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# SOME REFINEMENTS OF HERMITE–HADAMARD INEQUALITY USING $k$–FRACTIONAL CAPUTO DERIVATIVES

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*Abstract.* In this work using $k$-fractional Caputo derivatives we obtain some versions of the Hadamard inequality for the function $f$ such that $f(n)$ is $(h,m)$-convex modified of the second type. Throughout the work, we show that some known results from the literature can be obtained as particular cases of the results presented here.

## 1. Introduction

In Mathematical Sciences, the notion of convex function plays a very prominent role, due to its multiple applications and its theoretical overlaps with various mathematical areas. Readers interested in this notion, can consult [27], where a panorama, practically complete, of these branches is presented.

A function $\psi : I \to \mathbb{R}$, $I := [\nu_1, \nu_2]$ is said to be convex if $\psi\big(\lambda x + (1-\lambda)y\big) \leqslant \lambda\psi(x) + (1-\lambda)\psi(y)$ holds for all $x, y \in I$ and $\lambda \in [0,1]$. If the above inequality is reversed, then the function $\psi$ will be the concave on $[\nu_1, \nu_2]$.

One of the most important inequalities, for convex functions, is the famous Hermite–Hadamard inequality:

$$\psi\left(\frac{\nu_1+\nu_2}{2}\right) \leqslant \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\psi(x)dx \leqslant \frac{\psi(\nu_1)+\psi(\nu_2)}{2} \tag{1}$$

holds for any function $\psi$ convex on the interval $[\nu_1, \nu_2]$. This inequality was published by Hermite ([20]) in 1883 and, independently, by Hadamard in 1893 ([19]). It gives an estimation of the mean value of a convex function, and it is important to note that it also provides a refinement to the Jensen inequality. Several results can be consulted in [1, 2, 3, 4, 15, 17, 18, 24, 28, 34] and references therein for more information and other extensions of the Hermite–Hadamard inequality.

In [3] we presented the following definitions.

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DEFINITION 1. Let $h:[0,1]\to\mathbb{R}$ be a nonnegative function, $h\neq 0$ and $\psi:I=[0,+\infty)\to[0,+\infty)$. If inequality

$$\psi(\tau\xi+m(1-\tau)\varsigma)\leqslant h^{s}(\tau)\psi(\xi)+m(1-h^{s}(\tau))\psi(\varsigma) \tag{2}$$

is fulfilled for all $\xi,\varsigma\in I$ and $\tau\in[0,1]$, where $m\in[0,1]$, $s\in[-1,1]$. Then a function $\psi$ is called a $(h,m)$-convex modified of the first type on $I$.

DEFINITION 2. Let $h:[0,1]\to\mathbb{R}$ nonnegative functions, $h\neq 0$ and $\psi:I=[0,+\infty)\to[0,+\infty)$. If inequality

$$\psi(\tau\xi+m(1-\tau)\varsigma)\leqslant h^{s}(\tau)\psi(\xi)+m(1-h(\tau))^{s}\psi(\varsigma) \tag{3}$$

is fulfilled for all $\xi,\varsigma\in I$ and $\tau\in[0,1]$, where $m\in[0,1]$, $s\in[-1,1]$. Then a function $\psi$ is called a $(h,m)$-convex modified of the second type on $I$.

REMARK 3. From Definitions 1 and 2 we can define $N^{s}_{h,m}[a,b]$, where $a,b\in[0,+\infty)$, as the set of functions $(h,m)$-convex modified, for which $\psi(a)\geqslant 0$, characterized by the triple $(h(\tau),m,s)$. Note that if:

1. $(h(\tau),0,0)$ we have the increasing functions ([7]).
2. $(\tau,0,s)$ we have the $s$-starshaped functions ([7]).
3. $(\tau,0,1)$ we have the starshaped functions ([7]).
4. $(\tau,1,1)$ then $\psi$ is a convex function on $[0,+\infty)$ ([7]).
5. $(1,1,s)$ then $\psi$ is a P-convex function on $[0,+\infty)$ ([11]).
6. $(\tau,m,1)$ then $\psi$ is a $m$-convex function on $[0,+\infty)$ ([35]).
7. $(\tau,1,s)$ $s\in(0,1]$ then $\psi$ is a $s$-convex function on $[0,+\infty)$ ([6, 21]).
8. $(\tau,1,s)$ $s\in[-1,1]$ then $\psi$ is a $s$-convex extended function on $[0,+\infty)$ ([36]).
9. $(\tau,m,s)$ $s\in(0,1]$ then $\psi$ is a $(s,m)$-convex extended function on $[0,+\infty)$ ([30]).
10. $(\tau^{a},1,s)$ with $a\in(0,1]$, then $\psi$ is a $(a,s)$-convex function on $[0,+\infty)$ ([5]).
11. $(\tau^{a},m,1)$ with $a\in(0,1]$, then $\psi$ is a $(a,m)$-convex function on $[0,+\infty)$ ([25]).
12. $(\tau^{a},m,s)$ with $a\in(0,1]$, then $\psi$ is a $s-(a,m)$-convex function on $[0,+\infty)$ ([37]).
13. $(h(\tau),m,1)$ then we have a variant of the $(h,m)$-convex function on $[0,+\infty)$ ([29]).

All through the work we utilize the functions $\Gamma$ (see [31, 32, 38, 39]) and $\Gamma_k$ (see [9]):

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t,\ \ \Re(z)>0, \tag{4}$$

$$\Gamma_k(z)=\int_0^\infty t^{z-1}e^{-t^k/k}\,\mathrm{d}t,\ \ k>0. \tag{5}$$

Unmistakably if $k\to 1$ we have $\Gamma_k(z)\to\Gamma(z)$, $\Gamma_k(z)=(k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k)=z\Gamma_k(z)$.

To encourage comprehension of the subject, we present the definition of Riemann-Liouville fractional integral (with $0\leqslant \nu_1<t<\nu_2\leqslant\infty$). The first is the classic Riemann-Liouville fractional integrals.

DEFINITION 4. Let $\psi\in L_1[\nu_1,\nu_2]$. Then the Riemann-Liouville fractional integrals of order $\alpha\in\mathbb{C}$, $\Re(\alpha)>0$ are defined by (right and left respectively):

$$I^\alpha_{\nu_1+}\psi(x)=\frac{1}{\Gamma(\alpha)}\int_{\nu_1}^x (x-t)^{\alpha-1}\psi(t)\,dt,\ \ x>\nu_1 \tag{6}$$

$$I^\alpha_{\nu_2-}\psi(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\nu_2} (t-x)^{\alpha-1}\psi(t)\,dt,\ \ x<\nu_2. \tag{7}$$

Now, we present the Caputo fractional derivatives that will be used in our work.

DEFINITION 5. Let $\alpha>0$, and $\alpha\neq 1,2,3,\ldots$, $n=[\alpha]+1$, $f\in AC^n[\nu_1,\nu_2]$, the space of functions having the nth derivatives absolutely continuous. The right-sided and left-sided Caputo fractional derivatives of order $\alpha$ are defined as follows:

$$\left({}^C D^\alpha_{\nu_1+}f\right)(x)=\frac{1}{\Gamma(n-\alpha)}\int_{\nu_1}^x \frac{f^{(n)}(t)dt}{(x-t)^{\alpha-n+1}},\ \ x>\nu_1$$

$$\left({}^C D^\alpha_{\nu_2-}f\right)(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^{\nu_2} \frac{f^{(n)}(t)dt}{(t-x)^{\alpha-n+1}},\ \ \nu_2>x.$$

DEFINITION 6. Let $\alpha>0$, and $\alpha\neq 1,2,3,\ldots$, $n=[\alpha]+1$, $f\in AC^n[a,b]$, the space of functions having the nth derivatives absolutely continuous. The right-sided and left-sided Caputo $k$-fractional derivatives of order $\alpha$ are defined as follows:

$$\left({}^C D^{\alpha,k}_{\nu_1+}f\right)(x)=\frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\int_{\nu_1}^x \frac{f^{(n)}(t)dt}{(x-t)^{\alpha-n+1}},\ \ x>\nu_1$$

$$\left({}^C D^{\alpha,k}_{\nu_2-}f\right)(x)=\frac{(-1)^n}{k\Gamma_k(n-\frac{\alpha}{k})}\int_x^{\nu_2} \frac{f^{(n)}(t)dt}{(t-x)^{\alpha-n+1}},\ \ \nu_2>x.$$

Some refinements and extensions of the Hermite-Hadamard Inequality using the Caputo-type fractional derivative can be found in [10, 12, 13, 14, 16, 22, 23, 26, 33, 40].

In this paper, we obtain different variants of the Hermite-Hadamard inequality, in the framework of the $(h,m)$-convex modified functions, via generalized operators of the Definition 6.

## 2. Hermite-Hadamard inequalities for Caputo $k$-fractional derivatives

A version of the Hermite-Hadamard inequality can be presented, using the Caputo $k$-fractional derivatives, as follows:

THEOREM 7. *Let $\psi$ be a positive function such that $\psi \in C^n[\nu_1,\nu_2]$, $\nu_1 < \nu_2$. If $\psi^{(n)}$ is a $(h,m)$-convex modified function of the second type with $m \in (0,1]$ and $0 < \nu_1 < m\nu_2 < +\infty$, then we have the following inequality:*

$$\begin{aligned}
&\psi^{(n)}\left(\frac{\nu_1+\nu_2}{2}\right)\\
&\leqslant \frac{k\Gamma_k(n-\alpha)(r+1)^{n-\alpha}}{2(\nu_2-\nu_1)^{n-\alpha}} {}^{C}D^{\alpha,k}_{\left(\frac{\nu_1+r\nu_2}{r+1}\right)+}\psi(\nu_2)\\
&+(-1)^n\frac{k\Gamma_k(n-\alpha)(r+1)^{n-\alpha}}{2(\nu_2-\nu_1)^{n-\alpha}} {}^{C}D^{\alpha,k}_{\left(\frac{r\nu_1+\nu_2}{r+1}\right)-}\psi(\nu_1)\\
&\leqslant \left(\frac{n-\alpha}{2}\right)\left\{\left[h^s\left(\frac{1}{2}\right)\psi^{(n)}(\nu_1)+\left(1-h\left(\frac{1}{2}\right)\right)^s\psi^{(n)}(\nu_2)\right]\right.\\
&\times\int_0^1 t^{n-\alpha-1}h^s\left(\frac{t}{r+1}\right)dt\\
&+m\left[h^s\left(\frac{1}{2}\right)\psi^{(n)}\left(\frac{\nu_2}{m}\right)+\left(1-h\left(\frac{1}{2}\right)\right)^s\psi^{(n)}\left(\frac{\nu_1}{m}\right)\right]\\
&\left.\times\int_0^1 t^{n-\alpha-1}\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt\right\}.
\end{aligned} \tag{8}$$

*Proof.* For $x,y \in [0,+\infty)$, $t=\frac{1}{2}$ and $m=1$, we have

$$\psi^{(n)}\left(\frac{x+y}{2}\right) \leqslant \psi^{(n)}\left(\frac{x+y}{2}\right) \leqslant h^s\left(\frac{1}{2}\right)\psi^{(n)}(x)+\left(1-h\left(\frac{1}{2}\right)\right)^s\psi^{(n)}(y),$$

If we choose $x=\frac{t}{r+1}\nu_1+(\frac{r+1-t}{r+1})\nu_2$ and $y=\frac{t}{r+1}\nu_2+(\frac{r+1-t}{r+1})\nu_1$, with $t\in[0,1]$, we get

$$\begin{aligned}
&\psi^{(n)}\left(\frac{\nu_1+\nu_2}{2}\right)\\
&\leqslant h^s\left(\frac{1}{2}\right)\psi^{(n)}\left(\frac{t}{r+1}\nu_1+\left(\frac{r+1-t}{r+1}\right)\nu_2\right)\\
&\quad+\left(1-h\left(\frac{1}{2}\right)\right)^s\psi^{(n)}\left(\frac{t}{r+1}\nu_2+\left(\frac{r+1-t}{r+1}\right)\nu_1\right).
\end{aligned} \tag{9}$$

Multiplying both members of the previous inequality by $t^{n-\frac{\alpha}{k}-1}$, integrating with respect to $t$ from $0$ to $1$, and changing variables we obtain the first inequality of (8).

From right member of (9) we obtain

$$
\begin{aligned}
& h^s\left(\frac{1}{2}\right)\psi^{(n)}\left(\frac{t}{r+1}\nu_1+\left(\frac{r+1-t}{r+1}\right)\nu_2\right)\\
& +\left(1-h\left(\frac{1}{2}\right)\right)^s\psi^{(n)}\left(\frac{t}{r+1}\nu_2+\left(\frac{r+1-t}{r+1}\right)\nu_1\right)\\
= & h^s\left(\frac{1}{2}\right)\psi^{(n)}\left(\frac{t}{r+1}\nu_1+m\left(\frac{r+1-t}{r+1}\right)\frac{\nu_2}{m}\right)\\
& +\left(1-h\left(\frac{1}{2}\right)\right)^s\psi^{(n)}\left(\frac{t}{r+1}\nu_2+m\left(\frac{r+1-t}{r+1}\right)\frac{\nu_1}{m}\right)\\
\leqslant & h^s\left(\frac{1}{2}\right)\left[\psi^{(n)}(\nu_1)h^s\left(\frac{t}{r+1}\right)+m\psi^{(n)}\left(\frac{\nu_2}{m}\right)\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s\right]\\
& +\left(1-h\left(\frac{1}{2}\right)\right)^s\left[\psi^{(n)}(\nu_2)h^s\left(\frac{t}{r+1}\right)+m\psi^{(n)}\left(\frac{\nu_1}{m}\right)\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s\right].
\end{aligned}
$$

Multiplying this by $t^{n-\frac{\alpha}{k}-1}$, integrating with respect to $t$, between 0 and 1 and with a simple but tedious algebraic work, we obtain the right member of (8). In this way the proof is completed. □

REMARK 8. If in the previous Theorem we make $r=0$ and consider functions $(h,m)$-convex, that is, $s=1$, we obtain Theorem 2.1 of [26]. If we put $r=1$, then they are particular cases of the previous Theorem, Theorem 6 of [12], Theorem 4 of [13] with $k=1$ and Theorem 2.2 of [23], all for convex functions, that is, $h(t)=t$, $m=s=1$.

REMARK 9. We must point out that the inequality obtained in Theorem 7 is not contradictory with the one presented in Theorem 2.2 of [1] obtained for $(h-k)$-convex functions.

The following result, although it transcends the frames of the Caputo fractional $k$-derivatives, since it is stated in a general way, it will be used later.

LEMMA 10. *Let $\psi$ be a real function defined on some interval $[\nu_1,\nu_2]\subset\mathbb{R}$, differentiable on $(\nu_1,\nu_2)$. If $\psi'\in L_1(\nu_1,\nu_2)$, and $w(t)$ is a differentiable function on $[\nu_1,\nu_2]$, then we have the following equality:*

$$
\begin{aligned}
& \left\{-w(1)\left(\psi^{(n)}\left(\frac{\nu_1+r\nu_2}{r+1}\right)+\psi^{(n)}(\frac{r\nu_1+\nu_2}{r+1})\right)+w(0)\left(\psi^{(n)}(\nu_1)+\psi^{(n)}(\nu_2)\right)\right\}\\
& +\frac{r+1}{\nu_2-\nu_1}\left(\int_{\nu_1}^{\frac{r\nu_1+\nu_2}{r+1}}w'\left[\frac{u-\nu_1}{\frac{\nu_2-\nu_1}{r+1}}\right]\psi^{(n)}(u)du+\int_{\frac{\nu_1+r\nu_2}{r+1}}^{\nu_2}w'\left[\frac{\nu_2-u}{\frac{\nu_2-\nu_1}{r+1}}\right]\psi^{(n)}(u)du\right)\\
= & \frac{\nu_2-\nu_1}{r+1}\int_0^1 w(t)\left[\psi^{(n+1)}\left(\frac{t}{r+1}\nu_1+\frac{r+1-t}{r+1}\nu_2\right)\right.\\
& \left.-\psi^{(n+1)}\left(\frac{t}{r+1}\nu_2+\frac{r+1-t}{r+1}\nu_1\right)\right]dt. \qquad (10)
\end{aligned}
$$

*Proof.* First note that

$$\int_0^1 w(t)\left[\psi^{(n+1)}\left(\frac{t}{r+1}\nu_1+\frac{r+1-t}{r+1}\nu_2\right)-\psi^{(n+1)}\left(\frac{t}{r+1}\nu_2+\frac{r+1-t}{r+1}\nu_1\right)\right]dt$$
$$=\int_0^1 w(t)\psi^{(n+1)}\left(\frac{t}{r+1}\nu_1+\frac{r+1-t}{r+1}\nu_2\right)dt$$
$$quad-\int_0^1 \psi^{(n+1)}\left(\frac{t}{r+1}\nu_2+\frac{r+1-t}{r+1}\nu_1\right)dt$$
$$=I_1-I_2.$$

Integrating by parts, we have

$$I_1=\frac{r+1}{\nu_2-\nu_1}\left[-w(1)\psi^{(n)}\left(\frac{\nu_1+r\nu_2}{r+1}\right)+w(0)\psi^{(n)}(\nu_2)\right]$$
$$+\frac{(r+1)^2}{(\nu_2-\nu_1)^2}\int_{\nu_1}^{\frac{r\nu_1+\nu_2}{r+1}} w'\left[\frac{u-\nu_1}{\frac{\nu_2-\nu_1}{r+1}}\right]\psi^{(n)}(u)du,$$

since

$$\int_0^1 w'(t)\psi^{(n)}\left(\frac{t}{r+1}\nu_1+\frac{r+1-t}{r+1}\nu_2\right)dt$$
$$=\frac{n+1}{\nu_2-\nu_1}\int_{\nu_1}^{\frac{r\nu_1+\nu_2}{r+1}} w'\left[\frac{u-\nu_1}{\frac{\nu_2-\nu_1}{r+1}}\right]\psi^{(n)}(u)du.$$

Analogously

$$I_2=\frac{r+1}{\nu_2-\nu_1}\left[w(1)\psi^{(n)}\left(\frac{r\nu_1+\nu_2}{r+1}\right)-w(0)\psi^{(n)}(\nu_1)\right]$$
$$-\frac{(r+1)^2}{(\nu_2-\nu_1)^2}\int_{\frac{\nu_1+r\nu_2}{r+1}}^{\nu_2} w'\left[\frac{\nu_2-u}{\frac{\nu_2-\nu_1}{r+1}}\right]\psi^{(n)}(u)du.$$

From $I_1-I_2$, and grouping appropriately, we have the required inequality. □

From the Lemma 10 we have the following result.

LEMMA 11. *Let $\psi$ be a real positive function defined on some interval $[\nu_1,\nu_2]\subset\mathbb{R}$, such that $\psi^{(n+1)}\in L_1(\nu_1,m\nu_2)$, then we have the following equality:*

$$-\left(\psi^{(n)}\left(\frac{\nu_1+r\nu_2}{r+1}\right)+\psi^{(n)}\left(\frac{r\nu_1+\nu_2}{r+1}\right)\right) \tag{11}$$
$$+\frac{(r+1)^{n-\frac{\alpha}{k}}k\Gamma_k(n-\frac{\alpha}{k}+1)}{(\nu_2-\nu_1)^{n-\frac{\alpha}{k}}}\left({}^{C}D^{\alpha,k}_{\left(\frac{r\nu_1+\nu_2}{r+1}\right)-}\psi(\nu_1)+(-1)^{n}{}^{C}D^{\alpha,k}_{\left(\frac{r\nu_1+\nu_2}{r+1}\right)+}\psi(\nu_2)\right)$$
$$=\frac{\nu_2-\nu_1}{r+1}\int_0^1 t^{n-\frac{\alpha}{k}}\left[\psi^{(n+1)}\left(\frac{t}{r+1}\nu_1+\frac{r+1-t}{r+1}\nu_2\right)\right.$$
$$\left.-\psi^{(n+1)}\left(\frac{t}{r+1}\nu_2+\frac{r+1-t}{r+1}\nu_1\right)\right]dt.$$

*Proof.* It is enough, put $w(t) = t^{n-\frac{\alpha}{k}}$ in (10) and taking into account

$$\frac{(r+1)^{n-\frac{\alpha}{k}}(n-\frac{\alpha}{k})}{(\nu_2-\nu_1)^{n-\frac{\alpha}{k}}}\left(\int_{\nu_1}^{\frac{r\nu_1+\nu_2}{r+1}}(u-\nu_1)^{n-\frac{\alpha}{k}-1}\psi^{(n)}(u)du\right.$$
$$\left.+\int_{\frac{\nu_1+r\nu_2}{r+1}}^{\nu_2}(\nu_2-u)^{n-\frac{\alpha}{k}-1}\psi^{(n)}(u)du\right)$$
$$=\frac{(r+1)^{n-\frac{\alpha}{k}}k\Gamma_k(n-\frac{\alpha}{k}+1)}{(\nu_2-\nu_1)^{n-\frac{\alpha}{k}}}\left({}^{C}D^{\alpha,k}_{\left(\frac{r\nu_1+\nu_2}{r+1}\right)-}\psi(\nu_1)+(-1)^{nC}D^{\alpha,k}_{\left(\frac{r\nu_1+\nu_2}{r+1}\right)+}\psi(\nu_2)\right). \quad \square$$

REMARK 12. It is easy to see that the Lemma 2.1 of [8] can be obtained from the previous result, putting $r=0$, considering convex functions and using only the second integral relative to function $\psi^{(n+1)}(tx+(1-t)y)$, with $\nu_1 \leqslant x < y \leqslant \nu_2$.

REMARK 13. Lemma 3.1 of [26], with $r=0$, for functions $(h,m)$-convex; Lemma 1 of [12], Lemma 2 of [13] and Lemma 3.1 of [23], with $r=1$ and $h(t)=t$, $m=s=1$, they can be derived from the previous Lemma, as particular cases.

Our first main result is the following.

THEOREM 14. *Let $\psi$ be a real positive function defined on some interval $[\nu_1,\nu_2]\subset\mathbb{R}$, such that $\psi^{(n+1)}\in L_1(\nu_1,m\nu_2)$, if $\left|\psi^{(n+1)}\right|$, is modified $(h,m)$-convex of the second type on $\left[\nu_1,\frac{\nu_2}{m}\right]$, we have the following inequality:*

$$\left|-\left(\psi^{(n)}\left(\frac{\nu_1+r\nu_2}{r+1}\right)+\psi^{(n)}\left(\frac{r\nu_1+\nu_2}{r+1}\right)\right)\right. \tag{12}$$
$$\left.+\mathbb{A}\left({}^{C}D^{\alpha,k}_{\left(\frac{r\nu_1+\nu_2}{r+1}\right)-}\psi(\nu_1)+(-1)^{nC}D^{\alpha,k}_{\left(\frac{r\nu_1+\nu_2}{r+1}\right)+}\psi(\nu_2)\right)\right|$$
$$\leqslant\frac{\nu_2-\nu_1}{r+1}\left\{\left(\left|\psi^{(n+1)}(\nu_1)\right|+\left|\psi^{(n+1)}(\nu_2)\right|\right)\mathbb{B}+m\left(\left|\psi^{(n+1)}\left(\frac{\nu_1}{m}\right)\right|+\left|\psi^{(n+1)}\left(\frac{\nu_2}{m}\right)\right|\right)\mathbb{C}\right\}$$

*with* $\mathbb{A}=\frac{(r+1)^{n-\frac{\alpha}{k}}k\Gamma_k(n-\frac{\alpha}{k}+1)}{(\nu_2-\nu_1)^{n-\frac{\alpha}{k}}}$, $\mathbb{B}=\int_0^1 t^{n-\frac{\alpha}{k}}h^s\left(\frac{t}{r+1}\right)dt$, *and* $\mathbb{C}=\int_0^1 t^{n-\frac{\alpha}{k}}\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt$.

*Proof.* From Lemma 11 we obtain

$$\left|\int_0^1 t^{n-\frac{\alpha}{k}}\left[\psi^{(n+1)}\left(\frac{t}{r+1}\nu_1+\frac{r+1-t}{r+1}\nu_2\right)-\psi^{(n+1)}\left(\frac{t}{r+1}\nu_2+\frac{r+1-t}{r+1}\nu_1\right)\right]dt\right|$$
$$\leqslant\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}\nu_1+\frac{r+1-t}{r+1}\nu_2\right)\right|dt$$
$$+\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}\nu_2+\frac{r+1-t}{r+1}\nu_1\right)\right|dt.$$

Using the modified $(h,m)$-convexity of $\left|\psi^{(n+1)}\right|$, we get

$$\begin{aligned}
&\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}\nu_1+\frac{r+1-t}{r+1}\nu_2\right)\right|dt\\
&\leqslant \int_0^1 t^{n-\frac{\alpha}{k}}\left[h^s\left(\frac{t}{r+1}\right)\left|\psi^{(n+1)}(\nu_1)\right|+m\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s\left|\psi^{(n+1)}\left(\frac{\nu_2}{m}\right)\right|\right]dt\\
&=\left|\psi^{(n+1)}(\nu_1)\right|\int_0^1 t^{n-\frac{\alpha}{k}}h^s\left(\frac{t}{r+1}\right)dt+m\left|\psi'\left(\frac{\nu_2}{m}\right)\right|\int_0^1 t^{n-\frac{\alpha}{k}}\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt.
\end{aligned} \tag{13}$$

In the same way

$$\begin{aligned}
&\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}\nu_2+\frac{r+1-t}{r+1}\nu_1\right)\right|dt\\
&\leqslant\left|\psi^{(n+1)}(\nu_2)\right|\int_0^1 t^{n-\frac{\alpha}{k}}h^s\left(\frac{t}{r+1}\right)dt+m\left|\psi'\left(\frac{\nu_1}{m}\right)\right|\int_0^1 t^{n-\frac{\alpha}{k}}\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt.
\end{aligned} \tag{14}$$

From (13) and (14) we easily obtain (12). In this way the theorem is proved. □

REMARK 15. Considering $r=0$, it is easy to check that the first part of Theorem 2.7 of [26] for functions $(h,m)$-convex; with $r=1$ the Theorem 7 (with $q=1$) of [12] for convex functions and Theorem 5 of [13], Theorem 3.2 of [23] (the case $q=1$ of both for convex functions), they are all particular cases of the previous theorem.

The above result can be improved, if we impose additional conditions on $\left|\psi^{(n+1)}\right|^q$.

THEOREM 16. *Let $\psi$ be a real positive function defined on some interval $[\nu_1,\nu_2]$ $\subset\mathbb{R}$, such that $\psi^{(n+1)}\in L_1(\nu_1,m\nu_2)$, if $\left|\psi^{(n+1)}\right|^q$ is modified $(h,m)$-convex of the second type on $\left[\nu_1,\frac{\nu_2}{m}\right]$, we have the following inequality:*

$$\begin{aligned}
&\Bigg|-\left(\psi^{(n)}\left(\frac{\nu_1+r\nu_2}{r+1}\right)+\psi^{(n)}\left(\frac{r\nu_1+\nu_2}{r+1}\right)\right)\\
&+\mathbb{A}\left({}^{C}D^{\alpha,k}_{\left(\frac{r\nu_1+\nu_2}{r+1}\right)-}\psi(\nu_1)+(-1)^{n}\,{}^{C}D^{\alpha,k}_{\left(\frac{r\nu_1+\nu_2}{r+1}\right)+}\psi(\nu_2)\right)\Bigg|\\
&\leqslant\frac{(\nu_2-\nu_1)B_q}{r+1}\left\{(p_1C_{11}+mp_{12}C_{12})^{\frac{1}{q}}+(p_2C_{11}+mp_{21}C_{12})^{\frac{1}{q}}\right\}
\end{aligned} \tag{15}$$

*with $\mathbb{A}$ as before, $B_p=\frac{1}{\left(p\left(n-\frac{\alpha}{k}\right)+1\right)^{\frac{1}{p}}}$, $p_1=\left|\psi^{(n+1)}(\nu_1)\right|^q$, $p_{12}=\left|\psi^{(n+1)}\left(\frac{\nu_2}{m}\right)\right|^q$, $p_2=\left|\psi^{(n+1)}(\nu_2)\right|^q$, $p_{21}=\left|\psi^{(n+1)}\left(\frac{\nu_1}{m}\right)\right|^q$, $C_{11}=\int_0^1 h^s\left(\frac{t}{r+1}\right)dt$, and $C_{12}=\int_0^1\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt$.*

*Proof.* As previous result, from Lemma 11 we obtain

$$\left|\int_0^1 t^{n-\frac{\alpha}{k}}\left[\psi^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)-\psi^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right]dt\right|$$
$$\leqslant \int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)\right|dt$$
$$+\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right|dt.$$

From Hölder's inequality, we obtain

$$\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)\right|dt$$
$$\leqslant \left(\int_0^1 t^{p(n-\frac{\alpha}{k})}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)\right|^q dt\right)^{\frac{1}{q}} \tag{16}$$

and

$$\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right|dt$$
$$\leqslant \left(\int_0^1 t^{p(n-\frac{\alpha}{k})}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right|^q dt\right)^{\frac{1}{q}} \tag{17}$$

for $\frac{1}{p}+\frac{1}{q}=1$. Using the $(h,m)$-convexity of the second type of $\left|\psi^{(n+1)}\right|^q$, we obtain from (16) and (17):

$$\int_0^1\left|\psi^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)\right|^q dt$$
$$\leqslant \left|\psi^{(n+1)}(v_1)\right|^q\int_0^1 h^s\left(\frac{t}{r+1}\right)dt+m\left|\psi^{(n+1)}\left(\frac{v_2}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt, \tag{18}$$

$$\int_0^1\left|\psi^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right|^q dt$$
$$\leqslant \left|\psi^{(n+1)}(v_2)\right|^q\int_0^1 h^s\left(\frac{t}{r+1}\right)dt+m\left|\psi^{(n+1)}\left(\frac{v_1}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt. \tag{19}$$

Denoting, for brevity $B_p=\left(\int_0^1 t^{p(n-\frac{\alpha}{k})}\,dt\right)^{\frac{1}{p}}=\frac{1}{\left(p(n-\frac{\alpha}{k})+1\right)^{\frac{1}{p}}}$, substituting (18), (19) in (16) and (17), we obtain the required inequality. □

REMARK 17. Theorem 8 $(q>1)$ of [12], Theorem 6 of [13], the second part of Theorem 2.7 of [26] and Theorem 3.2 $(q>1)$ of [23] can be derived from the previous result, for different values of $r$ and different notions of convexity.

THEOREM 18. *Let $\psi$ be a real positive function defined on some interval $[\nu_1,\nu_2]\subset\mathbb{R}$, such that $\psi^{(n+1)}\in L_1(\nu_1,m\nu_2)$, if $\left|\psi^{(n+1)}\right|^q$, $q>1$, is modified $(h,m)$-convex of the second type on $\left[\nu_1,\frac{\nu_2}{m}\right]$, we have the following inequality:*

$$\begin{aligned}&\Bigg|-\left(\psi^{(n)}\left(\frac{\nu_1+r\nu_2}{r+1}\right)+\psi^{(n)}\left(\frac{r\nu_1+\nu_2}{r+1}\right)\right)\\&+\mathbb{A}\left({}^{C}D^{\alpha,k}_{\left(\frac{r\nu_1+\nu_2}{r+1}\right)-}\psi(\nu_1)+(-1)^{n\,C}D^{\alpha,k}_{\left(\frac{r\nu_1+\nu_2}{r+1}\right)+}\psi(\nu_2)\right)\Bigg|\\ \leqslant&\frac{(\nu_2-\nu_1)B_q}{r+1}\left\{(p_1C_1+mp_{12}C_2)^{\frac{1}{q}}+(p_2C_1+mp_{21}C_2)^{\frac{1}{q}}\right\}\end{aligned}\tag{20}$$

*with $\mathbb{A}$, $p_1$, $p_2$, $p_{12}$ and $p_{21}$ as before, $B_q=\frac{1}{\left(n-\frac{\alpha}{k}+1\right)^{1-\frac{1}{q}}}$, $C_1=\int_0^1 t^{n-\frac{\alpha}{k}}h^s\left(\frac{t}{r+1}\right)dt$, and $C_2=\int_0^1 t^{n-\frac{\alpha}{k}}\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt$.*

*Proof.* As before, from the Lemma 11 we have:

$$\begin{aligned}&\Bigg|\int_0^1 t^{n-\frac{\alpha}{k}}\Bigg[\psi^{(n+1)}\left(\frac{t}{r+1}\nu_1+\frac{r+1-t}{r+1}\nu_2\right)\\&-\psi^{(n+1)}\left(\frac{t}{r+1}\nu_2+\frac{r+1-t}{r+1}\nu_1\right)\Bigg]dt\Bigg|\\ \leqslant&\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}\nu_1+\frac{r+1-t}{r+1}\nu_2\right)\right|dt\\&+\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}\nu_2+\frac{r+1-t}{r+1}\nu_1\right)\right|dt.\end{aligned}$$

and using well known power mean inequality, we have

$$\begin{aligned}&\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}\nu_1+\frac{r+1-t}{r+1}\nu_2\right)\right|dtdt\\ \leqslant&\left(\int_0^1 t^{n-\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}\nu_1+\frac{r+1-t}{r+1}\nu_2\right)\right|^q dt\right)^{\frac{1}{q}}\end{aligned}\tag{21}$$

and

$$\begin{aligned}&\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}\nu_2+\frac{r+1-t}{r+1}\nu_1\right)\right|dt\\ \leqslant&\left(\int_0^1 t^{n-\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}\nu_2+\frac{r+1-t}{r+1}\nu_1\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}\tag{22}$$

Using the modified $(h,m)$-convexity of $\left|\psi^{(n+1)}\right|^q$, we get

$$\begin{aligned}
&\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)\right|^q dt\\
&\leqslant \int_0^1 t^{n-\frac{\alpha}{k}}\left[h^s\left(\frac{t}{n+1}\right)\left|\psi^{(r+1)}(v_1)\right|^q+m\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s\left|\psi^{(n+1)}\left(\frac{v_2}{m}\right)\right|^q\right]dt\\
&=\left|\psi^{(n+1)}(v_1)\right|^q\int_0^1 t^{n-\frac{\alpha}{k}}h^s\left(\frac{t}{r+1}\right)dt\\
&\quad+m\left|\psi^{(n+1)}\left(\frac{v_2}{m}\right)\right|^q\int_0^1 t^{n-\frac{\alpha}{k}}\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt. \qquad (23)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\int_0^1 t^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right|^q dt\\
&\leqslant \left|\psi^{(n+1)}(v_2)\right|^q\int_0^1 t^{n-\frac{\alpha}{k}}h^s\left(\frac{t}{r+1}\right)dt\\
&\quad+m\left|\psi^{(n+1)}\left(\frac{v_1}{m}\right)\right|^q\int_0^1 t^{n-\frac{\alpha}{k}}\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt. \qquad (24)
\end{aligned}$$

If we put (23) and (24), in (21) and in (22), it allows us to obtain the inequality (20). In this way the proof is completed. □

REMARK 19. This last result covers Theorem 7 $(q>1)$ of [12], Theorem 5 from [13] $(q>1)$ and Theorem 3.3 from [23], for different values of $r$ and different definitions of convexity.

## 3. Conclusions

In this paper, we have obtained different variants of the well-known Hermite-Hadamard Inequality in the framework of the Caputo fractional derivative, the generality of our results has been demonstrated by showing that they contain as particular cases, several known from the literature.

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