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FRACTIONAL CALCULUS AND FAMILIES OF GENERALIZED LEGENDRE–LAGUERRE–APPELL POLYNOMIALS

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Abstract. In this article, new families of the generalized Legendre-Laguerre-Appell polynomials are introduced using a combination of operational definitions and integral representations. The integral transformations and the appropriate operational rules are used to obtain the explicit summation equations, determinant definitions, and recurrence relations for the generalised Legendre-Laguerre-Appell polynomials. For the generalized Legendre-Laguerre-Bernoulli, Legendre-Laguerre-Euler, and Legendre-Laguerre-Genocchi polynomials, an equivalent investigation of these findings is offered. Additionally, a number of identities for these polynomials are derived by using suitable operational definitions.

1. Introduction and preliminaries

The use of integral transforms on fractional derivatives was introduced by Riemann and Liouville [18, 3]. To deal with fractional derivatives, a potent tool is provided by the combination of integral transformations and special polynomials, see for example [10, 4].

The class of Appell polynomial sequences [21], which appears in several problems of applied mathematics, theoretical physics, approximation theory, and several other mathematical branches, is one of the significant classes of polynomial sequences. The set of Appell sequences is closed according to the operation of umbral composition of polynomial sequences. Under this operation the set of Appell sequences forms an abelian group. The Appell sequences are defined by the following generating function:

$$R(y,t) := R(t)e^{yt} = \sum_{n=0}^{\infty} R_n(y)\frac{t^n}{n!}. \tag{1.1}$$

The power series $R(t)$ is then given by

$$R(t) = \sum_{n=0}^{\infty} R_n \frac{t^n}{n!}, \quad R_0 \neq 0, \tag{1.2}$$

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with R_i $(i = 0, 1, 2, \cdots)$ being real coefficients and the function $R(t)$ is an analytic function at $t = 0$. Numerous classical polynomial sequences, including the Bernoulli, Euler, Genocchi, Hermite and Laguerre polynomials *etc*, are included in the class of Appell sequences.

The generating function for the Bernoulli polynomials $B_n(y)$ is given by [1, p. 36]

$$\left(\frac{t}{e^t - 1}\right) e^{yt} = \sum_{n=0}^{\infty} B_n(y) \frac{t^n}{n!}, \qquad |t| < 2\pi, \tag{1.3}$$

where $B_k := B_k(0)$ is the k^{th} Bernoulli number.

The generating function for the Euler polynomials $E_n(y)$ is given by [1, p. 40]

$$\left(\frac{2}{e^t + 1}\right) e^{yt} = \sum_{n=0}^{\infty} E_n(y) \frac{t^n}{n!}, \qquad |t| < \pi, \tag{1.4}$$

where $E_k := 2^k E_k\left(\frac{1}{2}\right)$ is the k^{th} Euler number.

The generating function for the Genocchi polynomials $G_n(y)$ is given by [17]

$$\left(\frac{2t}{e^t + 1}\right) e^{yt} = \sum_{n=0}^{\infty} G_n(y) \frac{t^n}{n!}, \qquad |t| < \pi, \tag{1.5}$$

where $G_k := G_k(0)$ is the k^{th} Genocchi number.

The Appell polynomials and related members are being characterized from different aspects, for this see [12, 2, 13, 14, 16, 15, 26, 20].

In recent years, a number of generalizations of the special functions of mathematical physics have seen a considerable evolution. This recent advancement in the theory of special functions offers a strong analytical foundation for the vast majority of mathematical physics problems that have been thoroughly studied and have numerous practical applications. A major advancement in the theory of generalised special functions is the introduction of multi-index and multi-variable special functions.

One of the particular examples is the 2-variable Laguerre polynomials (2VLP) $L_n(x,y)$ [7, 8] and 2-variable Legendre polynomials $S_n(x,y)$, introduced by Dattoli and Ricci [9] and their convolutions with Appell polynomials Introduced by Subuhi and co-authors [22, 23]. These polynomials are of intrinsic mathematical importance and also have applications in physics.

To give an example, here we consider the Legendre-Laguerre Appell polynomials ${}_{S L}R_n(x,y,z)$, introduced by Subuhi *et al.* [24]. These polynomials are important in many areas of approximation theory as well as other fields of mathematics by virtue of their inherent mathematical importance. The generating equation for the Legendre-Laguerre Appell polynomials polynomials is given by

$$R(t) e^{yt} C_0(xt) C_0(-zt^2) = \sum_{n=0}^{\infty} {}_{S L}R_n(x,y,z) \frac{t^n}{n!}. \tag{1.6}$$

or, equivalently

$$R(t)e^{yt}J_0(2\sqrt{xt})J_0(2\sqrt{-zt^2}) = \sum_{n=0}^{\infty} {}_{S}L R_n(x,y,z)\frac{t^n}{n!}. \tag{1.7}$$

or

$$R(t)e^{(y-D_x^{-1})t+D_z^{-1}t^2} = \sum_{n=0}^{\infty} {}_{S}L R_n(x,y,z)\frac{t^n}{n!}. \tag{1.8}$$

We note that

$$\exp(-\alpha D_x^{-1}) = J_0(2\sqrt{\alpha x}), \qquad D_x^{-n}\{1\} := \frac{x^n}{n!}, \tag{1.9}$$

is the inverse derivative operator.

The polynomials ${}_{S}L R_n(x,y,z)$ are also defined by the following operational rule:

$$\exp\left(D_z^{-1}\frac{\partial^2}{\partial y^2} - D_x^{-1}\frac{\partial}{\partial y}\right)\{R_n(y)\} = {}_{S}L R_n(x,y,z). \tag{1.10}$$

The Euler's integral serves as the basis for further generalizations of special polynomials. In [10], Dattoli *et al.* used the Euler's integral to establish the operational definitions and generating relations for the generalised and novel forms of special polynomials.

The Euler's integral is given by [13, p. 218]

$$a^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-at}t^{\nu-1}dt, \quad \min\{\mathrm{Re}(\nu), \mathrm{Re}(a)\} > 0, \tag{1.11}$$

which consequently yields the following [10]:

$$\begin{aligned}\left(\alpha - \frac{\partial}{\partial x}\right)^{-\nu} f(x) &= \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}\, e^{t\frac{\partial}{\partial x}} f(x)dt \\ &= \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}\, f(x+t)dt.\end{aligned} \tag{1.12}$$

For the second order derivatives, we have the following formula:

$$\left(\alpha - \frac{\partial^2}{\partial x^2}\right)^{-\nu} f(x) = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}\, e^{t\frac{\partial^2}{\partial x^2}} f(x)dt. \tag{1.13}$$

This article introduces and explores the generalized Legendre-Laguerre Appell polynomials via fractional operators.For the generalised Legendre-Laguerre Appell polynomials, explicit summing formulae, determinant definitions, and recurrence relations are derived. The corresponding results for the generalized Legendre-Laguerre-Bernoulli, Euler and Genocchi polynomials are also deduced.

2. Generalized Legendre-Laguerre Appell polynomials

We establish the following findings in order to provide the generating equation and operational rule for the generalized Legendre-Laguerre Appell polynomials:

THEOREM 2.1. *For the generalized Legendre-Laguerre Appell Polynomials* ${}_{\nu SL}R_n(x,y,z;\alpha)$, *the following operational definition holds true:*

$$\left(\alpha - D_z^{-1}\frac{\partial^2}{\partial y^2} + D_x^{-1}\frac{\partial}{\partial y}\right)^{-\nu}\{R_n(y)\} = {}_{\nu SL}R_n(x,y,z;\alpha). \tag{2.1}$$

Proof. Replacing a by $\left(\alpha - D_z^{-1}\frac{\partial^2}{\partial y^2} + D_x^{-1}\frac{\partial}{\partial y}\right)$ in integral (1.11) and operating it on $R_n(y)$, we find

$$\begin{aligned}&\left(\alpha - D_z^{-1}\frac{\partial^2}{\partial y^2} + D_x^{-1}\frac{\partial}{\partial y}\right)^{-\nu}\{R_n(y)\}\\ &= \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\exp\left(tD_z^{-1}\frac{\partial^2}{\partial y^2} - tD_x^{-1}\frac{\partial}{\partial y}\right)R_n(y)dt,\end{aligned} \tag{2.2}$$

which in view of equation (1.10) becomes

$$\left(\alpha - D_z^{-1}\frac{\partial^2}{\partial y^2} + D_x^{-1}\frac{\partial}{\partial y}\right)^{-\nu}R_n(y) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_{sL}R_n(xt,zt,y)dt. \tag{2.3}$$

The transform on the r.h.s of equation (2.3) defines a new family of polynomials. Denoting this special family of polynomials by ${}_{\nu SL}R_n(x,y,z;\alpha)$ and naming it as the generalized Legendre-Laguerre Appell polynomials, so that we have

$${}_{\nu SL}R_n(x,y,z;\alpha) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_{sL}R_n(xt,zt,y)dt. \tag{2.4}$$

In view of equations (2.3) and (2.4), assertion (2.1) follows. □

THEOREM 2.2. *For the generalized Legendre-Laguerre Appell polynomials* ${}_{\nu SL}R_n(x,y,z;\alpha)$, *the following generating function holds true:*

$$\frac{A(u)\,C_0(xu)\,C_0(-zu^2)}{(\alpha - yu)^\nu} = \sum_{n=0}^{\infty}{}_{\nu SL}R_n(x,y,z;\alpha)\frac{u^n}{n!}, \tag{2.5}$$

or, equivalently

$$\frac{A(u)\,e^{(-D_x^{-1}u + D_z^{-1}u^2)}}{(\alpha - yu))^\nu} = \sum_{n=0}^{\infty}{}_{\nu SL}R_n(x,y,z;\alpha)\frac{u^n}{n!}. \tag{2.5a}$$

Proof. Multiplying both sides of equation (2.4) by $\frac{u^n}{n!}$ and summing over n, we find

$$\sum_{n=0}^{\infty} {}_{\nu SL}R_n(x,y,z;\alpha)\frac{u^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\nu)} \int_0^{\infty} e^{-\alpha t} t^{\nu-1} {}_{SL}R_n(xt,y,zt)\frac{u^n}{n!}\, dt. \tag{2.6}$$

Using equivalent generating functions (1.6) and (1.8) in the r.h.s. of equation (2.6), it follows that

$$\sum_{n=0}^{\infty} {}_{\nu SL}R_n(x,y,z;\alpha)\frac{u^n}{n!} = \frac{A(u)\, \exp(yu)}{\Gamma(\nu)} \int_0^{\infty} e^{-\left(\alpha-(-D_z^{-1}u^2+D_x^{-1}u)\right)t} t^{\nu-1} dt, \tag{2.7}$$

which on use of integral (1.11) in the r.h.s. yields assertion (2.5). □

REMARK 2.1. For $R(u)=1$ and $R_n(y)=y^n$, the following generating function and operation rule for the generalized Legendre-Laguerre polynomials ${}_{\nu SL_n}(x,y,z;\alpha)$ holds true:

$$\left(\alpha - D_z^{-1}\frac{\partial^2}{\partial y^2} + D_x^{-1}\frac{\partial}{\partial y}\right)^{-\nu} \{y^n\} = {}_{\nu SL_n}(x,y,z;\alpha), \tag{2.8}$$

and

$$\frac{C_0(xu)\, C_0(-zu^2)}{(\alpha - yu)^{\nu}} = \sum_{n=0}^{\infty} {}_{\nu SL_n}(x,y,z;\alpha)\frac{u^n}{n!}, \tag{2.9}$$

respectively.

Next, we derive an explicit summation formula for the generalized Legendre-Laguerre Appell polynomials ${}_{\nu SL}R_n(x,y,z;\alpha)$ by proving the following result:

THEOREM 2.3. *For the generalized Legendre-Laguerre Appell polynomials ${}_{\nu SL}R_n(x,y,z;\alpha)$, the following explicit summation formula in terms of the generalized Legendre-Laguerre polynomials ${}_{\nu SL_n}(x,y,z;\alpha)$ and Appell polynomials $R_n(y)$ holds true:*

$${}_{\nu SL}R_n(x,y,z;\alpha) = \sum_{k=0}^{n}\sum_{r=0}^{n-k} \binom{n}{k}\binom{n-k}{r}(-1)^k\, w^k\, R_r(w)\; {}_{\nu SL_{n-k-r}}(x,y,z;\alpha). \tag{2.10}$$

Proof. Consider the product of generating functions (1.1) and (2.9) in the following form:

$$R(t)e^{wt}\,(\alpha - yt))^{-\nu}\; C_0(xt)\, C_0(-zt^2)) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} R_r(w)\; {}_{\nu SL_n}(x,y,z;\alpha)\frac{t^{n+r}}{n!\, r!}. \tag{2.11}$$

Replacing n by $n-r$ in the r.h.s. of equation (2.11) and shifting the first exponential to the r.h.s., it follows that

$$\begin{aligned} &R(t)\,(\alpha - yt))^{-\nu}\; C_0(xt)\, C_0(-zt^2)) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n} \binom{n}{r}(-1)^k w^k R_r(w)\; {}_{\nu SL_n}(x,y,z;\alpha)\frac{t^{n+k}}{n!\, k!}, \end{aligned} \tag{2.12}$$

which on replacing n by $n-k$ gives

$$\begin{aligned} & R(t)\,(\alpha - yt))^{-\nu}\ C_0(xt)\ C_0(-zt^2)) \\ & = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-1)^k\ w^k R_r(w)\ {}_{\nu SL_n}(x,y,z;\alpha)\frac{t^n}{n!}. \end{aligned} \tag{2.13}$$

Finally, using generating function (2.5) in the l.h.s. of equation (2.13) and then equating the coefficients of like powers of t in the resultant equation, assertion (2.10) follows. □

REMARK 2.2. By taking $A(u) = \left(\frac{u}{e^u-1}\right)$ and $A_n(u) = B_n(u)$ in equations (2.1), (2.5) and (2.10), we find that for the generalized Legendre-Laguerre Bernoulli polynomials ${}_{\nu SL}B_n(x,y,z;\alpha)$, the following operational rule, generating equation and explicit summation formula hold true:

$$\left(\alpha - D_z^{-1}\frac{\partial^2}{\partial y^2} + D_x^{-1}\frac{\partial}{\partial y}\right)^{-\nu}\{B_n(y)\} = {}_{\nu SL}B_n(x,y,z;\alpha), \tag{2.14}$$

$$\left(\frac{u}{e^u-1}\right)\frac{C_0(xu)\ C_0(-zu^2)}{(\alpha-yu)^{\nu}} = \sum_{n=0}^{\infty}{}_{\nu SL}B_n(x,y,z;\alpha)\frac{u^n}{n!}, \tag{2.15}$$

$${}_{\nu SL}B_n(x,y,z;\alpha) = \sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-1)^k\ w^k\ B_r(w)\ {}_{\nu SL n-k-r}(x,y,z;\alpha). \tag{2.16}$$

REMARK 2.3. Taking $A(u) = \left(\frac{2}{e^u+1}\right)$ and $A_n(u) = E_n(u)$ in equations (2.1), (2.5) and (2.10), we find that for the generalized Legendre-Laguerre Euler polynomials ${}_{\nu SL}E_n(x,y,z;\alpha)$, the following operational rule, generating equation and explicit summation formula hold true:

$$\left(\alpha - D_z^{-1}\frac{\partial^2}{\partial y^2} + D_x^{-1}\frac{\partial}{\partial y}\right)^{-\nu}\{E_n(y)\} = {}_{\nu SL}E_n(x,y,z;\alpha), \tag{2.17}$$

$$\left(\frac{2}{e^u+1}\right)\frac{C_0(xu)\ C_0(-zu^2)}{(\alpha-yu)^{\nu}} = \sum_{n=0}^{\infty}{}_{\nu SL}E_n(x,y,z;\alpha)\frac{u^n}{n!}, \tag{2.18}$$

$${}_{\nu SL}E_n(x,y,z;\alpha) = \sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-1)^k\ w^k\ E_r(w)\ {}_{\nu SL n-k-r}(x,y,z;\alpha). \tag{2.19}$$

REMARK 2.4. Taking $A(u) = \left(\frac{2u}{e^u+1}\right)$ and $A_n(u) = G_n(u)$ in equations in equations (2.1), (2.5) and (2.10), we find that for the generalized Legendre-Laguerre Genocchi polynomials ${}_{\nu SL}G_n(x,y,z;\alpha)$, the following operational rule, generating equation and explicit summation formula hold true:

$$\left(\alpha - D_z^{-1}\frac{\partial^2}{\partial y^2} + D_x^{-1}\frac{\partial}{\partial y}\right)^{-\nu}\{G_n(y)\} = {}_{\nu SL}G_n(x,y,z;\alpha), \tag{2.20}$$

$$\left(\frac{2u}{e^u+1}\right)\frac{C_0(xu)\ C_0(-zu^2)}{(\alpha-yu)^{\nu}} = \sum_{n=0}^{\infty}{}_{\nu SL}G_n(x,y,z;\alpha)\frac{u^n}{n!}, \tag{2.21}$$

$$ {}_{vSL}G_n(x,y,z;\alpha) = \sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-1)^k\, w^k\, G_r(w)\, {}_{vSL}{}_{n-k-r}(x,y,z;\alpha). \tag{2.22} $$

In the following section, we obtain the determinant forms and recurrence relations for the generalised Legendre-Laguerre Appell polynomials and associated members.

3. Determinant forms and recurrence relations

To express the generalized Legendre-Laguerre Appell polynomials via determinant, we prove the following result:

THEOREM 3.1. *For the generalized Legendre-Laguerre Appell polynomials ${}_{vSL}R_n(x,y,z;\alpha)$, the following determinant form holds true:*

$$ {}_{vSL}R_0(x,y,z;\alpha) = \frac{1}{\beta_0}, $$

$$ {}_{vSL}{}_n(x,y,z;\alpha) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & {}_{vSL}{}_1(x,y,z;\alpha) & {}_{vSL}{}_2(x,y,z;\alpha) & \cdots & {}_{vSL}{}_{n-1}(x,y,z;\alpha) & {}_{vSL}{}_n(x,y,z;\alpha) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, $$

where $n = 1,2,\cdots$; $\beta_0,\ \beta_1,\cdots,\beta_n \in \mathbb{R}$; $\beta_0 \neq 0$ *and*

$$ \beta_n = -\tfrac{1}{A_0}\Big(\sum_{k=1}^{n}\tbinom{n}{k}A_k\,\beta_{n-k}\Big),\ n = 1,2,\cdots . \tag{3.3} $$

Proof. We consider the following determinant definition for the Appell polynomials [6, p.1533]:

$$ R_0(y) = \frac{1}{\beta_0}, \tag{3.4} $$

$$ R_n(y) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & y & y^2 & \cdots & y^{n-1} & y^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}. \tag{3.5} $$

Taking $n=0$ in formula (2.10) and then using equations (3.3) and (3.4), we get assertion (3.2).

Further, expanding determinant (3.5) with respect to the first row and then operating $\left(\alpha - D_z^{-1}\frac{\partial^2}{\partial y^2} + D_x^{-1}\frac{\partial}{\partial y}\right)^{-\nu}$ on both sides of the resulting equation and then using equations (2.1) and (2.8), we find

$$
\begin{aligned}
{}_{\nu S}L_n(x,y,z;\alpha) = {} & \frac{(-1)^n\, {}_{\nu S}L_0(x,y,z;\alpha)}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix} - \frac{(-1)^n\, {}_{\nu S}L_1(x,y,z;\alpha)}{(\beta_0)^{n+1}} \\
& \begin{vmatrix} \beta_0 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix} + \frac{(-1)^n\, {}_{\nu S}L_2(x,y,z;\alpha)}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix} + \cdots \\
& + \frac{(-1)^{2n-1}\, {}_{\nu S}L_{n-1}(x,y,z;\alpha)}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n}{2}\beta_{n-2} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \binom{n}{n-1}\beta_1 \end{vmatrix} \\
& + \frac{{}_{\nu S}L_n(x,y,z;\alpha)}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \binom{n}{n-1}\beta_1 \end{vmatrix}.
\end{aligned} \tag{3.6}
$$

Combining the terms in the r.h.s. of equation (3.6), we are lead to assertion (3.2). □

• Taking $\beta_0 = 1$ and $\beta_i = \frac{1}{i+1}$ $(i = 1,2,\ldots,n)$ (for which the determinant form of the Appell polynomials $R_n(y)$ reduce to the Bernoulli polynomials $B_n(y)$ [5, 6]) in equations (3.2) and (3.2), the determinant definition of the generalized Legendre-Laguerre Bernoulli polynomials ${}_{\nu SL}B_n(x,y,z;\alpha)$ can be obtained.

• Taking $\beta_0 = 1$ and $\beta_i = \frac{1}{2}$ $(i = 1,2,\ldots,n)$ (for which the determinant form of the Appell polynomials $R_n(y)$ reduce to the Euler polynomials $E_n(y)$ [6] in equations (3.2) and (3.2), the determinant definition of the generalizedLegendre-Laguerre Euler polynomials ${}_{\nu SL}E_n(x,y,z;\alpha)$ can be obtained.

• Taking $\beta_0 = 1$ and $\beta_i = \frac{1}{2(i+1)}$ $(i = 1,2,\ldots,n)$ (for which the determinant form of the Appell polynomials $R_n(y)$ reduce to the Genocchi polynomials $G_n(y)$ in equations (3.2) and (3.2), the determinant definition of the generalized Legendre-Laguerre Genocchi polynomials ${}_{\nu SL}G_n(x,y,z;\alpha)$ can be obtained.

Next, we derive the recurrence relations for the generalized Legendre-Laguerre Appell polynomials ${}_{\nu SL_n}(x,y,z;\alpha)$ by considering their generating equation (2.5a).

On differentiating generating function (2.5a), with respect to y, D_x^-, D_z^{-1} and α, we find the following recurrence relations for the generalized Legendre-Laguerre Appell polynomials ${}_{\nu SL}R_n(x,y,z;\alpha)$:

$$\begin{aligned}
&\frac{\partial}{\partial y}\Big({}_{\nu SL_n}(x,y,z;\alpha)\Big) = \nu\, n\; {}_{\nu SL}R_{n-1}(x,y,z;\alpha),\\
&\frac{\partial}{\partial D_x^{-1}}\Big({}_{\nu SL_n}(x,y,z;\alpha)\Big) = -n\; {}_{\nu SL}R_{n-1}(x,y,z;\alpha),\\
&\frac{\partial}{\partial D_z^{-1}}\Big({}_{\nu SL_n}(x,y,z;\alpha)\Big) = n(n-1)\; {}_{\nu SL}R_{n-2}(x,y,z;\alpha),\\
&\frac{\partial}{\partial \alpha}\Big({}_{\nu SL_n}(x,y,z;\alpha)\Big) = -\nu\; {}_{\nu+1\, SL}R_n(x,y,z;\alpha).
\end{aligned} \tag{3.7}$$

In the following section, which includes specific identities for the Legendre-Laguerre Appell polynomial family and some of its members, are the applications of the findings in section 2.

4. Applications

In literature, there are several identities involving Appell polynomials and associated members. The appropriate identities involving generalized Legendre-Laguerre Appell and related polynomials can be obtained using the operational formalism developed in the previous section. To achieve this, we perform the following operation:

$(\mathscr{O})$ operating $\left(\alpha - D_z^{-1}\frac{\partial^2}{\partial y^2} + D_x^{-1}\frac{\partial}{\partial y}\right)^{-\nu}$ $\{E_n(y)\}$ on both sides of a given relation.

First, we consider the following results for the Appell polynomials $R_n(y)$ [6, (31-32) p.1534]:

$$R_n(y) = \frac{1}{\beta_0}\Big(y^n - \sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k}\, R_k(y)\Big), \quad n = 1,2,\cdots\ , \tag{4.1}$$

$$y^n = \sum_{k=0}^{n}\binom{n}{k}\beta_{n-k}\, R_k(y), \quad n = 0,1,\cdots\ . \tag{4.2}$$

Performing operation $(\mathscr{O})$ on both sides of the above equations and then using operational definitions (2.1) and (2.8), we obtain the following identities involving generalized Legendre-Laguerre Appell polynomials ${}_{\nu SL}R_n(x,y,z;\alpha)$:

$$ {}_{\nu SL}R_n(x,y,z;\alpha) = \frac{1}{\beta_0}\Big({}_{\nu SL_n}(x,y,z;\alpha) - \sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k}\ {}_{\nu SL}R_k(x,y,z;\alpha)\Big), \quad n=1,2,\cdots\ , \tag{4.3}$$

$$ {}_{\nu SL_n}(x,y,z;\alpha) = \sum_{k=0}^{n}\binom{n}{k}\beta_{n-k}\ {}_{\nu SL}R_k(x,y,z;\alpha), \quad n=0,1,\cdots\ . \tag{4.4}$$

Next, we recall the following functional equations involving Bernoulli polynomials $B_n(y)$ [25, p.26]:

$$B_n(y+1) - \mathscr{B}_n(y) = n\, y^{n-1}, \quad n=0,1,2\ldots\ , \tag{4.5}$$

$$\sum_{m=0}^{n-1}\binom{n}{m}B_m(y) = ny^{n-1}, \quad n=2,3,4\ldots\ , \tag{4.6}$$

$$B_n(my) = m^{n-1}\sum_{k=0}^{m-1}B_n\left(y+\frac{k}{m}\right), \quad n=0,1.2,\ldots; \quad m=1,2,3\ldots \tag{4.7}$$

Again, performing operation $(\mathscr{O})$ on both sides of the above equations and then using operational definitions (2.1) and (2.8), the following identities involving generalized Legendre-Laguerre Bernoulli polynomials ${}_{\nu SL}B_n(x,y,z;\alpha)$ are obtained:

$$ {}_{\nu SL}B_n(x,y+1,z;\alpha) - {}_{\nu SL}B_n(x,y,z;\alpha) = n\ {}_{\nu SL_{n-1}}(x,y,z;\alpha), \quad n=0,1,2\cdots, \tag{4.8}$$

$$\sum_{m=0}^{n-1}\binom{n}{m}{}_{\nu SL}B_m(x,y,z;\alpha) = n\ {}_{\nu}SL_{n-1}(x,y,z;\alpha), \quad n=2,3,4\cdots, \tag{4.9}$$

$$ {}_{\nu SL}B_n(mx,my,mz;\alpha) = m^{n-1}\sum_{k=0}^{m-1}{}_{\nu S}B_n\left(x,y+\frac{k}{m},z;\alpha\right), \quad n=0,1,2,\ldots; \ m=1,2,3\cdots. \tag{4.10}$$

Further, performing operation $(\mathscr{O})$ with use of operational rules (2.1) and (2.8) on the following functional equations involving Euler polynomials $E_n(y)$ [25, p. 30] and Genocchi polynomials $G_n(y)$ [11, p. 1038, (42)]:

$$E_n(y+1) + E_n(y) = 2y^n,$$

$$E_n(my) = m^n\sum_{k=0}^{m-1}(-1)^kE_n\left(y+\frac{k}{m}\right) \quad n=0,1,2\ldots; \ m \text{ odd},$$

$$G_{n+1}(y) + G_n(y) = 2ny^{n-1},$$

yields the following identities involving the generalized Legendre-Laguerre Euler polynomials ${}_{vSL}E_n(x,y,z;\alpha)$ and generalized Legendre-Laguerre Genocchi polynomials ${}_{vSL}G_n(x,y,z;\alpha)$:

$$ {}_{vSL}E_n(x,y+1,z;\alpha) + {}_{vSL}E_n(x,y,z;\alpha) = 2\ {}_{vSL_n}(x,y,z;\alpha), \tag{4.11} $$

$$ {}_{vSL}E_n(mx,my,mz;\alpha) = m^n \sum_{k=0}^{m-1} (-1)^k\, {}_{vSL}E_n\left(x, y+\frac{k}{m}, z;\alpha\right), \quad n = 0,1.2,\ldots;\ m \text{ odd}, \tag{4.12} $$

$$ {}_{vSL}G_{n+1}(x,y,z;\alpha) + {}_{vSL}G_n(x,y,z;\alpha) = 2n\ {}_{vSL_{n-1}}(x,y,z;\alpha). \tag{4.13} $$

Finally, considering the following connection formulae involving the Bernoulli and Euler polynomials [25, pp. 29–30]:

$$ B_n(y) = 2^{-n} \sum_{m=0}^{n} \binom{n}{m} B_{n-m} E_m(2y), \quad n = 0,1,2\ldots, \tag{4.14} $$

$$ E_n(y) = \frac{2^{n+1}}{n+1}\left[B_{n+1}\left(\frac{y+1}{2}\right) - B_{n+1}\left(\frac{y}{2}\right)\right], \quad n = 0,1,2\ldots, \tag{4.15} $$

$$ E_n(my) = -\frac{2^{m^n}}{n+1} \sum_{k=0}^{m-1} (-1)^k B_{n+1}\left(\frac{y+k}{m}\right), \quad n = 0,1,2\ldots, m \text{ even}, \tag{4.16} $$

which on performing operation $(\mathscr{O})$ and then using appropriate operational definitions yields the following connection formulae involving the generalized Legendre-Laguerre Bernoulli and Legendre-Laguerre Euler polynomials:

$$ {}_{vSL}B_n(x,y,z;\alpha) = 2^{-n} \sum_{m=0}^{n} \binom{n}{m} B_{n-m}\ {}_{vSL}E_n(2x,2y,2z;\alpha), \quad n = 0,1,2\ldots, \tag{4.17} $$

$$ {}_{vSL}E_n(x,y,z;\alpha) = \frac{2^{n+1}}{n+1}\left[{}_{vSL}E_{n+1}\left(\frac{x}{2}, \frac{y+1}{2}, \frac{z}{2};\alpha\right) - {}_{vSL}E_{n+1}\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2};\alpha\right)\right], \quad n = 0,1,2\ldots, \tag{4.18} $$

$$ {}_{vSL}E_n(mx,my,mz;\alpha) = -\frac{2m^n}{n+1} \sum_{k=0}^{m-1} (-1)^k\, {}_{vSL}B_{n+1}\left(\frac{x}{m}, \frac{y+k}{m}, \frac{z}{m};\alpha\right), \quad n = 0,1,2,\ldots;\ m \text{ even}. \tag{4.19} $$

Competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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