}ds\\
&\quad+\int_0^{t}\Big(\frac{u(s)\sin(t-s)}{8}+\frac{t-s}{8}\Big)ds,\ \ t\in[0,2] \qquad (3.17)\\
u(0) &= 2 \qquad (3.18)\\
\Delta u\Big(\frac{1}{2}\Big) &= 1, \qquad (3.19)
\end{aligned}$$

Put

$$\begin{aligned}
\mathscr{F}(t,x,y) &= \frac{\tan h(u+v)}{16(1+t^2)},\ \ T=2,\ \ \alpha=0.7,\ \ \lambda=\frac{1}{3},\ \ q=\frac{1}{5},\\
\mathscr{G}_1(t,s,x) &= \frac{x}{1+32\sqrt{(t-s)}},\ \ \mathscr{G}_2(t,s,x)=\frac{x\sin(t-s)}{8}+\frac{t-s}{8},\\
a(t) &= \frac{1}{16(1+t^2)},\ \ a_1(t,s)=\frac{1}{32\sqrt{(t-s)}},\ \ a_2(t,s)=\frac{t-s}{8},\\
|\mathscr{I}_k(u)-\mathscr{I}_k(v)| &= 0.
\end{aligned}$$

Then

$$\begin{aligned}
\big|\mathscr{F}(t,x,y)-\mathscr{F}(t,z,w)\big| &\leqslant a(t)\big(|x-z|+|y-w|\big),\\
\big|\mathscr{G}_1(t,s,x)\big| &\leqslant a_1(t-s)|x|,\ \ i=1,2,\\
\big|\mathscr{G}_2(t,s,x)\big| &\leqslant a_2(t-s)\big(1+|x|\big)\\
\big\|\mathscr{I}_k(u)-\mathscr{I}_k(v)\big\| &\leqslant \delta_k\|u-v\|
\end{aligned}$$

$$b_1(t) = \int_0^{qt} a_1(t,s)ds = \left(\frac{1-\sqrt{\frac{4}{5}}}{16} \right) \sqrt{t},$$
$$b_2(t) = \int_0^{t} a_2(t,s)ds = \frac{t^2}{16},$$
$$\frac{T^\alpha}{\Gamma(\alpha+1)} \Big(\mathfrak{A}(t) + 2\|a\|_{\mathfrak{C}} + \|b_1+b_2\|_{\mathfrak{C}} \Big) + \sum_{k=1}^{m} \rho \|u\| \approx 0.6685 < 1.$$

The above inequalities and calculations show that the assumptions $(H1)-(H7)$ are satisfied. Hence using theorem 3.3, problem (3.17)–(3.19) has at least one solution in $\mathfrak{C}[0,2]$.

3.2. Example

Consider the following nonlinear fractional integro-differential equation

$$\begin{aligned} {}^{c}\mathscr{D}^{0.8}u(t) &= \frac{t^2(u(t)+u(\frac{1}{2}t))}{16(1+(u(t)+u(\frac{1}{2}t))^2)} \\ &\quad + \int_0^{\frac{1}{4}} (t+s)u(s)ds + \int_0^{t} \frac{\ln(1+u^2(s))}{8(1+t+s)} ds, \quad t \in [0,1], \qquad (3.20) \\ u(0) &= -1. \end{aligned}$$

Put

$$\mathscr{F}(t,x,y) = \frac{t^2(x+y)}{16(1+(x+y)^2)}, \quad T=1, \quad \alpha = 0.8, \quad \lambda = \frac{1}{2}, \quad q = \frac{1}{5},$$
$$\mathscr{G}_1(t,s,x) = (t+s)x, \quad \mathscr{G}_2(t,s,x) = \frac{\ln(1+x^2)}{8(1+t+s)},$$
$$a(t) = \frac{t^2}{8}, \quad a_1(t,s) = t+s, \quad a_2(t,s) = \frac{1}{4(1+t+s)},$$
$$|\mathscr{I}_k(u) - \mathscr{I}_k(v)| = 0.$$

Then

$$|\mathscr{F}(t,x,y) - \mathscr{F}(t,z,w)| \leqslant a(t)(|x-z| + |y-w|).$$
$$|\mathscr{G}_i(t,s.x) - \mathscr{G}_i(t,s,y)| \leqslant a_i(t,s)|x-y|, \quad i = 1,2,$$
$$b_1(t) = \int_0^{qt} a_1(t,s)ds = \frac{t^2}{5} + \frac{t^2}{50},$$
$$b_2(t) = \int_0^{t} a_2(t,s)ds = \frac{\ln(1+2t)}{4},$$
$$\frac{T^\alpha}{\Gamma(\alpha+1)} \Big(2\|a\|_{\mathfrak{C}} + \|b_1+b_2\|_{\mathfrak{C}} \Big) \approx 0.89146 < 1.$$

According to the above calculation, it is easy to see that all assumptions of Theorem 3.4 are satisfied. Then problem has a unique solution in $\mathfrak{C}[0,1]$.

4. Conclusion

We proved the existence and uniqueness of solutions for impulsive fractional neutral integro-differential equation of pantograph type. The results are obtained by using fractional calculus and fixed point theorems. We provided two examples to illustrate the obtained results.

REFERENCES

[1] R. P. AGARWAL, B. AHMAD, *Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions*, Comput. Math. Appl. **62**, 1200–1214 (2011).

[2] A. AGHAJANI, Y. JALILIAN, J. J. TRUJILLO, *On the existence of solutions of fractional integro-differential equations*, Fract. Calc. Appl. Anal. **15** (1), 44–69 (2012).

[3] A. ANGURAJ, A. VINODKUMAR, K. MALAR, *Existence and Stability results for random impulsive fractional Pantograph equations*, Filomat **30** (14), 2016, 3839–3854.

[4] S. ABBASBANDY, S. KAZEM, M. ALHUTHALI, H. ALSULAMI, *Application of the operational matrix of fractional-order Legendre functions for solving the time-fractional convection-diffusion equation*, Appl. Math. Comput. **266**, 31–40 (2015).

[5] R. ALMEIDA, R. KAMOCKI, A. B. MALINOWSKA, T. ODZIJEWICZ, *Optimal leader-following consensus of fractional opinion formation models*, Journal of Compu. and Applied Math. May 2020.

[6] R. ALMEIDA, A. M. C. BRITO DA CRUZ, N. MARTINS, M.T.T. MONTEIRO, *An epidemiological MSEIR model described by the Caputo fractional derivative*, Int. Journal of Dynam. and Control **7** (2), 776–784 (2019).

[7] M. BENCHOHRA, J. HENDERSON AND S. K. NTOUYAS, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, vol. **2**, New York, 2006.

[8] D. A. BENSON, *The fractional advection-dispersion equation, development and application*, Ph.D. thesis, University of Nevada at Reno (1998).

[9] K. BALACHANDRAN AND R. SAKTHIVEL, *Existence of solutions of neutral functional integro-differential equations in Banach spaces*, Proc. Indian Acad. Sci. Math. Sci. **109** (1999), 325–332.

[10] K. BALACHANDRAN, S. KIRUTHIKA, *Existence of solutions of nonlinear fractional pantograph equations*, Acta Mathematica Scientia **33B** (3), 2013, 712–720.

[11] D. N. CHALISHAJAR, K. MALAR, R. ILAVARASI, *Existence and Controllability results of impulsive fractional neutral integro-diffetential equation with sectorial operator and infinite delay*, Sixth Inter. Confer. on New Trends in the Applications of Diff. Equ. in Sciences, AIP Conference Proceedings **2159**, 030006, (2019).

[12] D. N. CHALISHAJAR, K. MALAR, R. ILAVARASI, *Existence and Controllability Results of impulsive Neutral Fractional integro-differential with Sectorial Operator and Infinite Delay*, DCDIS Series A : Mathematical Analysis **28**, (2021), 77–106.

[13] A. GRANAS, J. DUGUNDJI, *Fixed Point Theory*, Springer-Verlag, New York (2003).

[14] J. R. GRAEF, J. HENDERSON AND A. OUAHAB, *Impulsive differential inclusions*, A fixed point apporach, De Gruyter Series in Nonlinear Analysis and Applications **20**, Berlin: de Gruyter, 2013.

[15] K. GUAN, Z. LUO, *Stability results for impulsive pantograph equations*, Appl. Math. Letters **26** (12), 1167–1174, 2013.

[16] J. K. HALE, *Partial neutral functional differential equations*, (English, English summary), Rev. Roumaine Math. Pures. Appl. **39**, 1994, 339–344.

[17] E. HERNANDEZ, H. R. HENRIQUEZ, M. A. MCKIBBEN, *Existence results for abstract impulsive seconder-order neutral functional differential equations*, Nonlinear Anal. TMA **70**, 2736–2751, (2009).

[18] A. ISERLES, *On the generalized pantograph functional-differential equations*, Eur. Jour. of Appl. Math. **4** (1), 1–38, 1993.

[19] A. ISERLES, *Exact and discretiezed stability of the pantograph equation*, Appl. Numer. Math. **24**, 295–308, (1997).

[20] A. ISERLES, Y. LIU, *On pantograph integro-differntial equations*, Jou. Integral Equa. Appl. **6**, (2), 213–237, (1994).

[21] B. S. H. KASHKARI, M. I. SYAM, *Fractional-order Legendre operational matrix of fractional integration for solving the Riccati-equation with fractional-order*, Appl. Math. Comput. **290**, 281–291, (2016).

[22] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, in North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam (2006).

[23] J. KLAFTER, S. C. LIM, R. METZLER, *Fractional Dynamics in Physics*, Recent Advances, World Scientific, Singapore (2011).

[24] V. LAKSHMIKANTHAM, D. D. BAINOV, P. S. SIMEONOV, *Theory of Impulsive Differential Equations*, World Scientific Press, Singapore (1989).

[25] R. MAGIN, M. ORTIGUEIRA, I. PODLUBNY, J. J. TRUJILLO, *On the fractional signals and systems*, Signal Processing **91**, 350–371, (2011).

[26] R. MAGIN, *Fractional calculus models of complex dynamics in biological tissues*, Comput. Math. Appl. **59**, 1586–1593, (2010).

[27] J. R. OCKENDON, A. B. TAYLOR, *The dynamics of a current collection system for an electric locomotive*, Proc. Royal. Soc. London. Ser. A, **322**, 447–468, (1971).

[28] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, New York (1999).

[29] S. G. SAMKO, A. A. KILBAS, O. I. MARICHEV, *Fractional Integrals and Derivatives*, vol 1, Yverdon-les-Bains, Switzerland, Theory and Applications, Gordon and Breach Science Publishers, Yverdon (1993).

[30] S. ROSA, D. F. M. TORRES, *Fractional- order modelling and optimal Control of Cholera transmission*, Fractal and Fract. **5**, 261, 2021.

[31] J. DA VANTERLER, C. SOUSA, E. CAPELAS DE OLIVEIRA, *On the Ψ-Hilfer fractional derivative*, Commun Nonlinear Sci. Numer. Simulat. **60**, 72–91 (2018).

[32] J. DA VANTERLER, C. SOUSA, N. MAGUN, N. DOS SANTOS, L. A. MAGNA, E. CAPELAS DE OLIVEIRA, *Validation of a fractional model for erythrocyte sedimentation rate*, Compu. and Appli. Math. **37** (6), (2018).

[33] S. YÜZBASI, N. SAHIN, M. SEZER, *A Bessel collection method for numerical solution of generalized pantograph equations*, Numerical Methods for Partial Differential Equations, vol. **28**, 1105–1123, (2012).

(Received September 25, 2021)

Ravi Ilavarasi
Department of Mathematics
Erode Arts and Science College
Erode – 638009, Tamil Nadu, India
e-mail: ilavarasi99ravi@gmail.com

Kandasamy Malar
Department of Mathematics
Erode Arts and Science College
Erode – 638009, Tamil Nadu, India
e-mail: malarganesaneac@gmail.com