

HÖLDER AND MINKOWSKI TYPE INEQUALITIES FOR PSEUDO-FRACTIONAL INTEGRAL

DANIELA S. OLIVEIRA* AND AZIZOLLAH BABAKHANI

(Communicated by H. Lopushanska)

Abstract. We first introduce the concept of fractional operators and pseudo-analysis. Then we present new versions of Hölder, Minkowski and reverse Minkowski inequalities via ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integral on a semiring $([a, b], \oplus, \odot)$.

1. Introduction

Pseudo-analysis is a generalization of the classical analysis, where the field of real numbers is replaced with a semiring, i.e., a real interval $[a, b] \subseteq [-\infty, \infty]$ with pseudo-addition \oplus and with pseudo-multiplication \odot , see [18, 19, 21].

Notice that pseudo-integrals were started to attract mathematicians' attentions in several applications, for example in the area of nonlinear partial differential equations occurring in different applied fields, see [31] as well as the edited volume [32]. Inequalities play a central and fundamental role in the fields of probability and measure theory, classical analysis, optimization theory, mathematical finance and economics.

Many authors have investigated important inequalities via the pseudo integrals [1, 10, 14, 20, 24, 30]. The study of inequalities via fractional pseudo-fractional integrals has been investigated in recent years [3, 11, 28].

The classical Hölder's and Minkowski's integral inequality hold [22] for u and v be measurable functions on X , with range in $[0, \infty]$:

(i) (Hölder's inequality)

$$\int_X uv \, d\mu \leq \left(\int_X u^p \, d\mu \right)^{\frac{1}{p}} \left(\int_X v^q \, d\mu \right)^{\frac{1}{q}},$$

(ii) (Minkowski's inequality)

$$\left(\int_X (u+v)^p \, d\mu \right)^{\frac{1}{p}} \leq \left(\int_X u^p \, d\mu \right)^{\frac{1}{p}} + \left(\int_X v^p \, d\mu \right)^{\frac{1}{p}},$$

Mathematics subject classification (2020): 26A33, 47GXX, 47AXX.

Keywords and phrases: Pseudo-addition, pseudo-multiplication, semiring, ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integral, Minkowski's inequality, Hölder's inequality.

* Corresponding author.

where p and q be conjugate exponents, $1 < p < \infty$ and X be a measure space, with measure μ .

Agahi et al. proved generalizations of Hölder’s and Minkowski’s inequality for pseudo-integral [4]:

THEOREM 1. (Hölder’s inequality for pseudo-integral) *Let p and q be conjugate exponents, $1 < p < \infty$. For a given measurable space (X, \mathcal{A}) let $u, v : X \rightarrow [a, b]$ be two measurable functions and let a generator $g : [a, b] \rightarrow [0, \infty]$ of the pseudo-addition \oplus and the pseudo-multiplication \odot is a decreasing function. Then for any σ - \oplus -measure μ it holds:*

$$\int_X^\oplus (u \odot v) \odot d\mu \leq \left(\int_X^\oplus u^{(p)} \odot d\mu \right)_{\odot}^{\left(\frac{1}{p}\right)} \odot \left(\int_X^\oplus v^{(q)} \odot d\mu \right)_{\odot}^{\left(\frac{1}{q}\right)}.$$

THEOREM 2. (Minkowski’s inequality for pseudo-integral) *Let p and q be conjugate exponents, $1 < p < \infty$. For a given measurable space (X, \mathcal{A}) let $u, v : X \rightarrow [a, b]$ be two measurable functions and let a generator $g : [a, b] \rightarrow [0, \infty]$ of the pseudo-addition \oplus and the pseudo-multiplication \odot be an increasing function. Then for any σ - \oplus -measure μ it holds:*

$$\left(\int_X^\oplus (u \oplus v)^{(p)} \odot d\mu \right)_{\odot}^{\left(\frac{1}{p}\right)} \leq \left(\int_X^\oplus u^{(p)} \odot d\mu \right)_{\odot}^{\left(\frac{1}{p}\right)} \oplus \left(\int_X^\oplus v^{(p)} \odot d\mu \right)_{\odot}^{\left(\frac{1}{p}\right)}.$$

The main motivation of this paper is to obtain a general version of the Hölder’s and Minkowski’s inequality and reverse Minkowski’s inequality using the ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integral. The paper has been organized as follows. Section 2 is devoted to the fractional and pseudo-fractional operators. Section 3 contains some of preliminaries on the pseudo-analysis. In Section 4, we prove generalizations of the Hölder and Minkowski inequalities using pseudo-fractional integrals. In Sections 5, we prove the reverse Minkowski pseudo-fractional integral inequality. In Section 6, other integral inequalities related to the Minkowski inequality are also proved. Concluding remarks close the paper.

2. Fractional operators

In this section, we present ψ -Riemann-Liouville [12, 23] and ψ -Riemann-Liouville-Mittag-Leffler fractional integrals [16, 29] and ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integrals [17] that will be used in this paper.

DEFINITION 1. [12, 23] Let $\alpha > 0$, $\Omega = [a, b]$ be a finite or infinite interval, f an integrable function defined on Ω and $\psi \in C^1(\Omega)$ an increasing function such that $\psi'(x) \neq 0$, for all $x \in \Omega$. The left- and right-sided ψ -Riemann-Liouville fractional integrals of order α of f on Ω are defined by

$$\mathbb{I}_{a+}^{\alpha; \psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} f(t) dt \tag{1}$$

and

$$\mathbb{I}_{b^-}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} f(t) dt, \tag{2}$$

respectively. For $\alpha \rightarrow 0$, we have

$$\mathbb{I}_{a^+}^{0;\psi} f(x) = \mathbb{I}_{b^-}^{0;\psi} f(x) = f(x).$$

DEFINITION 2. [16, 29] Let $\alpha, \gamma, \rho, \omega \in \mathbb{R}$ with $\alpha > 0$ and $\rho > 0$ and let $\Omega = [a, b]$ be a finite or infinite interval of the real axis \mathbb{R} , f an integrable function defined on Ω and $\psi \in C^1(\Omega)$ an increasing function such that $\psi'(x) \neq 0$, for all $x \in \Omega$. The left- and right-sided ψ -Riemann-Liouville-Mittag-Leffler fractional integrals are defined by

$$\mathbb{E}_{\rho, \alpha, \omega; a^+}^{\gamma; \psi} f(x) = \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] f(t) dt \tag{3}$$

and

$$\mathbb{E}_{\rho, \alpha, \omega; b^-}^{\gamma; \psi} f(x) = \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(t) - \psi(x))^{\rho}] f(t) dt,$$

respectively, where $E_{\rho, \alpha}^{\gamma}(\cdot)$ is the three-parameters Mittag-Leffler function. In particular, if $\gamma = 0$ we have the fractional integrals given by Eq.(1) and Eq.(2), this is,

$$\mathbb{E}_{\rho, \alpha, \omega; a^+}^{0; \psi} f(x) = \mathbb{I}_{a^+}^{\alpha; \psi} f(x) \quad \text{and} \quad \mathbb{E}_{\rho, \alpha, \omega; b^-}^{0; \psi} f(x) = \mathbb{I}_{b^-}^{\alpha; \psi} f(x). \tag{4}$$

If $\alpha \rightarrow 0$ and $\gamma = 0$, we have

$$\mathbb{E}_{\rho, 0, \omega; a^+}^{0; \psi} f(x) = f(x) \quad \text{and} \quad \mathbb{E}_{\rho, 0, \omega; b^-}^{0; \psi} f(x) = f(x). \tag{5}$$

DEFINITION 3. [17] Let $\alpha, \gamma, \rho, \omega \in \mathbb{R}$ with $\alpha > 0$ and $\rho > 0$ and let $\Omega = [a, b]$ be a finite or infinite interval of the real axis \mathbb{R} . Let a generator $g : \Omega \rightarrow [0, \infty]$ of the pseudo-addition \oplus and the pseudo-multiplication \odot be an increasing function. Also let f an integrable function defined on Ω , ψ be an increasing and positive function on $(a, b]$, having a continuous derivative ψ' on (a, b) with $\psi'(x) \neq 0$. The ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integral (left-sided and right-sided) are defined as

$$\begin{aligned} & \mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a^+}^{\gamma; \psi} f(x) \\ & := g^{-1} \left(\mathbb{E}_{\rho, \alpha, \omega; a^+}^{\gamma; \psi} g(f(x)) \right) \\ & = \int_{[a, x]}^{\oplus} [g^{-1} (\psi'(t) (\psi'(x) - \psi'(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}]) \odot f(t)] dt \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; b-}^{\gamma; \Psi} f(x) \\ & := g^{-1} \left(\mathbb{E}_{\rho, \alpha, \omega; b-}^{\gamma; \Psi} g(f(x)) \right) \\ & = \int_{[x, b]}^{\oplus} \left[g^{-1} \left(\Psi'(t) (\Psi'(t) - \Psi'(x))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\Psi(t) - \Psi(x))^{\rho}] \odot f(t) \right) \right] dt. \end{aligned}$$

In particular, if $\gamma = 0$ we have fractional integrals given by Definition 3.1 of [7] i.e.,

$$\begin{aligned} \mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{0; \Psi} f(x) & = g^{-1} \left(\mathbb{E}_{\rho, \alpha, \omega; a+}^{0; \Psi} g(f(x)) \right) = g^{-1} \left(\mathbb{I}_{a+}^{\alpha; \Psi} g(f(x)) \right) \\ & = \mathbb{I}_{\oplus, \odot, a+}^{\alpha; \Psi} f(x) \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; b-}^{0; \Psi} f(x) & = g^{-1} \left(\mathbb{E}_{\rho, \alpha, \omega; b-}^{0; \Psi} g(f(x)) \right) = g^{-1} \left(\mathbb{I}_{b-}^{\alpha; \Psi} g(f(x)) \right) \\ & = \mathbb{I}_{\oplus, \odot, b-}^{\alpha; \Psi} f(x). \end{aligned}$$

We assume $g : [a, b] \rightarrow [0, \infty]$ of the pseudo-addition \oplus , the pseudo-multiplication \odot be a generator, an increasing function and $f : [a, b] \rightarrow [a, b]$ is a measurable function.

3. Pseudo-analysis

In this section, we summarize some properties of the pseudo-analysis [18, 20, 21]. Let $[a, b] \subset [-\infty, +\infty]$. The full order on $[a, b]$ will be denoted by \preceq .

DEFINITION 4. [20] A binary operation \oplus on $[a, b]$ is pseudo-addition if it is commutative, non-decreasing (with respect to \preceq), continuous, associative, and with a zero (neutral) element denoted by $\mathbf{0}$. Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$. A binary operation \odot on $[a, b]$ is pseudo-multiplication if it is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and with a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b]$, $\mathbf{1} \odot x = x$. Also, $\mathbf{0} \odot x = \mathbf{0}$ and that \odot is distributive over \oplus , i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a, b], \oplus, \odot)$ is a semiring (see [13]).

DEFINITION 5. [21] An important class of pseudo-operations \oplus and \odot is when these are defined by a monotone and continuous function $g : [a, b] \rightarrow [0, \infty]$, i.e., pseudo-operations \oplus and \odot are given with

$$x \oplus y = g^{-1}(g(x) + g(y)) \quad \text{and} \quad x \odot y = g^{-1}(g(x)g(y)).$$

DEFINITION 6. [21] Let X be a non-empty set and \mathcal{A} be a σ -algebra of subsets of a set X . A set $\mu : \mathcal{A} \rightarrow [a, b]$ is called a σ - \oplus -measure if it satisfies the following conditions:

1. $\mu(\emptyset) = \mathbf{0}$;
2. $\mu : \left(\bigcup_{i=1}^{\infty} \mathcal{A}_i \right) = \bigoplus_{i=1}^{\infty} \mu(\mathcal{A}_i)$ holds for any sequence $\{\mathcal{A}_i\}_{i \in \mathbb{N}}$ of pairwise disjoint sets from \mathcal{A} .

DEFINITION 7. [21] Let pseudo-operations \oplus and \odot are defined by a monotone and continuous function $g : [a, b] \rightarrow [0, \infty]$. The g -integral for a measurable function $f : [c, d] \rightarrow [a, b]$ is given by

$$\int_{[c,d]}^{\oplus} f \odot dx = g^{-1} \left(\int_c^d g(f(x)) dx \right).$$

DEFINITION 8. [18] Let g be the additive generator of the strict pseudo-addition \oplus on $[a, b]$ such that g is continuously differentiable on (a, b) . The corresponding pseudo-multiplication \odot will always be defined as $u \odot v = g^{-1}(g(u)g(v))$. If the function f is differentiable on (c, d) and has the same monotonicity as the function g , then the g -derivative of f at the point $x \in (c, d)$ is defined by

$$\frac{d^{\oplus} f(x)}{dx} = g^{-1} \left(\frac{d}{dx} g(f(x)) \right).$$

Also, if there exists the n - g -derivative of f , then

$$\frac{d^{(n)\oplus} f(x)}{dx} = g^{-1} \left(\frac{d^n}{dx^n} g(f(x)) \right).$$

DEFINITION 9. [15] Let g be a generator of a pseudo-addition \oplus on interval $[-\infty, +\infty]$. Binary operation \ominus and \oslash on $[-\infty, +\infty]$ defined by the formulas:

$$x \ominus y = g^{-1}(g(x) - g(y)) \quad x \oslash y = g^{-1} \left(\frac{g(x)}{g(y)} \right),$$

if expressions $g(x) - g(y)$ and $\frac{g(x)}{g(y)}$ have sense are said to be the pseudo-subtraction and pseudo-division consistent with the pseudo-addition \oplus .

DEFINITION 10. [15] Let $g : [-\infty, +\infty] \rightarrow [-\infty, +\infty]$ be a continuous, strictly increasing and odd function such that $g(0) = 0$, $g(1) = 1$, $g(+\infty) = +\infty$. The system of pseudo-arithmetical operations $\{\oplus, \ominus, \odot, \oslash\}$ generated by this function is said to be the consistent system.

4. Hölder and Minkowski inequalities

Our first result is the following Hölder inequality using ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integral.

THEOREM 3. (Hölder’s inequality) *Let $\alpha, \gamma, \rho, \omega \in \mathbb{R}$, with $\alpha > 0$ and $\rho > 0$. Also let p and q be conjugate exponents, $1 < p < \infty$. Assume that there exist two functions f, h are they both measurable functions for $x > a$ and ψ be an increasing function on $C^1(\Omega)$ such that $\psi'(t) \neq 0$ for all $t \in \Omega$. Let a generator $g : \Omega \rightarrow [0, \infty]$ of the pseudo-addition \oplus and the pseudo-multiplication \odot . Then the Hölder’s inequality for ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integral holds:*

$$\left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} (f \odot h)(x) \right] \leq \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} f^p(x) \right]_{\odot}^{\frac{1}{p}} \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} h^q(x) \right]_{\odot}^{\frac{1}{q}}. \tag{6}$$

Proof. Taking into account the Young’s inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{7}$$

with $a = f(t)/A$ and $b = h(t)/B$, where $A, B \neq 0$, we have

$$f(t)h(t) \leq AB \left(\frac{f^p(t)}{pA^p} + \frac{h^q(t)}{qB^q} \right). \tag{8}$$

Since ψ be an increasing function such that $\psi'(t) \neq 0$ for all $t \in \Omega$, therefore $\psi'(t) > 0$, for all $t \in \Omega$, thus $\psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}]$ is a positive function where $t \in (a, x)$. Multiplying both sides of (8) by

$$\psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}]$$

and integrating the result with respect t over (a, x) , we get

$$\begin{aligned} & \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] f(t)h(t) dt \\ & \leq AB \left[\frac{1}{pA^p} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] f^p(t) dt \right. \\ & \quad \left. + \frac{1}{qB^q} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] h^q(t) dt \right]. \end{aligned}$$

Since function g is increasing function and g^{-1} is also increasing function and applying the compositions $g \circ f$ and $g \circ h$ to previous inequality, we obtain

$$\begin{aligned}
 &g^{-1} \left\{ \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)(t) (g \circ h)(t) dt \right\} \quad (9) \\
 &\leq g^{-1} \left\{ AB \left[\frac{1}{pA^p} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)^p(t) dt \right. \right. \\
 &\quad \left. \left. + \frac{1}{qB^q} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h)^q(t) dt \right] \right\}.
 \end{aligned}$$

Let

$$A^p = \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)^p(t) dt$$

and

$$B^q = \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h)^q(t) dt.$$

Thus the inequality (9) becomes

$$\begin{aligned}
 &g^{-1} \left\{ \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] g \left(g^{-1} \left((g \circ f)(g \circ h)(t) \right) \right) dt \right\} \\
 &\leq g^{-1} \left\{ \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] g \left(g^{-1} \left((g \circ f)^p(t) \right) \right) dt \right]^{\frac{1}{p}} \right. \\
 &\quad \left. \times \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] g \left(g^{-1} \left((g \circ h)^q(t) \right) \right) dt \right]^{\frac{1}{q}} \right\}
 \end{aligned}$$

or, equivalently,

$$\begin{aligned}
 &g^{-1} \left\{ \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] g(f \circ h)(t) dt \right\} \\
 &\leq g^{-1} \left\{ g \left(g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f_{\odot}^p)(t) dt \right)^{\frac{1}{p}} \right) \right) \right. \\
 &\quad \left. \times g \left(g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h_{\odot}^q)(t) dt \right)^{\frac{1}{q}} \right) \right) \right\}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 & g^{-1} \left\{ \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma [\omega (\psi(x) - \psi(t))^\rho] g(f \circ h)(t) dt \right\} \\
 & \leq g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma [\omega (\psi(x) - \psi(t))^\rho] (g \circ f_\odot^p)(t) dt \right)^{\frac{1}{p}} \right) \\
 & \quad \odot g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma [\omega (\psi(x) - \psi(t))^\rho] (g \circ h_\odot^q)(t) dt \right)^{\frac{1}{q}} \right) \Bigg\},
 \end{aligned}$$

this is,

$$\begin{aligned}
 & g^{-1} \left\{ \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma [\omega (\psi(x) - \psi(t))^\rho] g(f \circ h)(t) dt \right\} \tag{10} \\
 & \leq g^{-1} \left\{ \left[g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma \right. \right. \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \left. \times [\omega (\psi(x) - \psi(t))^\rho] (g \circ f_\odot^p)(t) dt \right) \right)^{\frac{1}{p}} \right] \right\} \\
 & \quad \odot g^{-1} \left\{ \left[g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma \right. \right. \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \left. \times [\omega (\psi(x) - \psi(t))^\rho] (g \circ h_\odot^q)(t) dt \right) \right)^{\frac{1}{q}} \right] \right\}.
 \end{aligned}$$

From (10) we derive the inequality (6). \square

REMARK 1. Taking $g(x) = x$ in (6), we obtain the Hölder’s inequality for ψ -Riemann-Liouville-Mittag-Leffler fractional integral, this is,

$$\left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \Psi} f(x) h(x) \right] \leq \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \Psi} f^p(x) \right]^{\frac{1}{p}} \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \Psi} h^q(x) \right]^{\frac{1}{q}}. \tag{11}$$

On the other hand, taking $g(x) = x$, $\psi(x) = x$, $\alpha = 1$ and $\gamma = 0$ in (6), we obtain Hölder’s inequality for pseudo-integral [4].

Our second result is the following Minkowski inequality using ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integral.

THEOREM 4. (Minkowski’s inequality) *Let $\alpha, \gamma, \rho, \omega \in \mathbb{R}$, with $\alpha > 0$ and $\rho > 0$. Also let p and q be conjugate exponents, $1 < p < \infty$. Assume that there exist two functions f, h are they both measurable functions for $x > a$ and ψ be an increasing*

function on $C^1(\Omega)$ such that $\psi'(t) \neq 0$ for all $t \in \Omega$. Let a generator $g : \Omega \rightarrow [0, \infty]$ of the pseudo-addition \oplus and the pseudo-multiplication \odot . Then the Minkowski's inequality for ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integral holds:

$$\left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)^p(x) \right]_{\odot}^{\frac{1}{p}} \leq \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f^p(x) \right]_{\odot}^{\frac{1}{p}} \oplus \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h^p(x) \right]_{\odot}^{\frac{1}{p}}. \tag{12}$$

Proof. To prove (9), we write

$$(f + h)^p(t) = f(t)(f + h)^{p-1}(t) + h(t)(f + h)^{p-1}(t) \tag{13}$$

Multiplying to both sides of (13) by $\psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}]$ and integrating the result with respect t on (a, x) , we get

$$\begin{aligned} \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} (f + h)^p(x) \right] &\leq \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} f(x)(f + h)^{p-1}(x) \right] \\ &\quad + \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} h(x)(f + h)^{p-1}(x) \right]. \end{aligned} \tag{14}$$

Hölder's inequality (Remark 1) gives

$$\begin{aligned} \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} (f + h)^p(x) \right] &\leq \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} f^p(x) \right]^{\frac{1}{p}} \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} (f + h)^{(p-1)q}(x) \right]^{\frac{1}{q}} \\ &\quad + \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} h^p(x) \right]^{\frac{1}{p}} \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} (f + h)^{(p-1)q}(x) \right]^{\frac{1}{q}}. \end{aligned}$$

Since $\frac{1}{p} + \frac{1}{q} = 1$ it follows that $(p - 1)q = p$, we have

$$\left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} (f + h)^p(x) \right]^{\frac{1}{p}} \leq \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} f^p(x) \right]^{\frac{1}{p}} + \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} h^p(x) \right]^{\frac{1}{p}} \tag{15}$$

or, equivalently,

$$\begin{aligned} &\left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (f + h)^p(t) dt \right)^{\frac{1}{p}} \\ &\leq \left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] f^p(t) dt \right)^{\frac{1}{p}} \\ &\quad + \left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] h^p(t) dt \right)^{\frac{1}{p}}. \end{aligned}$$

Since function g is increasing function, then g^{-1} is also increasing function and we have

$$\begin{aligned}
 &g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ f + g \circ h)^p(t) dt \right)^{\frac{1}{p}} \right) \\
 &\leq g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ f)^p(t) dt \right)^{\frac{1}{p}} \right. \\
 &\quad \left. + \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ h)^p(t) dt \right)^{\frac{1}{p}} \right). \tag{16}
 \end{aligned}$$

For the left-hand side of the inequality (16), we obtain

$$\begin{aligned}
 &g^{-1} \left(\left(g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \right. \\
 &\quad \left. \left. \left. \left. \times \left(g \left(g^{-1} ((g \circ f + g \circ h)(t)) \right)^p dt \right) \right) \right) \right)^{\frac{1}{p}} \right) \\
 &= \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] \right. \right. \\
 &\quad \left. \left. \times g \left(g^{-1} (g(f \oplus h)(t))^p dt \right) \right) \right)^{\frac{1}{p}}_{\odot} \\
 &= \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^p(x) \right]^{\frac{1}{p}}_{\odot}. \tag{17}
 \end{aligned}$$

For the right-hand side of the inequality (16), we get

$$\begin{aligned}
 &g^{-1} \left\{ g \left[g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] g \left(g^{-1} (g \circ f)^p(t) dt \right)^{\frac{1}{p}} \right) \right] \right. \\
 &\quad \left. + g \left[g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] g \left(g^{-1} (g \circ h)^p(t) dt \right)^{\frac{1}{p}} \right) \right] \right\} \\
 &= \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ f_{\odot}^p)(t) dt \right)^{\frac{1}{p}} \right) \\
 &\quad \oplus \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ h_{\odot}^p)(t) dt \right)^{\frac{1}{p}} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= g^{-1} \left(\left(g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \right. \\
 &\qquad \qquad \qquad \left. \left. \left. \left. \times (g \circ f_{\odot}^p)(t) dt \right) \right) \right) \right)^{\frac{1}{p}} \\
 &\oplus g^{-1} \left(\left(g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \right. \\
 &\qquad \qquad \qquad \left. \left. \left. \left. \times (g \circ h_{\odot}^p)(t) dt \right) \right) \right) \right)^{\frac{1}{p}} \\
 &= \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) \right]_{\odot}^{\frac{1}{p}} \oplus \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) \right]_{\odot}^{\frac{1}{p}}. \tag{18}
 \end{aligned}$$

Hence (17) and (18) yield the result in (12), which completes the proof of theorem. \square

REMARK 2. Taking $g(x) = x$ in (12), we obtain the Mikowski’s inequality for ψ -Riemann-Liouville-Mittag-Leffler fractional integral:

$$\left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} (f + h)^p(x) \right]_{\odot}^{\frac{1}{p}} \leq \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} f^p(x) \right]_{\odot}^{\frac{1}{p}} + \left[\mathbb{E}_{\rho, \alpha, \omega; a+}^{\gamma; \psi} h^p(x) \right]_{\odot}^{\frac{1}{p}}. \tag{19}$$

On the other hand, taking $g(x) = x$, $\psi(x) = x$, $\alpha = 1$ and $\gamma = 0$ in (12), we obtain Minkowski’s inequality for pseudo-integral [4].

5. Reverse Minkowski pseudo-fractional integral inequality

In this section, we present reverse Minkowski inequality using ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integral.

THEOREM 5. Let $\alpha, \gamma, \rho, \omega \in \mathbb{R}$, with $\alpha > 0$ and $\rho > 0$. Also let $p \geq 1$ and ψ be an increasing function on $C^1(\Omega)$ such that $\psi'(t) \neq 0$ for all $t \in \Omega$. Assume that there exist two positive functions f, h are they both measurable functions for $x > a$ satisfying $\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) < \infty$ and $\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) < \infty$. Let a generator $g : \Omega \rightarrow [0, \infty]$ of the pseudo-addition \oplus and the pseudo-multiplication \odot be an increasing function. If the following condition satisfied $0 < m \leq \frac{f(t)}{h(t)} \leq M$ for all $t \in [a, x]$ and $m, M \in \mathbb{R}_{+}^*$, then the inequality holds for ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integral:

$$\begin{aligned}
 &\left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) \right]_{\odot}^{\frac{1}{p}} \oplus \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) \right]_{\odot}^{\frac{1}{p}} \\
 &\leq g^{-1} \left(\left[\frac{1 + M(m + 2)}{(M + 1)(m + 1)} \right] \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^p(x) \right]_{\odot}^{\frac{1}{p}}. \tag{20}
 \end{aligned}$$

Proof. Using the condition $\frac{f(t)}{h(t)} \leq M$, $t \in [a, x]$, $x > a$, we have

$$f^p(t) \leq \left(\frac{M}{M+1} \right)^p (f+h)^p(t). \quad (21)$$

Since ψ be an increasing function such that $\psi'(t) \neq 0$ for all $t \in \Omega$, therefore $\psi'(t) > 0$, for all $t \in \Omega$, thus $\psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}]$ is a positive function where $t \in (a, x)$. Multiplying both sides of (21) by

$$\psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}]$$

and integrating the result with respect t over (a, x) , we get

$$\begin{aligned} & \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] f^p(t) dt \\ & \leq \left(\frac{M}{M+1} \right)^p \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (f+h)^p(t) dt. \end{aligned} \quad (22)$$

Now, using the condition $m \leq \frac{f(t)}{h(t)}$, $t \in [a, x]$, $x > a$ we have

$$h^p(t) \leq \left(\frac{1}{m+1} \right)^p (f+h)^p(t). \quad (23)$$

Multiplying both sides of (23) by $\psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}]$ and integrating the result with respect t over (a, x) , we get

$$\begin{aligned} & \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] h^p(t) dt \\ & \leq \left(\frac{1}{m+1} \right)^p \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (f+h)^p(t) dt. \end{aligned} \quad (24)$$

Applying the compositions $g \circ f$ and $g \circ h$ to inequalities (22) and (24) we obtain, respectively,

$$\begin{aligned} & \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)^p(t) dt \\ & \leq \left(\frac{M}{M+1} \right)^p \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f + g \circ h)^p(t) dt \end{aligned} \quad (25)$$

and

$$\begin{aligned} & \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h)^p(t) dt \\ & \leq \left(\frac{1}{m+1} \right)^p \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f + g \circ h)^p(t) dt \end{aligned} \quad (26)$$

or, equivalently,

$$\left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho](g \circ f)^P(t) dt \right]^{\frac{1}{p}} \tag{27}$$

$$\leq \left(\frac{M}{M+1} \right) \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho](g \circ f + g \circ h)^P(t) dt \right]^{\frac{1}{p}}.$$

and

$$\left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho](g \circ h)^P(t) dt \right]^{\frac{1}{p}} \tag{28}$$

$$\leq \left(\frac{1}{m+1} \right) \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho](g \circ f + g \circ h)^P(t) dt \right]^{\frac{1}{p}}.$$

Adding the inequalities (27) and (28) yields

$$\left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho](g \circ f)^P(t) dt \right]^{\frac{1}{p}}$$

$$+ \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho](g \circ h)^P(t) dt \right]^{\frac{1}{p}}$$

$$\leq \left(\frac{M}{M+1} \right) \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho](g \circ f + g \circ h)^P(t) dt \right]^{\frac{1}{p}}$$

$$+ \left(\frac{1}{m+1} \right) \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho](g \circ f + g \circ h)^P(t) dt \right]^{\frac{1}{p}}.$$

Since function g is increasing function, then g^{-1} is also increasing function and we have

$$g^{-1} \left\{ \left[\left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho](g \circ f)^P(t) dt \right)^{\frac{1}{p}} \right] \right.$$

$$\left. + \left[\left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho](g \circ h)^P(t) dt \right)^{\frac{1}{p}} \right] \right\}$$

$$\leq g^{-1} \left\{ \left(\frac{M}{M+1} \right) \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho](g \circ f + g \circ h)^P(t) dt \right]^{\frac{1}{p}} \right.$$

$$\left. + \left(\frac{1}{m+1} \right) \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho](g \circ f + g \circ h)^P(t) dt \right]^{\frac{1}{p}} \right\}.$$

Hence,

$$\begin{aligned}
& g^{-1} \left\{ \left[\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)^{\rho}(t) dt \right)^{\frac{1}{p}} \right] \right. \\
& \quad \left. + \left[\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h)^{\rho}(t) dt \right)^{\frac{1}{p}} \right] \right\} \\
& = g^{-1} \left\{ g \left[g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \\
& \qquad \qquad \qquad \left. \left. \left. \times g \left(g^{-1} \left((g \circ f)^{\rho}(t) \right) dt \right)^{\frac{1}{p}} \right) \right) \right] \right. \\
& \quad \left. + g \left[g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \right. \\
& \qquad \qquad \qquad \left. \left. \left. \times g \left(g^{-1} \left((g \circ h)^{\rho}(t) \right) dt \right)^{\frac{1}{p}} \right) \right) \right] \right\} \\
& = g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f_{\odot}^{\rho})(t) dt \right)^{\frac{1}{p}} \right) \\
& \quad \oplus g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h_{\odot}^{\rho})(t) dt \right)^{\frac{1}{p}} \right) \\
& = g^{-1} \left\{ \left[g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \right. \\
& \qquad \qquad \qquad \left. \left. \left. \times (g \circ f_{\odot}^{\rho})(t) dt \right) \right)^{\frac{1}{p}} \right\} \\
& \quad \oplus g^{-1} \left\{ \left[g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \right. \\
& \qquad \qquad \qquad \left. \left. \left. \times (g \circ h_{\odot}^{\rho})(t) dt \right) \right)^{\frac{1}{p}} \right\} \\
& = \left[g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f_{\odot}^{\rho})(t) dt \right) \right]_{\odot}^{\frac{1}{p}} \\
& \quad \oplus \left[g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h_{\odot}^{\rho})(t) dt \right) \right]_{\odot}^{\frac{1}{p}} \\
& = \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi, f_{\odot}^{\rho}}(x) \right]_{\odot}^{\frac{1}{p}} \oplus \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi, h_{\odot}^{\rho}}(x) \right]_{\odot}^{\frac{1}{p}}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 & g^{-1} \left\{ \left(\frac{M}{M+1} \right) \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho] (g \circ f + g \circ h)^p(t) dt \right]^{\frac{1}{p}} \right. \\
 & \quad \left. + \left(\frac{1}{m+1} \right) \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho] (g \circ f + g \circ h)^p(t) dt \right]^{\frac{1}{p}} \right\} \\
 &= g^{-1} \left\{ g \left[g^{-1} \left(\left(\frac{M}{M+1} \right) \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho] \right. \right. \right. \right. \\
 & \quad \left. \left. \left. \left. \times (g \circ f + g \circ h)^p(t) dt \right)^{\frac{1}{p}} \right) \right] \right. \\
 & \quad \left. + g \left[g^{-1} \left(\left(\frac{1}{m+1} \right) \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho] \right. \right. \right. \right. \right. \\
 & \quad \left. \left. \left. \left. \times (g \circ f + g \circ h)^p(t) dt \right)^{\frac{1}{p}} \right) \right] \right\} \\
 &= g^{-1} \left(\left(\frac{M}{M+1} \right) \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho] \right. \right. \\
 & \quad \left. \left. \times (g \circ f + g \circ h)^p(t) dt \right)^{\frac{1}{p}} \right) \\
 & \quad \oplus g^{-1} \left(\left(\frac{1}{m+1} \right) \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho] \right. \right. \\
 & \quad \left. \left. \times (g \circ f + g \circ h)^p(t) dt \right)^{\frac{1}{p}} \right) \\
 &= g^{-1} \left\{ g \left[g^{-1} \left(\left(\frac{M}{M+1} \right) \right) \right] g \left[g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma \right. \right. \right. \right. \\
 & \quad \left. \left. \left. \left. \times [\omega(\psi(x) - \psi(t))^\rho] (g(g^{-1}((g \circ f + g \circ h)(t)))) \right)^p dt \right)^{\frac{1}{p}} \right] \right\} \\
 & \quad \oplus g \left[g^{-1} \left(\left(\frac{1}{m+1} \right) \right) \right] g \left[g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^\gamma \right. \right. \right. \right. \\
 & \quad \left. \left. \left. \left. \times [\omega(\psi(x) - \psi(t))^\rho] (g(g^{-1}((g \circ f + g \circ h)(t)))) \right)^p dt \right)^{\frac{1}{p}} \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
&= g^{-1} \left(\frac{M}{M+1} \right) \odot g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \\
&\quad \left. \left. \times (g(f \oplus h)(t))^p dt \right)^{\frac{1}{p}} \right) \\
&\oplus g^{-1} \left(\frac{1}{m+1} \right) \odot g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \\
&\quad \left. \left. \times (g(f \oplus h)(t))^p dt \right)^{\frac{1}{p}} \right) \\
&= g^{-1} \left(\frac{M}{M+1} \right) \odot g^{-1} \left(\left(g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} \right. \right. \right. \right. \\
&\quad \left. \left. \times E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] g \left(g^{-1} (g(f \oplus h)(t))^p dt \right) \right) \right)^{\frac{1}{p}} \right) \oplus g^{-1} \left(\frac{1}{m+1} \right) \\
&\odot g^{-1} \left(\left(g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \right. \\
&\quad \left. \left. \times g \left(g^{-1} (g(f \oplus h)(t))^p dt \right) \right) \right)^{\frac{1}{p}} \right) \\
&= g^{-1} \left(\frac{M}{M+1} \right) \odot \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \\
&\quad \left. \left. \times g \left((f \oplus h)_{\odot}^p(t) dt \right) \right)^{\frac{1}{p}} \right) \odot \\
&\oplus g^{-1} \left(\frac{1}{m+1} \right) \odot \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \\
&\quad \left. \left. \times g \left((f \oplus h)_{\odot}^p(t) dt \right) \right)^{\frac{1}{p}} \right) \odot \\
&= \left[g^{-1} \left(\frac{M}{M+1} \right) \oplus g^{-1} \left(\frac{1}{m+1} \right) \right] \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^p(x) \right]_{\odot}^{\frac{1}{p}} \\
&= g^{-1} \left[g \left(g^{-1} \left(\frac{M}{M+1} \right) \right) + g \left(g^{-1} \left(\frac{1}{m+1} \right) \right) \right] \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^p(x) \right]_{\odot}^{\frac{1}{p}} \\
&= g^{-1} \left(\left[\frac{M}{M+1} + \frac{1}{m+1} \right] \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^p(x) \right]_{\odot}^{\frac{1}{p}} \\
&= g^{-1} \left(\left[\frac{1 + M(m+2)}{(M+1)(m+1)} \right] \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^p(x) \right]_{\odot}^{\frac{1}{p}},
\end{aligned}$$

which completes the proof. \square

REMARK 3. Taking $g(x) = x$ and $\gamma = 0$ in (20), we obtain the reverse Mikowski’s inequality for ψ -Riemann-Liouville fractional integral [5]:

$$\left[\mathbb{I}_{a+}^{\alpha; \psi} f^p(x) \right]^{\frac{1}{p}} + \left[\mathbb{I}_{a+}^{\alpha; \psi} h^p(x) \right]^{\frac{1}{p}} \leq \left[\frac{1 + M(m + 2)}{(M + 1)(m + 1)} \right] \left[\mathbb{I}_{a+}^{\alpha; \psi} (f + h)^p(x) \right]^{\frac{1}{p}}. \tag{29}$$

THEOREM 6. Let $\alpha, \gamma, \rho, \omega \in \mathbb{R}$, with $\alpha > 0$ and $\rho > 0$. Also let $p \geq 1$ and ψ be an increasing function on $C^1(\Omega)$ such that $\psi'(t) \neq 0$ for all $t \in \Omega$. Assume that there exist two positive functions f, h are they both measurable functions for $x > a$ satisfying $\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) < \infty$ and $\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) < \infty$. Let a generator $g : \Omega \rightarrow [0, \infty]$ of the pseudo-addition \oplus and the pseudo-multiplication \odot be an increasing function. If $0 < m \leq \frac{f(t)}{h(t)} \leq M, t \in [a, x]$ and $m, M \in \mathbb{R}_+^*$, then we have

$$\begin{aligned} & g^{-1} \left(\frac{(m + 1)(M + 1)}{M} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) \right]^{\frac{1}{p}} \\ & \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) \right]^{\frac{1}{p}} \\ & \leq \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) \right]^{\frac{2}{p}} \oplus \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) \right]^{\frac{2}{p}} \\ & \oplus \left\{ 2 \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) \right]^{\frac{1}{p}} \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) \right]^{\frac{1}{p}} \right\}. \end{aligned} \tag{30}$$

Proof. By multiplication of inequalities (27) and (28), we obtain

$$\begin{aligned} & \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ f)^p(t) dt \right]^{\frac{1}{p}} \\ & \times \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ h)^p(t) dt \right]^{\frac{1}{p}} \\ & \leq \left(\frac{M}{M + 1} \right) \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ f + g \circ h)^p(t) dt \right]^{\frac{1}{p}} \\ & \times \left(\frac{1}{m + 1} \right) \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ f + g \circ h)^p(t) dt \right]^{\frac{1}{p}}. \end{aligned}$$

Since function g is increasing function, then g^{-1} is also increasing function and we have

$$\begin{aligned}
 &g^{-1} \left\{ \left(\frac{(M+1)(m+1)}{M} \right) \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma \right. \right. \\
 &\qquad \qquad \qquad \left. \left. \times [\omega(\psi(x) - \psi(t))^\rho](g \circ f)^P(t) dt \right]^{\frac{1}{p}} \right. \\
 &\qquad \left. \times \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho](g \circ h)^P(t) dt \right]^{\frac{1}{p}} \right\} \quad (31) \\
 &\leq g^{-1} \left(\left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho](g \circ f + g \circ h)^P(t) dt \right)^{\frac{2}{p}} \right).
 \end{aligned}$$

Then the left-hand side in (31):

$$\begin{aligned}
 &g^{-1} \left\{ g \left[g^{-1} \left(\frac{(m+1)(M+1)}{M} \right) \right] g \left[g^{-1} \left(\left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} \right. \right. \right. \right. \\
 &\qquad \qquad \qquad \left. \left. \left. \times E_{\rho,\alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho] g \left(g^{-1}(g \circ f)^P(t) \right) dt \right)^{\frac{1}{p}} \right. \right. \\
 &\qquad \left. \left. \times \left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho] g \left(g^{-1}(g \circ h)^P(t) \right) dt \right)^{\frac{1}{p}} \right) \right] \right\} \\
 &= g^{-1} \left(\frac{(m+1)(M+1)}{M} \right) \odot g^{-1} \left(\left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} \right. \right. \\
 &\qquad \qquad \qquad \left. \left. \times E_{\rho,\alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho] (g \circ f_\odot^P)(t) dt \right)^{\frac{1}{p}} \right. \\
 &\qquad \left. \times \left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho] (g \circ h_\odot^P)(t) dt \right)^{\frac{1}{p}} \right) \\
 &= g^{-1} \left(\frac{(m+1)(M+1)}{M} \right) \odot g^{-1} \left\{ g \left[g^{-1} \left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} \right. \right. \right. \\
 &\qquad \qquad \qquad \left. \left. \left. \times E_{\rho,\alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho] (g \circ f_\odot^P)(t) dt \right)^{\frac{1}{p}} \right] \right. \\
 &\qquad \left. \times g \left[g^{-1} \left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma [\omega(\psi(x) - \psi(t))^\rho] (g \circ h_\odot^P)(t) dt \right)^{\frac{1}{p}} \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
&= g^{-1} \left(\frac{(m+1)(M+1)}{M} \right) \odot g^{-1} \left(\left(g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. \times E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f_{\odot}^{\rho})(t) dt \right) \right) \right) \right)^{\frac{1}{p}} \\
&\quad \odot g^{-1} \left(\left(g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. \times [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h_{\odot}^{\rho})(t) dt \right) \right) \right) \right)^{\frac{1}{p}} \\
&= g^{-1} \left(\frac{(m+1)(M+1)}{M} \right) \\
&\quad \odot \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f_{\odot}^{\rho})(t) dt \right) \right)^{\frac{1}{p}} \\
&\quad \odot \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h_{\odot}^{\rho})(t) dt \right) \right)^{\frac{1}{p}} \\
&= g^{-1} \left(\frac{(m+1)(M+1)}{M} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} f_{\odot}^{\rho}(x) \right]^{\frac{1}{p}} \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} h_{\odot}^{\rho}(x) \right]^{\frac{1}{p}}.
\end{aligned} \tag{32}$$

The right-hand side in (31):

$$\begin{aligned}
&g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \\
&\quad \left. \left. \times \left(g \left(g^{-1} \left((g \circ f + g \circ h)(t) \right) \right) \right)^p dt \right)^{\frac{2}{p}} \right) \\
&= g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \\
&\quad \left. \left. \times g \left(g^{-1} \left(g(f \oplus h)(t) \right)^p dt \right)^{\frac{2}{p}} \right) \right) \\
&= g^{-1} \left(\left(g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. \times g \left((f \oplus h)_{\odot}^{\rho}(t) dt \right) \right) \right) \right)^{\frac{2}{p}} \right)
\end{aligned}$$

$$\begin{aligned}
 &= \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] g \left((f \oplus h)_{\odot}^p(t) \right) dt \right) \right)_{\odot}^{\frac{2}{p}} \\
 &= \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^p(x) \right]_{\odot}^{\frac{2}{p}}. \tag{33}
 \end{aligned}$$

From (32) and (33), we obtain

$$\begin{aligned}
 &g^{-1} \left(\frac{(m+1)(M+1)}{M} \right)_{\odot} \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) \right]_{\odot}^{\frac{1}{p}} \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) \right]_{\odot}^{\frac{1}{p}} \\
 &\leq \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^p(x) \right]_{\odot}^{\frac{2}{p}}. \tag{34}
 \end{aligned}$$

Applying Minkowski’s inequality to the right-hand side of (34), we get

$$\begin{aligned}
 &\left[\left(\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^p(x) \right)_{\odot}^{\frac{1}{p}} \right]_{\odot}^2 \\
 &\leq \left[\left(\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) \right)_{\odot}^{\frac{1}{p}} \oplus \left(\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) \right)_{\odot}^{\frac{1}{p}} \right]_{\odot}^2 \\
 &= \left[\left(\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) \right)_{\odot}^{\frac{1}{p}} \right]_{\odot}^2 \oplus \left[\left(\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) \right)_{\odot}^{\frac{1}{p}} \right]_{\odot}^2 \\
 &\quad \oplus 2 \odot \left(\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) \right)_{\odot}^{\frac{1}{p}} \odot \left(\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) \right)_{\odot}^{\frac{1}{p}}. \tag{35}
 \end{aligned}$$

By (34) and (35) we obtain (30). \square

6. Other related inequalities via ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integral

Now, in this section, we provide others inequalities using ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integral.

THEOREM 7. *Let $\alpha, \gamma, \rho, \omega \in \mathbb{R}$, with $\alpha > 0$ and $\rho > 0$. Also let $p, q \geq 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ and ψ be an increasing function on $C^1(\Omega)$ such that $\psi'(t) \neq 0$ for all $t \in \Omega$. Assume that there exist two positive functions f, h are they both measurable functions for $x > a$ satisfying $\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) < \infty$ and $\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) < \infty$. Let a generator $g : \Omega \rightarrow [0, \infty]$ of the pseudo-addition \oplus and the pseudo-multiplication \odot be an increasing function. If $0 < m \leq \frac{f(t)}{h(t)} \leq M, t \in [a, x]$ and $m, M \in \mathbb{R}_+^*$, then we*

have

$$\begin{aligned} & \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} \left(f_{\odot}^{\frac{1}{p}} \odot h_{\odot}^{\frac{1}{q}} \right) (x) \right] \\ & \geq g^{-1} \left[\left(\frac{m}{M} \right)^{\frac{1}{pq}} \right]_{\odot} \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} f(x) \right]_{\odot}^{\frac{1}{p}} \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} h(x) \right]_{\odot}^{\frac{1}{q}}. \end{aligned} \tag{36}$$

Proof. By the conditions $\frac{f(t)}{h(t)} \leq M$ and $m \leq \frac{f(t)}{h(t)}$, $t \in [a, x]$, $x > a$, we obtain, respectively,

$$f^{\frac{1}{p}}(t) \cdot h^{\frac{1}{q}}(t) \geq M^{-\frac{1}{q}} f(t) \tag{37}$$

and

$$f^{\frac{1}{p}}(t) \cdot h^{\frac{1}{q}}(t) \geq m^{\frac{1}{p}} h(t). \tag{38}$$

Multiplying both sides of (36) and (38) by $\psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}]$ and integrating the results with respect t over (a, x) , we get, respectively,

$$\begin{aligned} & \int_a^t \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] f^{\frac{1}{p}}(t) \cdot h^{\frac{1}{q}}(t) dt \\ & \geq M^{-\frac{1}{q}} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] f(t) dt \end{aligned} \tag{39}$$

and

$$\begin{aligned} & \int_a^t \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] f^{\frac{1}{p}}(t) \cdot h^{\frac{1}{q}}(t) dt \\ & \geq m^{\frac{1}{p}} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] h(t) dt. \end{aligned} \tag{40}$$

Applying the compositions $g \circ f$ and $g \circ h$ to the previous inequalities, we obtain

$$\begin{aligned} & \int_a^t \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)^{\frac{1}{p}}(t) \cdot (g \circ h)^{\frac{1}{q}}(t) dt \\ & \geq M^{-\frac{1}{q}} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)(t) dt \end{aligned} \tag{41}$$

and

$$\begin{aligned} & \int_a^t \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)^{\frac{1}{p}}(t) \cdot (g \circ h)^{\frac{1}{q}}(t) dt \\ & \geq m^{\frac{1}{p}} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h)(t) dt \end{aligned} \tag{42}$$

or, equivalently,

$$\begin{aligned} & \left[\int_a^t \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)^{\frac{1}{p}}(t) \cdot (g \circ h)^{\frac{1}{q}}(t) dt \right]^{\frac{1}{p}} \\ & \geq M^{-\frac{1}{pq}} \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)(t) dt \right]^{\frac{1}{p}} \end{aligned} \tag{43}$$

and

$$\left[\int_a^t \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}](g \circ f)^{\frac{1}{p}}(t) \cdot (g \circ h)^{\frac{1}{q}}(t) dt \right]^{\frac{1}{q}} \quad (44)$$

$$\geq m^{\frac{1}{pq}} \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}](g \circ h)(t) dt \right]^{\frac{1}{q}}.$$

Multiplying the inequalities (43) and (44) and considering the condition $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\left[\int_a^t \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}](g \circ f)^{\frac{1}{p}}(t) \cdot (g \circ h)^{\frac{1}{q}}(t) dt \right]$$

$$\geq \left(\frac{m}{M} \right)^{\frac{1}{pq}} \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}](g \circ f)(t) dt \right]^{\frac{1}{p}}$$

$$\times \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}](g \circ h)(t) dt \right]^{\frac{1}{q}}.$$

Since function g is increasing function, then g^{-1} is also increasing function and we have

$$g^{-1} \left(\int_a^t \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}](g \circ f)^{\frac{1}{p}}(t) \cdot (g \circ h)^{\frac{1}{q}}(t) dt \right)$$

$$\geq g^{-1} \left\{ \left(\frac{m}{M} \right)^{\frac{1}{pq}} \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}](g \circ f)(t) dt \right]^{\frac{1}{p}} \right.$$

$$\left. \times \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}](g \circ h)(t) dt \right]^{\frac{1}{q}} \right\}. \quad (45)$$

For the left side of the inequality (45), we have

$$g^{-1} \left(\int_a^t \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] \right.$$

$$\left. \times g \left\{ g^{-1} \left[g \left(g^{-1} \left((g \circ f)^{\frac{1}{p}}(t) \right) \right) g \left(g^{-1} \left((g \circ h)^{\frac{1}{q}}(t) \right) \right) \right] \right\} dt \right)$$

$$= g^{-1} \left(\int_a^t \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] \right.$$

$$\left. \times g \left[g^{-1} \left((g \circ f)^{\frac{1}{p}}(t) \right) \odot g^{-1} \left((g \circ h)^{\frac{1}{q}}(t) \right) \right] dt \right)$$

$$= g^{-1} \left(\int_a^t \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] g \left(f_{\odot}^{\frac{1}{p}} \odot h_{\odot}^{\frac{1}{q}} \right) (t) dt \right)$$

$$= \mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega, a+}^{\gamma; \psi} \left(f_{\odot}^{\frac{1}{p}} \odot h_{\odot}^{\frac{1}{q}} \right) (x). \quad (46)$$

For the right side of the inequality (45), we get

$$\begin{aligned}
 & g^{-1} \left\{ g \left[g^{-1} \left(\left(\frac{m}{M} \right)^{\frac{1}{pq}} \right) \right] g \left[g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} \right. \right. \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \left. \times [\omega(\psi(x) - \psi(t))^\rho] (g \circ f)(t) dt \right)^{\frac{1}{p}} \right. \right. \\
 & \left. \left. \times \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^\rho] (g \circ h)(t) dt \right)^{\frac{1}{q}} \right) \right] \right\} \\
 & \times g^{-1} \left[\left(\frac{m}{M} \right)^{\frac{1}{pq}} \right] \odot g^{-1} \left\{ g \left[g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} \right. \right. \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \left. \times [\omega(\psi(x) - \psi(t))^\rho] (g \circ f)(t) dt \right)^{\frac{1}{p}} \right) \right. \\
 & \left. \times g \left[g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^\rho] (g \circ h)(t) dt \right)^{\frac{1}{q}} \right) \right] \right\} \\
 & = g^{-1} \left[\left(\frac{m}{M} \right)^{\frac{1}{pq}} \right] \odot g^{-1} \left\{ \left[g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} \right. \right. \right. \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \left. \times [\omega(\psi(x) - \psi(t))^\rho] (g \circ f)(t) dt \right)^{\frac{1}{p}} \right) \right] \right\} \\
 & \odot g^{-1} \left\{ \left[g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} \right. \right. \right. \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \left. \times [\omega(\psi(x) - \psi(t))^\rho] (g \circ h)(t) dt \right)^{\frac{1}{q}} \right) \right] \right\} \\
 & = g^{-1} \left[\left(\frac{m}{M} \right)^{\frac{1}{pq}} \right] \odot \left[g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \times [\omega(\psi(x) - \psi(t))^\rho] (g \circ f)(t) dt \right)^{\frac{1}{p}} \right] \odot \\
 & \left[g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^\rho] (g \circ h)(t) dt \right)^{\frac{1}{q}} \right] \odot \\
 & = g^{-1} \left[\left(\frac{m}{M} \right)^{\frac{1}{pq}} \right] \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} \right]_{\odot}^{\frac{1}{p}} \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h(x) \right]_{\odot}^{\frac{1}{q}}. \tag{47}
 \end{aligned}$$

From (46) and (47), we obtain (36). \square

THEOREM 8. *Let $\alpha, \gamma, \rho, \omega \in \mathbb{R}$, with $\alpha > 0$ and $\rho > 0$. Also let $p, q \geq 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ and ψ be an increasing function on $C^1(\Omega)$ such that $\psi'(t) \neq 0$ for all $t \in \Omega$. Assume that there exist two positive functions f, h are they both measurable functions for $x > a$ satisfying $\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} f(x) < \infty$ and $\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} h(x) < \infty$. Let a generator $g : \Omega \rightarrow [0, \infty]$ of the pseudo-addition \oplus and the pseudo-multiplication \odot be an increasing function. If $0 < m \leq \frac{f(t)}{h(t)} \leq M, t \in [a, x]$ and $m, M \in \mathbb{R}_+^*$, then we have*

$$\begin{aligned} & \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} (f \odot h)(x) \right] \\ & \leq g^{-1} \left[\frac{2^{p-1} M^p}{p(M+1)^p} \right] \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} (f \odot \oplus h \odot)(x) \right] \\ & \quad \oplus g^{-1} \left[\frac{2^{q-1}}{q(m+1)^q} \right] \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} (f \odot \oplus h \odot)(x) \right]. \end{aligned} \tag{48}$$

Proof. From (27), we get

$$\begin{aligned} & \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ f)^p(t) dt \right] \\ & \leq \left(\frac{M}{M+1} \right)^p \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ f + g \circ h)^p(t) dt \right] \end{aligned}$$

or, equivalently,

$$\left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} f \odot (x) \right] \leq g^{-1} \left[\left(\frac{M}{M+1} \right)^p \right] \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} (f \oplus h) \odot (x) \right]. \tag{49}$$

On the other hand, using the condition $m \leq \frac{f(t)}{h(t)}, t \in [a, x], x > a$, we have

$$(m+1)^q h^q(t) \leq (f+h)^q(t). \tag{50}$$

Applying the compositions $g \circ f$ and $g \circ h$ to (50), multiplying both sides of this inequality by $\psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}]$ and integrating the result with respect t over (a, x) , we obtain

$$\begin{aligned} & (m+1)^q \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ h)^q(t) dt \\ & \leq \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ f + g \circ h)^q(t) dt. \end{aligned} \tag{51}$$

Since function g is increasing function, then g^{-1} is also increasing function and we have

$$\begin{aligned}
 & g^{-1} \left\{ \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ h)^q(t) dt \right\} \\
 & \leq g^{-1} \left\{ \frac{1}{(m+1)^q} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ f + g \circ h)^q(t) dt \right\} \\
 & \quad \times g^{-1} \left\{ \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] g \left[g^{-1} \left((g \circ h)^q(t) \right) \right] dt \right\} \\
 & \leq g^{-1} \left\{ g \left[g^{-1} \left(\frac{1}{(m+1)^q} \right) \right] g \left[g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} \right. \right. \right. \\
 & \quad \left. \left. \left. \times E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] \left(g \left(g^{-1} \left((g \circ f + g \circ h)(t) \right) \right) \right)^q dt \right) \right] \right\} \\
 & \quad \times g^{-1} \left\{ \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] \left(g \circ h_{\odot}^q \right) (t) dt \right\} \\
 & \leq g^{-1} \left(\frac{1}{(m+1)^q} \right) \odot g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] \right. \\
 & \quad \left. \times g \left(g^{-1} \left(g(f \oplus h)(t) \right)^q \right) dt \right) \\
 & \quad \times g^{-1} \left\{ \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] \left(g \circ h_{\odot}^q \right) (t) dt \right\} \\
 & \leq g^{-1} \left(\frac{1}{(m+1)^q} \right) \odot g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] \right. \\
 & \quad \left. \times g \left((f \oplus h)_{\odot}^q(t) \right) dt \right) \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^q(x) \right] \\
 & \leq g^{-1} \left(\frac{1}{(m+1)^q} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^q(x) \right]. \tag{52}
 \end{aligned}$$

Considering Young’s inequality, we have

$$f(t)h(t) \leq \frac{f^p(t)}{p} + \frac{h^q(t)}{q}. \tag{53}$$

Applying the compositions $g \circ f$ and $g \circ h$ to (53), multiplying both sides of this inequality by $\psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\alpha, \rho}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}]$ and integrating the result

with respect t over (a, x) , we obtain

$$\begin{aligned} & \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] f(t) h(t) dt \\ & \leq \frac{1}{p} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] f^p(t) dt \\ & \quad + \frac{1}{q} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] h^p(t) dt. \end{aligned}$$

Hence,

$$\begin{aligned} & g^{-1} \left\{ \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)(t)(g \circ h)(t) dt \right\} \\ & = g^{-1} \left\{ \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] \right. \\ & \quad \left. \times g \left(g^{-1} \left((g \circ f)(t)(g \circ h)(t) \right) \right) dt \right\} \\ & = g^{-1} \left\{ \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (f \circ h)(t) dt \right\} \\ & = \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} (f \circ h)(x) \right]. \end{aligned} \tag{54}$$

On the other hand, we have

$$\begin{aligned} & g^{-1} \left\{ \frac{1}{p} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)^p(t) dt \right. \\ & \quad \left. + \frac{1}{q} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h)^p(t) dt \right\} \\ & = g^{-1} \left\{ g \left[g^{-1} \left(\frac{1}{p} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \\ & \quad \left. \left. \times g \left(g^{-1} \left((g \circ f)^p(t) \right) \right) (t) dt \right) \right] \right. \\ & \quad \left. + g \left[g^{-1} \left(\frac{1}{q} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \\ & \quad \left. \left. \times g \left(g^{-1} \left((g \circ h)^q(t) \right) \right) (t) dt \right) \right] \right\} \\ & \times g^{-1} \left\{ \frac{1}{p} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f_{\odot}^p)(t) dt \right\} \end{aligned}$$

$$\begin{aligned} & \oplus g^{-1} \left\{ \frac{1}{q} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] (g \circ h_{\odot}^p)(t) dt \right\} \\ &= g^{-1} \left(\frac{1}{p} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) \right] \oplus g^{-1} \left(\frac{1}{q} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) \right]. \end{aligned} \tag{55}$$

From (54) and (55), we get

$$\begin{aligned} \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \odot h)(x) \right] &\leq g^{-1} \left(\frac{1}{p} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) \right] \oplus g^{-1} \left(\frac{1}{q} \right) \\ &\odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) \right]. \end{aligned} \tag{56}$$

Using the inequalities (49) and (52), the inequality (56) takes the following form

$$\begin{aligned} \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \odot h)(x) \right] &\leq g^{-1} \left(\frac{M^p}{p(M+1)^p} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^p(x) \right] \\ &\oplus g^{-1} \left(\frac{1}{q(m+1)^q} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^q(x) \right]. \end{aligned} \tag{57}$$

By using the inequality $(s+r)^p \leq 2^{p-1}(s^p+r^p)$, $p > 1$ and $r, s \geq 0$, we can write

$$\left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^p(x) \right] \leq g^{-1}(2^{p-1}) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f_{\odot}^p \oplus h_{\odot}^p)(x) \right] \tag{58}$$

and

$$\left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^q(x) \right] \leq g^{-1}(2^{q-1}) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f_{\oplus}^q \oplus h_{\oplus}^q)(x) \right]. \tag{59}$$

Applying the inequalities (58) and (59) in (57), we get

$$\begin{aligned} & \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \odot h)(x) \right] \\ & \leq g^{-1} \left(\frac{M^p}{p(M+1)^p} \right) \odot g^{-1}(2^{p-1}) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f_{\odot}^p \oplus h_{\odot}^p)(x) \right] \\ & \oplus g^{-1} \left(\frac{1}{q(m+1)^q} \right) \odot g^{-1}(2^{q-1}) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f_{\oplus}^q \oplus h_{\oplus}^q)(x) \right] \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \odot h)(x) \right] \\ & \leq g^{-1} \left(\frac{2^{p-1}M^p}{p(M+1)^p} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f_{\odot}^p \oplus h_{\odot}^p)(x) \right] \\ & \oplus g^{-1} \left(\frac{2^{q-1}}{q(m+1)^q} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f_{\oplus}^q \oplus h_{\oplus}^q)(x) \right]. \quad \square \end{aligned} \tag{60}$$

THEOREM 9. *Under the assumptions of Theorem 8 and using the condition $0 < c < m \leq \frac{f(t)}{h(t)} \leq M$, $t \in [a, x]$ and $m, M \in \mathbb{R}_+^*$, we have*

$$\begin{aligned}
 & g^{-1} \left(\frac{M+1}{M-c} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} (f \ominus ch)_{\odot}^p(t) \right]_{\odot}^{\frac{1}{p}} \\
 & \leq \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} f_{\odot}^p(t) \right]_{\odot}^{\frac{1}{p}} \oplus \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} h_{\odot}^p(t) \right]_{\odot}^{\frac{1}{p}} \\
 & \leq g^{-1} \left(\frac{m+1}{m-c} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} (f \ominus ch)_{\odot}^p(t) \right]_{\odot}^{\frac{1}{p}}.
 \end{aligned} \tag{61}$$

Proof. Using the condition $0 < c < m \leq \frac{f(t)}{h(t)} \leq M$, $t \in [a, x]$, $x > a$, we have

$$\frac{(f(t) - ch(t))^p}{(M-c)^p} \leq h^p(t) \leq \frac{(f(t) - ch(t))^p}{(m-c)^p} \tag{62}$$

and

$$\frac{M^p (f(t) - ch(t))^p}{(M-c)^p} \leq f^p(t) \leq \frac{m^p (f(t) - ch(t))^p}{(m-c)^p}. \tag{63}$$

Multiplying by $\psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}]$ both sides of each of the inequalities (62) and (63) and integrating the results with respect t over (a, x) , we get, respectively,

$$\begin{aligned}
 & \left(\frac{1}{M-c} \right) \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (f(t) - ch(t))^p dt \right]^{\frac{1}{p}} \\
 & \leq \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] h^p(t) dt \right]^{\frac{1}{p}} \\
 & \leq \left(\frac{1}{m-c} \right) \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (f(t) - ch(t))^p dt \right]^{\frac{1}{p}}
 \end{aligned} \tag{64}$$

and

$$\begin{aligned}
 & \left(\frac{M}{M-c} \right) \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (f(t) - ch(t))^p dt \right]^{\frac{1}{p}} \\
 & \leq \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] f^p(t) dt \right]^{\frac{1}{p}} \\
 & \leq \left(\frac{m}{m-c} \right) \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (f(t) - ch(t))^p dt \right]^{\frac{1}{p}},
 \end{aligned} \tag{65}$$

Adding the inequalities (64) and (65), we obtain

$$\begin{aligned}
 & \left(\frac{M+1}{M-c} \right) \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (f(t) - ch(t))^p dt \right]^{\frac{1}{p}} \\
 & \leq \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] h^p(t) dt \right]^{\frac{1}{p}} \\
 & \quad + \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] f^p(t) dt \right]^{\frac{1}{p}} \\
 & \leq \left(\frac{m+1}{m-c} \right) \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (f(t) - ch(t))^p dt \right]^{\frac{1}{p}},
 \end{aligned}$$

Since g is increasing function, then g^{-1} is also increasing function and we have

$$\begin{aligned}
 & g^{-1} \left\{ \left(\frac{M+1}{M-c} \right) \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \\
 & \quad \left. \left. \times ((g \circ f)(t) - c(g \circ h)(t))^p dt \right]^{\frac{1}{p}} \right\} \\
 & \leq g^{-1} \left\{ \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h)^p(t) dt \right]^{\frac{1}{p}} \right. \\
 & \quad \left. + \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)^p(t) dt \right]^{\frac{1}{p}} \right\} \\
 & \leq g^{-1} \left\{ \left(\frac{m+1}{m-c} \right) \left[\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \\
 & \quad \left. \left. \times ((g \circ f)(t) - c(g \circ h)(t))^p dt \right]^{\frac{1}{p}} \right\} \tag{66}
 \end{aligned}$$

which yields

$$\begin{aligned}
 & g^{-1} \left\{ g \left[g^{-1} \left(\frac{M+1}{M-c} \right) \right] g \left[g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \right. \\
 & \quad \left. \left. \left. \times \left(g \left(g^{-1} ((g \circ f)(t) - (g \circ ch)(t)) \right) \right)^p dt \right)^{\frac{1}{p}} \right] \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
&\leq g^{-1} \left\{ g \left[g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \\
&\quad \left. \left. \left. \times g \left(g^{-1} ((g \circ h)^{\rho}(t)) \right) dt \right)^{\frac{1}{p}} \right) \right] \right. \\
&\quad \left. + g \left[g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \right. \\
&\quad \left. \left. \left. \times g \left(g^{-1} ((g \circ f)^{\rho}(t)) \right) dt \right)^{\frac{1}{p}} \right) \right] \right\} \\
&\leq g^{-1} \left\{ g \left[g^{-1} \left(\frac{m+1}{m-c} \right) \right] g \left[g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \right. \\
&\quad \left. \left. \left. \times \left(g \left(g^{-1} ((g \circ f)(t) - (g \circ ch)(t)) \right) \right)^{\rho} dt \right)^{\frac{1}{p}} \right) \right] \right\}
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&g^{-1} \left(\frac{M+1}{M-c} \right) \odot g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \\
&\quad \left. \left. \times g \left(g^{-1} ((g(f \ominus ch)(t))^{\rho}) \right) dt \right)^{\frac{1}{p}} \right) \\
&\leq g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h_{\odot}^{\rho})(t) dt \right)^{\frac{1}{p}} \right) \\
&\quad \oplus g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f_{\odot}^{\rho})(t) dt \right)^{\frac{1}{p}} \right) \\
&\leq g^{-1} \left(\frac{m+1}{m-c} \right) \odot g^{-1} \left(\left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \\
&\quad \left. \left. \times g \left(g^{-1} ((g(f \ominus ch)(t))^{\rho}) \right) dt \right)^{\frac{1}{p}} \right).
\end{aligned}$$

The last inequality takes the form

$$\begin{aligned}
&g^{-1} \left(\frac{M+1}{M-c} \right) \odot \left(\left(g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \right. \\
&\quad \left. \left. \left. \times g \left((f \ominus ch)_{\odot}^{\rho}(t) \right) dt \right) \right) \right)^{\frac{1}{p}} \right)
\end{aligned}$$

$$\begin{aligned} &\leq \left(\left(g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h_{\odot}^p)(t) dt \right) \right) \right)^{\frac{1}{p}} \right) \\ &\quad \oplus \left(\left(g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \\ &\quad \quad \quad \left. \left. \left. \times (g \circ f_{\odot}^p)(t) dt \right) \right) \right)^{\frac{1}{p}} \right) \\ &\leq g^{-1} \left(\frac{m+1}{m-c} \right) \odot \left(\left(g \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \right. \right. \\ &\quad \quad \quad \left. \left. \left. \times g \left((f \ominus ch)_{\odot}^p(t) dt \right) \right) \right) \right)^{\frac{1}{p}} \right). \end{aligned}$$

Hence

$$\begin{aligned} &g^{-1} \left(\frac{M+1}{M-c} \right) \odot \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \\ &\quad \quad \quad \left. \left. \times g \left((f \ominus ch)_{\odot}^p(t) dt \right) \right) \right)^{\frac{1}{p}} \\ &\leq \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ h_{\odot}^p)(t) dt \right) \right)^{\frac{1}{p}} \\ &\quad \oplus \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f_{\odot}^p)(t) dt \right) \right)^{\frac{1}{p}} \\ &\leq g^{-1} \left(\frac{m+1}{m-c} \right) \odot \left(g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \\ &\quad \quad \quad \left. \left. \times g \left((f \ominus ch)_{\odot}^p(t) dt \right) \right) \right)^{\frac{1}{p}}. \tag{67} \end{aligned}$$

It follows from (67) that the inequality in (61) holds. \square

THEOREM 10. *Let $\alpha, \gamma, \rho, \omega \in \mathbb{R}$, with $\alpha > 0$ and $\rho > 0$. Also let $p \geq 1$ and ψ be an increasing function on $C^1(\Omega)$ such that $\psi'(t) \neq 0$ for all $t \in \Omega$. Assume that there exist two positive functions f, h are they both measurable functions for $x > a$ satisfying $\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f_{\odot}^p(x) < \infty$ and $\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h_{\odot}^p(x) < \infty$. Let a generator $g : \Omega \rightarrow [0, \infty]$ of the pseudo-addition \oplus and the pseudo-multiplication \odot be an increasing function. If $0 < a \leq f(t) \leq A$ and $0 < b \leq h(t) \leq B$, $t \in [a, x]$ and $a, A, b, B \in \mathbb{R}_+^*$, then the*

following inequality holds

$$\begin{aligned} & \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} f_{\odot}^p(x) \right]^{\frac{1}{p}} \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} h_{\odot}^p(x) \right]^{\frac{1}{p}} \\ & \leq g^{-1} \left[\frac{A(a+B) + B(A+b)}{(A+b)(aB)} \right] \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma, \psi} (f \oplus h)_{\odot}^p(x) \right]^{\frac{1}{p}}. \end{aligned} \tag{68}$$

Proof. By using the given, realizing the product between $a \leq f(t) \leq A$ and $\frac{1}{B} \leq \frac{1}{h(t)} \leq \frac{1}{b}$, we have

$$\frac{a}{B} \leq \frac{f(t)}{h(t)} \leq \frac{A}{b}. \tag{69}$$

From inequality (69), we can write

$$h^p(t) \leq \left(\frac{B}{a+B} \right)^p (f+h)^p(t) \tag{70}$$

and

$$f^p(t) \leq \left(\frac{A}{A+b} \right)^p (f+h)^p(t). \tag{71}$$

Multiplying by $\psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}]$ both sides of each of (70) and (71) and integrating the results with respect t over (a, x) , we obtain respectively

$$\begin{aligned} & \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] h^p(t) dt \\ & \leq \left(\frac{B}{a+B} \right)^p \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (f+h)^p(t) dt \end{aligned}$$

and

$$\begin{aligned} & \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] f^p(t) dt \\ & \leq \left(\frac{A}{A+b} \right)^p \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (f+h)^p(t) dt \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] h^p(t) dt \right] \\ & \leq \left(\frac{B}{a+B} \right) \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (f+h)^p(t) dt \right]^{\frac{1}{p}} \end{aligned} \tag{72}$$

and

$$\left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho] f^p(t) dt \right]^{\frac{1}{p}} \tag{73}$$

$$\leq \left(\frac{A}{A+b} \right) \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho] (f+h)^p(t) dt \right]^{\frac{1}{p}}.$$

Adding the inequalities (72) and (73), we obtain

$$\left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho] f^p(t) dt \right]$$

$$+ \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho] h^p(t) dt \right]$$

$$\leq \left[\frac{A(a+B) + B(A+b)}{(A+b)(a+B)} \right]$$

$$\times \left[\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho,\alpha}^\gamma[\omega(\psi(x) - \psi(t))^\rho] (f+h)^p(t) dt \right]^{\frac{1}{p}}.$$

From here the proof is analogous that of Theorem 5. \square

THEOREM 11. *Let $\alpha, \gamma, \rho, \omega \in \mathbb{R}$, with $\alpha > 0$ and $\rho > 0$. Also let $p \geq 1$ and ψ be an increasing function on $C^1(\Omega)$ such that $\psi'(t) \neq 0$ for all $t \in \Omega$. Assume that there exist two positive functions f, h are they both measurable functions for $x > a$ satisfying $\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} f(x) < \infty$ and $\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} h(x) < \infty$. Let a generator $g : \Omega \rightarrow [0, \infty]$ of the pseudo-addition \oplus and the pseudo-multiplication \odot be an increasing function. If $0 < m \leq f(t) \leq M, t \in [a, x]$ and $m, M \in \mathbb{R}_+^*$, then the following inequality holds*

$$g^{-1} \left(\frac{1}{M} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \odot h)(x) \right]$$

$$\leq g^{-1} \left[\frac{1}{(m+1)(M+1)} \right] \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^2(x) \right] \tag{74}$$

$$\leq g^{-1} \left(\frac{1}{m} \right) \odot \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \odot h)(x) \right].$$

Proof. Using the condition $m \leq \frac{f(t)}{h(t)} \leq M$, we have

$$(m+1)h(t) \leq f(t) + h(t) \leq (M+1)h(t) \tag{75}$$

and

$$\left(\frac{M+1}{M} \right) f(t) \leq f(t) + h(t) \leq \left(\frac{m+1}{m} \right) f(t). \tag{76}$$

Multiplying the inequalities (75) and (76), we can write

$$\frac{f(t)h(t)}{M} \leq \frac{(f+h)^2(t)}{(M+1)(m+1)} \leq \frac{f(t)h(t)}{m}. \quad (77)$$

Now, multiplying both sides of (77) by $\psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}]$ and integrating the result with respect to t on (a, x) , we get

$$\begin{aligned} & \frac{1}{M} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] f(t)h(t) dt \\ & \leq \frac{1}{(M+1)(m+1)} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (f+h)^2(t) dt \\ & \leq \frac{1}{m} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] f(t)h(t) dt. \end{aligned}$$

Since function g is increasing function, then g^{-1} is also increasing function and we have

$$\begin{aligned} & g^{-1} \left[\frac{1}{M} \left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)(t)(g \circ h)(t) dt \right) \right] \\ & \leq g^{-1} \left[\frac{1}{(M+1)(m+1)} \left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] \right. \right. \\ & \quad \left. \left. \times (g \circ f + g \circ h)^2(t) dt \right) \right] \\ & \leq g^{-1} \left[\frac{1}{m} \left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)(t)(g \circ h)(t) dt \right) \right] \end{aligned}$$

or, equivalently,

$$\begin{aligned} & g^{-1} \left(\frac{1}{M} \right) \odot g^{-1} \left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] \right. \\ & \quad \left. \times (g \circ f)(t)(g \circ h)(t) dt \right) \quad (78) \\ & \leq g^{-1} \left(\frac{1}{(M+1)(m+1)} \right) \\ & \quad \odot g^{-1} \left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f + g \circ h)^2(t) dt \right) \\ & \leq g^{-1} \left(\frac{1}{m} \right) \odot g^{-1} \left(\int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma}[\omega(\psi(x) - \psi(t))^{\rho}] \right. \\ & \quad \left. \times (g \circ f)(t)(g \circ h)(t) dt \right). \end{aligned}$$

From the left-hand side inequality of Theorem 3, we have

$$\begin{aligned} & g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f)(t) (g \circ h)(t) dt \right) \\ &= \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \odot h)(x) \right]. \end{aligned} \quad (79)$$

From the right-hand side inequality of Theorem 6, we obtain

$$\begin{aligned} & g^{-1} \left(\int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma} [\omega(\psi(x) - \psi(t))^{\rho}] (g \circ f + g \circ h)^2(t) dt \right) \\ &= \left[\mathbb{E}_{\oplus, \odot, \rho, \alpha, \omega; a+}^{\gamma; \psi} (f \oplus h)_{\odot}^2(x) \right]. \end{aligned} \quad (80)$$

Combining the inequalities (78), (79) and (80), we obtain (74). \square

Concluding remarks

The results obtained generalize some results presented earlier by other authors. In this paper have introduced the generalization of the Hölder and Minkowski pseudo-fractional integral inequalities. In the case of generated pseudo-operations, we recover the g -integral of Pap [18], which can be seen as a generalization of the Lebesgue integral. This type of integrals was shown to be extremely useful in the advanced investigation and applications of nonlinear partial differential equations [31]. We also proved some general version of the reverse Minkowski inequalities and other inequalities via ψ -Riemann-Liouville-Mittag-Leffler pseudo-fractional integral. Our generalizations may be explored for other fractional operators to generalize some inequalities.

For further investigations we propose to consider the following problems:

OPEN PROBLEM. What can be told for the pseudo- ψ -fractional integral inequality type of this manuscript when variational pseudo- ψ -fractional integral [33] are considered?

OPEN PROBLEM. Is there a general version of pseudo- ψ -fractional integral inequality type of Minkowski's inequality when set-valued functions [34] are considered?

REFERENCES

- [1] S. ABBASZADEH, A. EBADIAN AND M. JADDI, *Hölder integral inequalities with different pseudo-operations*, Asian-Eur. J. Math. **12** 3 (2019) 1950032-1–1950032-15.
- [2] H. AGAHI AND M. ALIPOUR, *On pseudo-Mittag-Leffler functions and applications*, Fuzzy Sets Syst. **327** (2017) 21–30.
- [3] H. AGAHI, A. BABAKHANI AND R. MESIAR, *Pseudo-fractional integral inequality of Chebyshev type*, Inf. Sci. **301** (2015) 161–168.
- [4] H. AGAHI, Y. OUYANG, R. MESIAR, E. PAP AND M. ŠTRBOJA, *Hölder and Minkowski type inequalities for pseudo-integral*, Appl. Math. Comput. **217** (2011) 8630–8639.

- [5] T. A. ALJAAIDI AND D. B. PACHPATTE, *The Minkowski's inequalities via ψ -Riemann-Liouville fractional integral operators*, Rend. Circ. Mat. Palermo, II. Ser **70** (2021) 893–906.
- [6] A. BABAKHANI, *Minkowski's inequality for variational fractional integrals*, Kyungpook Math. J. **60** (2020) 289–295.
- [7] A. BABAKHANI, M. YADOLLAHZADEH AND A. NEAMATY, *Some properties of pseudo-fractional operators*, J. Pseudo-Differ. Oper. Appl. **9** (2018) 677–700.
- [8] A. BABAKHANI, H. AGAHI AND R. MESIAR, *A $(*,*)$ -based Minkowski inequality for Sugeno fractional integral of order $\alpha > 0$* , Fract. Calc. Appl. Anal. **18** 4 (2015), 862–874.
- [9] Z. DAHMANI, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal. **1** 1 (2010) 51–58.
- [10] B. DARABY, *Generalization of the Stolarsky type inequality for pseudo-integrals*, Fuzzy Sets Syst. **194** (2012) 90–96.
- [11] M. HOSSEINI, A. BABAKHANI, H. AGAHI AND S. H. RASOULI, *On pseudo-fractional integral inequalities related to Hermite-Hadamard type*, Soft Comp. **20** (2016) 2521–2529.
- [12] A. A. KILBAS, H. M. SRIVASTAVA AND J. TRUJILLO, *Theory and Applications of the Fractional Differential Equations*, vol. **204**, Elsevier Amsterdam (2006).
- [13] W. KUICH, *Semirings, Automata, Languages*, Springer, Berlin (1986).
- [14] D. Q. LI, X. Q. SONG, T. YUE, Y. Z. SONG, *Generalization of the Lyapunov type inequality for pseudo-integrals*, Appl. Math. Comput. **241** (2014) 64–69.
- [15] R. MESIAR AND J. RYBÁRIK, *Pseudo-arithmetical operations*, Tatra Mt. Math. Publ. **2** (1993) 185–192.
- [16] D. S. OLIVEIRA, *Properties of ψ -Mittag-Leffler fractional integrals*, Rend. Circ. Mat. Palermo, II. Ser (2021) 14 pages.
- [17] D. S. OLIVEIRA, *ψ -Mittag-Leffler pseudo-fractional operators*, J. Pseudo-Differ. Oper. Appl. **12** (2021) Article number: 40.
- [18] E. PAP, *g-calculus*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. Ser. Mat. **23** (1993) 145–156.
- [19] E. PAP, *Pseudo-additive measures and their applications*, in: E. Pap (Ed.), Handbook of Measure Theory, 1403–1465, Elsevier, Amsterdam (2002).
- [20] E. PAP AND M. ŠTRBOJA, *Generalization of the Jensen inequality for pseudo-integral*, Infor. Sci. **180** (2010) 543–548.
- [21] E. PAP, M. ŠTRBOJA AND I. RUDAS, *Pseudo- L^p space and convergence*, Fuzzy Sets Syst., **238** (2014) 113–128.
- [22] W. RUDIN, *Real and Complex Analysis*, Third edition, McGraw-Hill, New York (1987).
- [23] S. G. SAMKO, A. A. KILBAS AND O. I. MARICHEV, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Switzerland (1993).
- [24] J. F. TIAN AND M. H. HA, *Extensions of Hölder's inequality via pseudo-integral*, Math. Probl. Eng. **2018** (2018) Article ID 4080619.
- [25] F. USTA, H. BUDAK, F. ERTUGRAL AND M. Z. SARIKAYA, *The Minkowski's inequalities utilizing newly defined generalized fractional integral operators*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68** 1 (2019) 686–701.
- [26] J. VANTERLER DA C. SOUSA AND E. CAPELAS DE OLIVEIRA, *The Minkowski's inequality by means of a generalized fractional integral*, AIMS Math. **3** 1 (2018) 131–147.
- [27] J. VANTERLER DA C. SOUSA, GASTÃO S. F. FREDERICO AND E. CAPELAS DE OLIVEIRA, *ψ -Hilfer pseudo-fractional operator: new results about fractional calculus*, Comput. Appl. Math. **39** (2020) Article number: 254.
- [28] M. YADOLLAHZADEH, A. BABAKHANI AND A. NEAMATY, *Hermite-Hadamard's inequality for pseudo-fractional integral operators*, Stoch. Anal. Appl. **37** 4 (2019) 620–635.
- [29] X. J. YANG, *General Fractional Derivatives: Theory, Methods and Applications*, CRC Press, Boca Raton (2019).
- [30] D. ZHANG AND E. PAP, *Fubini theorem and generalized Minkowski inequality for the pseudo-integral*, Int. J. Approx. Reason. **122** (2020) 9–23.
- [31] E. PAP, *Applications of the generated pseudo-analysis on nonlinear partial differential equations*, Litvinov GL, Maslov VP (eds.) Proceedings of the conference on idempotent mathematics and mathematical physics, contemporary mathematics 377, pp. 239–259, American Mathematical Society, Providence, RI.

- [32] V. P. MASLOV, S. N. SAMBORSKIJ (eds.), *Idempotent analysis*, Advances in soviet mathematics 13, American Mathematical Society, Providence (1992).
- [33] O. P. AGRAWAL, *Generalized Variational Problems and Euler Lagrange equations*, Computers and Mathematics with Applications **59** (2010) 1852–1864.
- [34] M. ŠTRBOJA, T. GRBIĆ, I. ŠTAJNER-PAPUGA, G. GRUJIĆ AND S. MEDIĆ, *Jensen and Chebyshev inequalities for pseudo-integrals of set-valued functions*, Fuzzy Sets Syst. **222** (2013) 18–32.

(Received August 8, 2022)

Daniela S. Oliveira
Universidade Tecnológica Federal do Paraná
85053-525, Guarapuava-PR, Brazil
e-mail: oliveiradaniela@utfpr.edu.br

Azizollah Babakhani
Department of Mathematics, Faculty of Basic Science
Babol Noshirvani University of Technology
Shariati Ave., 47148-71167 Babol, Iran
e-mail: babakhani@nit.ac.ir