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A NEW GENERALIZATION OF q–HERMITE–HADAMARD TYPE INTEGRAL INEQUALITIES FOR p, $(p-s)$ AND MODIFIED $(p-s)$–CONVEX FUNCTIONS

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Abstract. In this study, we develop three new quantum Hermite-Hadamard inequalities for the class of p, $(p-s)$ and modified type $(p-s)$-convex functions by utilizing left and right quantum integral. As special cases of these inequalities, we get known and new Hermite-Hadamard type inequality for modified type $(p-s)$-convex functions. The ideas and techniques of this article may be the starting point for further research in this field.

1. Introduction

The primitive notion of 'convexity' is very well fitted in modern Mathematics. The theory of convexity is not only important within itself, but it also has applications in almost every area of mathematics. Intensive research was carried out during the 20th century and significant results were achieved in the field of convex analysis. The book of G. H. Hardy, J. E. Littlewood and G. Polya [23], on inequalities played a considerably important role in the popularity of the subject of convex analysis. Over years idea of convex sets and hence convex functions is largely generalized. Today the study of convex functions evolved into a broader theory of functions including quasi-convex functions [3, 17, 25], log-convex functions [22], co-ordinated convex functions [18, 26], harmonically convex functions [27], GA-convex function [30, 35], m-convex function. A significant generalization of classical convex functions is called p-convex functions, which were introduced by Zhang and Wan [37]. A wide class of inequalities have been derived via convex functions, see [33, 31, 32]. A function $\digamma : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$ is called convex, if it satisfies the inequality,

$$\digamma(\tau\varkappa + (1-\tau)y) \leqslant \tau\digamma(\varkappa) + (1-\tau)\digamma(y),$$

where $\varkappa, y \in I$ and $\tau \in [0,1]$. It is also well known that $\digamma$ is convex if and only if it satisfies the Hermite-Hadamard's inequality, stated below:

$$\digamma\left(\frac{\sigma+\kappa}{2}\right) \leqslant \frac{1}{\kappa-\sigma}\int_{\sigma}^{\kappa} \digamma(\varkappa)d\varkappa \leqslant \frac{\digamma(\sigma)+\digamma(\kappa)}{2},$$

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where $\sigma, \kappa \in I$ with $\sigma < \kappa$.

Convexity naturally gives rise to inequalities, Hermite-Hadamard inequality is the first consequence of convex function. Convexity is mixed with other mathematical concepts like; optimization [10], time scale [13, 9], quantum and post quantum calculus [6].

Quantum calculus or q-calculus is a approach pertinent to the classic study of calculus but it is mainly based on the idea of derivation of q -analogous results without the use of limits. We get the initial mathematical formulas in q-calculus as q approaches 1. The idea of q calculus was first introduced by Euler who started his study in the eighteen Century. In (1910) F. Jackson further developed this field by defining a quantum integral known as the q-Jackson integral, see [16, 19, 29, 14]. In q-calculus, the classical derivative is replaced by the q-difference operator in order to deal with non-differentiable functions, for more details see [4, 15]. Therefore q-calculus bridges a connection between mathematics and physics. q-calculus applications can be found in a variety of fields of mathematics and physics, and the interested reader is directed to [7, 28, 36]. Many integral inequalities have been investigated by utilizing quantum integrals for different types of convex functions. For example, [1, 2, 6, 5, 11, 12], the authors used, ${}^{\kappa}D_q$-derivatives and q_σ, q_κ-integrals to prove H-H integral inequalities and their left-right estimates for convex and coordinated convex functions. Inspired by the ongoing studies, we prove some new Hermite-Hadamard type inequalities, for p, $(p-s)$ and modified type $(p-s)$-convex function. Special cases of these inequalities can yield well-known results in literature.

Before we begin our main results, the following definitions and concepts need to be clarified.

2. Preliminaries and definitions of q-calculus

The quantum number $[n]_q$ is expressed as

$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}$ with $q \in (0,1)$, see, [14].

DEFINITION 1. [37] Let I be a p-convex set. A function $\mathcal{F} : I \to \mathbb{R}$ is said to be a p-convex function, if

$$\mathcal{F}\left((\tau\sigma^p + (1-\tau)\kappa^p)^{\frac{1}{p}}\right) \leqslant \tau\mathcal{F}(\sigma) + (1-\tau)\mathcal{F}(\kappa) \tag{1}$$

for all $\sigma, \kappa \in I$, $p \in R \setminus \{0\}$ and for $\tau \in [0,1]$. If the inequality in (1) is reversed, then $\mathcal{F}$ is said to be p-concave.

DEFINITION 2. [24] Let I be a s-convex set. A function $\mathcal{F} : I \to \mathbb{R}$ is said to be a s-convex function, if

$$\mathcal{F}(\tau\sigma + (1-\tau)\kappa) \leqslant \tau^s\mathcal{F}(\sigma) + (1-\tau)^s\mathcal{F}(\kappa) \tag{2}$$

for all $\sigma, \kappa \in I$, and for $\tau \in [0,1], s \in (0,1]$. If the inequality in (2) is reversed, then $\mathcal{F}$ is said to be s-concave.

DEFINITION 3. [20] Let $h: J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative and non-zero function. A function $\digamma: I \to \mathbb{R}$, where I is p-convex set in $\mathbb{R}$ is called $(p-h)$-convex function, if $\digamma$ is non-negative and

$$\digamma\left((\tau\sigma^p + (1-\tau)\kappa^p)^{\frac{1}{p}}\right) \leqslant h(\tau)\digamma(\sigma) + h(1-\tau)\digamma(\kappa) \tag{3}$$

for all $\sigma, \kappa \in I$ and $\tau \in (0,1)$, where $p > 0$. Similarly, if the inequality sign in (3) is reversed, then $\digamma$ is said to be (p,h)-concave function.

If $h(\tau) = \tau^s$ in Definition (3), then we have definition of $(p-s)$-convex functions.

DEFINITION 4. Let I be a $(p-s)$-convex set. A function $\digamma: I \to \mathbb{R}$ is said to be a $(p-s)$-convex function, if

$$\digamma\left((\tau\sigma^p + (1-\tau)\kappa^p)^{\frac{1}{p}}\right) \leqslant \tau^s\digamma(\sigma) + (1-\tau)^s\digamma(\kappa), \tag{4}$$

For all $\sigma, \kappa \in I$, $p > 0$ and for $s \in (0,1]$, $\tau \in [0,1]$. If the inequality in (4) is reversed, then $\digamma$ is said to be $(p-s)$-concave.

Of course, if we put $s = 1$, $(p-s)$-convexity reduces to ordinary p-convexity of function.

If we put $p = 1$, then $(p-s)$-convexity reduces to s-convexity in the second sense.

DEFINITION 5. [21] Let $h: J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative and non-zero function. A function $\digamma: I \to R$, where I is p-convex set in $\mathbb{R}$, is called modified $(p-h)$-convex function, if $\digamma$ is non-negative and

$$\digamma\left((\tau\sigma^p + (1-\tau)\kappa^p)^{\frac{1}{p}}\right) \leqslant h(\tau)\digamma(\sigma) + (1-h(\tau))\digamma(\kappa) \tag{5}$$

for all $\sigma, \kappa \in I$ and $\tau \in (0,1)$. Similarly, if the inequality sign in (5) is reversed, then $\digamma$ is said to be a (p,h)-concave function.

If $h(\tau) = \tau^s$ in Definition (5), then, we have definition of modified $(p-s)$-convex functions.

DEFINITION 6. Let I be a $(p-s)$-convex set. A function $\digamma: I \to \mathbb{R}$ is said to be a modified $(p-s)$-convex function, if

$$\digamma\left((\tau\sigma^p + (1-\tau)\kappa^p)^{\frac{1}{p}}\right) \leqslant \tau^s\digamma(\sigma) + (1-\tau^s)\digamma(\kappa) \tag{6}$$

for all $\sigma, \kappa \in I$, $s, \tau \in [0,1]$, where $p > 0$. If the inequality in (6) is reversed, then $\digamma$ is said to be $(p-s)$-concave.

Of course, if we put $s = 1$, modified $(p-s)$-convexity reduces to ordinary p-convexity of function.

If we put $p = 1$, $s = 1$, then modified $(p-s)$-convexity reduces to ordinary-convexity.

DEFINITION 7. [34] Let $F : I \to \mathbb{R}$ be a continuous function and let $\varkappa \in I$. Then the q_σ derivative on I of F at $\varkappa$ is defined as

$$ {}_\sigma D_q F(\varkappa) = \frac{F(\varkappa) - F(q\varkappa + (1-q)\sigma)}{(1-q)(\varkappa - \sigma)} \quad \varkappa \neq \sigma, \tag{7} $$

$$ {}_\sigma D_q F(\sigma) = \lim_{\varkappa \to \sigma} {}_\sigma D_q F(\varkappa). $$

DEFINITION 8. If $\sigma = 0$ in (7), then we get classical q-derivative of $F(\varkappa)$ at $\varkappa \in I$, given by

$$ {}_0 D_q F(\varkappa) = D_q F(\varkappa) = \frac{F(\varkappa) - F(q\varkappa)}{(1-q)\varkappa}. $$

DEFINITION 9. [8] Let $F : I \to \mathbb{R}$ be a continuous function and let $\varkappa \in I$. Then the q^κ derivative on I of F at $\varkappa$ is defined as

$$ {}^\kappa D_q F(\varkappa) = \frac{F(\varkappa) - F(q\varkappa + (1-q)\kappa)}{(1-q)(\varkappa - \kappa)} \quad \varkappa \neq \kappa, $$

$$ {}^\kappa D_q F(\kappa) = \lim_{\varkappa \to \kappa} {}^\kappa D_q F(\varkappa). $$

DEFINITION 10. [34] Let $F : I \to \mathbb{R}$ be a continuous function. Then the q_σ-integral on I is defined as

$$ \int_\sigma^\varkappa F(\tau)\, {}_\sigma d_q \tau = (1-q)(\varkappa - \sigma) \sum_{n=0}^{\infty} q^n F(q^n \varkappa + (1-q^n)\sigma) \tag{8} $$

for $\varkappa \in I$. If $\sigma = 0$ in (8), then

$$ \int_0^\varkappa F(\tau)\, {}_0 d_q \tau = \int_0^\varkappa F(\tau) d_q \tau, $$

where $\int_0^\varkappa F(\tau) d_q \tau$ is familiar q-definite integral on $[0, \varkappa]$ defined by the expression

$$ \int_0^\varkappa F(\tau)\, {}_0 d_q \tau = \int_0^\varkappa F(\tau) d_q \tau = (1-q)\varkappa \sum_{n=0}^{\infty} q^n F(q^n \varkappa). $$

Moreover, if $c \in (\sigma, \varkappa)$, then the q-integral on I is defined as

$$ \int_c^\varkappa F(\tau)\, {}_\sigma d_q \tau = \int_\sigma^\varkappa F(\tau)\, {}_\sigma d_q \tau - \int_\sigma^c F(\tau)\, {}_\sigma d_q \tau. $$

DEFINITION 11. [8] Let $F : I \to \mathbb{R}$ be a continuous function. Then the q^κ-integral on I is defined as

$$ \int_\varkappa^\kappa F(\tau)\, {}^\kappa d_q \tau = (1-q)(\kappa - \varkappa) \sum_{n=0}^{\infty} q^n F(q^n \varkappa + (1-q^n)\kappa) \tag{9} $$

for $\varkappa \in I$. If $\kappa = 1$ in (9), then

$$\int_{\varkappa}^{1} \digamma(\tau)^1 d_q\tau = \int_{\varkappa}^{1} \digamma(\tau) d_q\tau.$$

In [20], if we take $h(\tau) = \tau$, then we have the following Theorem.

THEOREM 1. *Let $\digamma : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I^o, $\sigma,\kappa \in I$ with $\sigma < \kappa$ and $p > 0$. If $\digamma' \in L[\sigma,\kappa]$, then*

$$\digamma\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right) \leqslant \frac{p}{\kappa^p-\sigma^p}\int_{\sigma}^{\kappa}\frac{\digamma(\varkappa)}{\varkappa^{1-p}}d\varkappa \leqslant \frac{\digamma(\sigma)+\digamma(\kappa)}{2}. \tag{10}$$

In [20], if we take $h(\tau) = \tau^s$, then we have the following Theorem.

THEOREM 2. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative and non-zero function. For a p-convex function $\digamma : I \to \mathbb{R}$, where I is p-convex set in R and $s \in [0,1)$, $p > 0$, then we have*

$$2^{s-1}\digamma\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right) \leqslant \frac{p}{(\kappa^p-\sigma^p)}\int_{\sigma}^{\kappa}\digamma(\varkappa)\varkappa^{p-1}d\varkappa \leqslant \frac{\digamma(\sigma)+\digamma(\kappa)}{s+1}. \tag{11}$$

3. Principal outcomes

In this section, we prove q-Hermite-Hadamard (H-H) type inequalities for the class of p, $(p-s)$ and modified type $(p-s)$-convex functions.

THEOREM 3. (q-H-H type inequality for p-convex functions) *Let $\digamma : I \subseteq (0,\infty) \to \mathbb{R}$ be a p-convex function on I^o, $\sigma^p,\kappa^p \in I$ with $\sigma^p < \kappa^p$ and $p > 0, q \in (0,1)$, then we have the following inequality*

$$\begin{aligned}\digamma\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right) &\leqslant \frac{1}{2(\kappa^p-\sigma^p)}\left[\int_{\sigma^p}^{\kappa^p}\digamma(\varkappa^{\frac{1}{p}})\,{}_{\sigma^p}d_q\varkappa + \int_{\sigma^p}^{\kappa^p}\digamma(\varkappa^{\frac{1}{p}})\,{}^{\kappa^p}d_q\varkappa\right]\\ &\leqslant \frac{\digamma(\sigma)+\digamma(\kappa)}{2}.\end{aligned} \tag{12}$$

Proof. By Definition of p-convexity

$$\digamma\left((\tau\varkappa^p+(1-\tau)y^p)^{\frac{1}{p}}\right) \leqslant \tau\digamma(\varkappa)+(1-\tau)\digamma(y).$$

If $\tau = \frac{1}{2}$, then

$$\digamma\left(\left(\frac{\varkappa^p+y^p}{2}\right)^{\frac{1}{p}}\right) \leqslant \frac{\digamma(\varkappa)+\digamma(y)}{2}. \tag{13}$$

Considering

$$\varkappa^p = \tau\kappa^p + (1-\tau)\sigma^p \ y^p = \tau\sigma^p + (1-\tau)\kappa^p.$$

in (13), we get

$$2F\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right) \leqslant F\left((\tau\kappa^p+(1-\tau)\sigma^p)^{\frac{1}{p}}\right) + F\left((\tau\sigma^p+(1-\tau)\kappa^p)^{\frac{1}{p}}\right).$$

q-integrating w.r.t τ over $[0,1]$, we have

$$2F\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right)\int_0^1 d_q\tau \leqslant \int_0^1 F\left((\tau\kappa^p+(1-\tau)\sigma^p)^{\frac{1}{p}}\right)d_q\tau + \int_0^1 F\left((\tau\sigma^p+(1-\tau)\kappa^p)^{\frac{1}{p}}\right)d_q\tau.$$

From Definitions (10) and (11), we have

$$\int_0^1 F\left((\tau\kappa^p+(1-\tau)\sigma^p)^{\frac{1}{p}}\right)d_q\tau = (1-q)\sum_{n=0}^{\infty} q^n F\left((q^n\kappa^p+(1-q^n)\sigma^p)^{\frac{1}{p}}\right) = \frac{1}{\kappa^p-\sigma^p}\int_{\sigma^p}^{\kappa^p} F(\varkappa^{\frac{1}{p}})\ {}_{\sigma^p}d_q\varkappa$$

$$\int_0^1 F\left((\tau\sigma^p+(1-\tau)\kappa^p)^{\frac{1}{p}}\right)d_q\tau = (1-q)\sum_{n=0}^{\infty} q^n F\left((q^n\sigma^p+(1-q^n)\kappa^p)^{\frac{1}{p}}\right) = \frac{1}{\kappa^p-\sigma^p}\int_{\sigma^p}^{\kappa^p} F(\varkappa^{\frac{1}{p}})\ {}^{\kappa^p}d_q\varkappa.$$

Thus, we get

$$2F\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right) \leqslant \frac{1}{\kappa^p-\sigma^p}\left[\int_{\sigma^p}^{\kappa^p} F(\varkappa^{\frac{1}{p}})\ {}_{\sigma^p}d_q\varkappa + \int_{\sigma^p}^{\kappa^p} F(\varkappa^{\frac{1}{p}})\ {}^{\kappa^p}d_q\varkappa\right]$$

and the first inequality (12) is proved.

To prove the second inequality, we use the p-convexity we have

$$F\left((\tau\kappa^p+(1-\tau)\sigma^p)^{\frac{1}{p}}\right) \leqslant \tau F(\kappa) + (1-\tau)F(\sigma) \tag{14}$$

$$F\left((\tau\sigma^p+(1-\tau)\kappa^p)^{\frac{1}{p}}\right) \leqslant \tau F(\sigma) + (1-\tau)F(\kappa). \tag{15}$$

By adding (14) and (15), from Definition (10) and (11), we have

$$\frac{1}{\kappa^p-\sigma^p}\left[\int_{\sigma^p}^{\kappa^p} F(\varkappa^{\frac{1}{p}})\ {}_{\sigma^p}d_q\varkappa + \int_{\sigma^p}^{\kappa^p} F(\varkappa^{\frac{1}{p}})\ {}^{\kappa^p}d_q\varkappa\right] \leqslant F(\sigma)+F(\kappa).$$

Thus, the proof is accomplished. □

REMARK 1. If we set $p=1$ in Theorem 3, then Theorem 3 reduces to [8, Theorem 20].

REMARK 2. In Theorem 3, if we take the limit as $q\to 1$, then we have

$$F\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right)\leqslant\frac{1}{(\kappa^p-\sigma^p)}\left[\int_{\sigma^p}^{\kappa^p}F(\varkappa^{\frac{1}{p}})d\varkappa\right]\leqslant\frac{F(\sigma)+F(\kappa)}{2},$$

Putting $\varkappa^{\frac{1}{p}}=x$, $d\varkappa=px^{p-1}dx$, we get

$$F\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right)\leqslant\frac{p}{(\kappa^p-\sigma^p)}\int_{\sigma}^{\kappa}\frac{F(x)}{x^{1-p}}dx\leqslant\frac{F(\sigma)+F(\kappa)}{2},$$

which is readily appears in (10).

THEOREM 4. (q-H-H type inequality for $(p-s)$-convex functions) *Let $F:I\subseteq(0,\infty)\to\mathbb{R}$ be a $(p-s)$-convex function on I^o, $\sigma^p,\kappa^p\in I$ with $\sigma^p<\kappa^p$ and $p>0$, $s\in(0,1]$, $q\in(0,1)$, then we have the following inequalities*

$$2^sF\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right)\leqslant\frac{1}{(\kappa^p-\sigma^p)}\left[\int_{\sigma^p}^{\kappa^p}F(\varkappa^{\frac{1}{p}})\,{}_{\sigma^p}d_q\varkappa+\int_{\sigma^p}^{\kappa^p}F(\varkappa^{\frac{1}{p}})\,{}^{\kappa^p}d_q\varkappa\right] \tag{16}$$

$$\leqslant[F(\sigma)+F(\kappa)]\left[\frac{1}{[s+1]_q}+\theta_1\right]$$

where

$$\theta_1=\int_0^1(1-\tau)^sd_q\tau.$$

Proof. From the Definition of $(p-s)$-convexity, we get

$$F\left((\tau\varkappa^p+(1-\tau)y^p)^{\frac{1}{p}}\right)\leqslant\tau^sF(\varkappa)+(1-\tau)^sF(y),$$

By $\tau=\frac{1}{2}$, we have

$$F\left(\left(\frac{\varkappa^p+y^p}{2}\right)^{\frac{1}{p}}\right)\leqslant\frac{F(\varkappa)+F(y)}{2^s}. \tag{17}$$

By taking

$$\varkappa^p=\tau\kappa^p+(1-\tau)\sigma^p \text{ and } y^p=\tau\sigma^p+(1-\tau)\kappa^p.$$

in (17), we get

$$2^sF\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\leqslant F\left((\tau\kappa^p+(1-\tau)\sigma^p)^{\frac{1}{p}}\right)+F\left((\tau\sigma^p+(1-\tau)\kappa^p)^{\frac{1}{p}}\right). \tag{18}$$

On q-integrating w.r.t τ over $[0,1]$, we have

$$2^s\mathcal{F}\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right)\int_0^1 d_q\tau \leqslant \int_0^1 \mathcal{F}\left((\tau\kappa^p+(1-\tau)\sigma^p)^{\frac{1}{p}}\right)d_q\tau + \int_0^1 \mathcal{F}\left((\tau\sigma^p+(1-\tau)\kappa^p)^{\frac{1}{p}}\right)d_q\tau.$$

From Definitions (10) and (11), we have

$$\int_0^1 \mathcal{F}\left((\tau\sigma^p+(1-\tau)\kappa^p)^{\frac{1}{p}}\right)d_q\tau = (1-q)\sum_{n=0}^{\infty}q^n\mathcal{F}\left((q^n\sigma^p+(1-q^n)\kappa^p)^{\frac{1}{p}}\right) = \frac{1}{\kappa^p-\sigma^p}\int_{\sigma^p}^{\kappa^p}\mathcal{F}(\varkappa)^{\frac{1}{p}}\ {}^{\kappa^p}d_q\varkappa$$

and

$$\int_0^1 \mathcal{F}\left((\tau\kappa^p+(1-\tau)\sigma^p)^{\frac{1}{p}}\right)d_q\tau = (1-q)\sum_{n=0}^{\infty}q^n\mathcal{F}\left((q^n\kappa^p+(1-q^n)\sigma^p)^{\frac{1}{p}}\right) = \frac{1}{\kappa^p-\sigma^p}\int_{\sigma^p}^{\kappa^p}\mathcal{F}(\varkappa)^{\frac{1}{p}}\ {}_{\sigma^p}d_q\varkappa.$$

Therefore, we get

$$2^s\mathcal{F}\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right) \leqslant \frac{1}{(\kappa^p-\sigma^p)}\left[\int_{\sigma^p}^{\kappa^p}\mathcal{F}(\varkappa)^{\frac{1}{p}}\ {}_{\sigma^p}d_q\varkappa+\int_{\sigma^p}^{\kappa^p}\mathcal{F}(\varkappa)^{\frac{1}{p}}\ {}^{\kappa^p}d_q\varkappa\right]$$

and the first inequality (16) is proved.

To prove the second inequality, we use the $(p-s)$-convexity we have

$$\mathcal{F}\left((\tau\kappa^p+(1-\tau)\sigma^p)^{\frac{1}{p}}\right) \leqslant \tau^s\mathcal{F}(\kappa)+(1-\tau)^s\mathcal{F}(\sigma) \tag{19}$$

$$\mathcal{F}\left((\tau\sigma^p+(1-\tau)\kappa^p)^{\frac{1}{p}}\right) \leqslant \tau^s\mathcal{F}(\sigma)+(1-\tau)^s\mathcal{F}(\kappa). \tag{20}$$

By adding (19) and (20), from Definitions (10) and (11), we have

$$\frac{1}{(\kappa^p-\sigma^p)}\left[\int_{\sigma^p}^{\kappa^p}\mathcal{F}(\varkappa^{\frac{1}{p}})\ {}_{\sigma^p}d_q\varkappa+\int_{\sigma^p}^{\kappa^p}\mathcal{F}(\varkappa^{\frac{1}{p}})\ {}^{\kappa^p}d_q\varkappa\right] \leqslant (\mathcal{F}(\sigma)+\mathcal{F}(\kappa))\left[\frac{1}{[s+1]_q}+\theta_1\right].$$

Thus, the proof is accomplished. □

REMARK 3. In Theorem 4, if we take the limit as $q\to 1$, then we have

$$2^s\mathcal{F}\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right) \leqslant \frac{2}{(\kappa^p-\sigma^p)}\int_{\sigma^p}^{\kappa^p}\mathcal{F}(\varkappa^{\frac{1}{p}})d\varkappa \leqslant [\mathcal{F}(\sigma)+\mathcal{F}(\kappa)]\left[\frac{2}{s+1}\right],$$

Putting $\varkappa^{\frac{1}{p}} = x$, $d\varkappa = px^{p-1}dx$, we get

$$2^{s-1}F\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right) \leqslant \frac{p}{(\kappa^p-\sigma^p)}\int_{\sigma}^{\kappa} F(x)x^{p-1}dx \leqslant \frac{F(\sigma)+F(\kappa)}{s+1},$$

which is readily appears in (11).

THEOREM 5. (q-H-H type inequality for modified $(p-s)$-convex functions) *Let $F : I \subseteq (0,\infty) \to \mathbb{R}$ be a modified type $(p-s)$-convex function on I^o, $\sigma^p, \kappa^\kappa \in I$ with $\sigma^p < \kappa^p$ and $p>0$, $s \in (0,1]$, $q \in (0,1)$, then we have the following inequality*

$$\begin{aligned} &2^{s}F\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right) \\ &\leqslant \frac{1}{(\kappa^p-\sigma^p)}\left[\int_{\sigma^p}^{\kappa^p} F(\varkappa^{\frac{1}{p}})\ {}_{\sigma^p}d_q\varkappa + (2^s-1)\int_{\sigma^p}^{\kappa^p} F(\varkappa^{\frac{1}{p}})\ {}^{\kappa^p}d_q\varkappa\right] \qquad (21) \\ &\leqslant \left(\frac{2}{[s+1]_q}-1\right)(F(\kappa)-F(\sigma)) + 2^s\left[\frac{F(\sigma)-F(\kappa)}{[s+1]_q}+F(\kappa)\right]. \end{aligned}$$

Proof. From the Definition of modified $(p-s)$-convexity

$$F\left((\tau\varkappa^p+(1-\tau)y^p)^{\frac{1}{p}}\right) \leqslant \tau^s F(\varkappa) + (1-\tau^s)F(y),$$

If we take $\tau = \frac{1}{2}$, then

$$F\left(\left(\frac{\varkappa^p+y^p}{2}\right)^{\frac{1}{p}}\right) \leqslant \frac{F(\varkappa)+(2^s-1)F(y)}{2^s}. \qquad (22)$$

Putting $\varkappa^p = \tau\kappa^p + (1-\tau)\sigma^p$ and $y^p = \tau\sigma^p + (1-\tau)\kappa^p$ in (22), we get

$$2^sF\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right) \leqslant F\left((\tau\kappa^p+(1-\tau)\sigma^p)^{\frac{1}{p}}\right) + (2^s-1)F\left((\tau\sigma^p+(1-\tau)\kappa^p)^{\frac{1}{p}}\right). \qquad (23)$$

q-integrating w.r.t τ over $[0,1]$, we have

$$\begin{aligned} 2^sF\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right)\int_0^1 d_q\tau \leqslant &\int_0^1 F\left((\tau\kappa^p+(1-\tau)\sigma^p)^{\frac{1}{p}}\right)d_q\tau \\ &+(2^s-1)\int_0^1 F\left((\tau\sigma^p+(1-\tau)\kappa^p)^{\frac{1}{p}}\right)d_q\tau. \end{aligned}$$

From Definitions (10) and (11), we have

$$\begin{aligned} \int_0^1 F\left((\tau\kappa^p+(1-\tau)\sigma^p)^{\frac{1}{p}}\right)d_q\tau &= (1-q)\sum_{n=0}^{\infty} q^n F\left((q^n\kappa^p+(1-q^n)\sigma^p)^{\frac{1}{p}}\right) \\ &= \frac{1}{\kappa^p-\sigma^p}\int_{\sigma^p}^{\kappa^p} F(\varkappa^{\frac{1}{p}})\ {}_{\sigma^p}d_q\varkappa \end{aligned}$$

and

$$\int_0^1 \digamma\left((\tau\sigma^p+(1-\tau)\kappa^p)^{\frac{1}{p}}\right)d_q\tau = (1-q)\sum_{n=0}^{\infty}q^n\digamma\left((q^n\sigma^p+(1-q^n)\kappa^p)^{\frac{1}{p}}\right)$$
$$= \frac{1}{\kappa^p-\sigma^p}\int_{\sigma^p}^{\kappa^p}\digamma(\varkappa^{\frac{1}{p}})\ {}^{\kappa^p}d_q\varkappa.$$

Then it follows that

$$2^s\digamma\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right) \leqslant \frac{1}{(\kappa^p-\sigma^p)}\left[\int_{\sigma^p}^{\kappa^p}\digamma(\varkappa^{\frac{1}{p}})\ {}^{\kappa^p}d_q\varkappa+(2^s-1)\int_{\sigma^p}^{\kappa^p}\digamma(\varkappa^{\frac{1}{p}})\ {}_{\sigma^p}d_q\varkappa\right]$$

and the first inequality (21) is proved.

To prove the second inequality, we use the modified $(p-s)$-convexity we have

$$\digamma\left((\tau\kappa^p+(1-\tau)\sigma^p)^{\frac{1}{p}}\right) \leqslant \tau^s\digamma(\kappa)+(1-\tau^s)\digamma(\sigma) \tag{24}$$

$$\digamma\left((\tau\sigma^p+(1-\tau)\kappa^p)^{\frac{1}{p}}\right) \leqslant \tau^s\digamma(\sigma)+(1-\tau^s)\digamma(\kappa). \tag{25}$$

By adding (24) and (25), from Definitions (10) and (11), we have

$$\frac{1}{(\kappa^p-\sigma^p)}\left[\int_{\sigma^p}^{\kappa^p}\digamma(\varkappa^{\frac{1}{p}})\ {}_{\sigma^p}d_q\varkappa+(2^s-1)\int_{\sigma^p}^{\kappa^p}\digamma(\varkappa^{\frac{1}{p}})\ {}^{\kappa^p}d_q\varkappa\right]$$
$$\leqslant \left(\frac{2}{[s+1]_q}-1\right)(\digamma(\kappa)-\digamma(\sigma))+2^s\left[\frac{\digamma(\sigma)-\digamma(\kappa)}{[s+1]_q}+\digamma(\kappa)\right].$$

Thus, the proof is accomplished. □

COROLLARY 1. *In Theorem 5, if $q\to 1$, then we get the following inequalities for the modified $(p-s)$-convex functions*

$$2^s\digamma\left(\left(\frac{\sigma^p+\kappa^p}{2}\right)^{\frac{1}{p}}\right) \leqslant \frac{2^s p}{\kappa^p-\sigma^p}\int_{\sigma}^{\kappa}\digamma(\tau)\tau^{p-1}d\tau \leqslant (1-s)(\digamma(\kappa)-\digamma(\sigma))$$
$$+2^s\left[\frac{\digamma(\sigma)-\digamma(\kappa)}{s+1}+\digamma(\kappa)\right].$$

REMARK 4. If we set $s=1$ in corollary 1, then we recapture the inequality (10).

REMARK 5. In Theorem 5, if we set $s=1$ and $p=1$, then Theorem 5 reduces to [8, Theorem 18].

4. Conclusions

In the current investigation, we considered the class of p, $(p-s)$ and modified $(p-s)$-convex functions. Then we derived three new q Hermite-Hadamard type inequalities for the class of $p,(p-s)$ and modified type $(p-s)$-convex functions. It is expected that the ideas and techniques presented in this paper will also be applicable to several other types of convexities.

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