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OPTIMAL (ω,c)–ASYMPTOTICALLY PERIODIC MILD SOLUTIONS TO SOME FRACTIONAL EVOLUTION EQUATIONS

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Abstract. In this article, we establish some new properties of the two-parameter Mittag-Leffler function and use them to prove that, mild solutions of the evolution equation ${}^{C}D_t^{\alpha}u(t) = Au(t) + f(t)$ $(t \in \mathbb{R})$ are (ω,c)-asymptotically periodic, where A is the generator of a strongly continuous semigroup $\{T(\theta)\}_{\theta \geqslant 0}$ (which is exponentially stable) on a Banach space $\mathbb{X}$ and ${}^{C}D_t^{\alpha}$ denotes the Caputo fractional derivative of order $0 < \alpha \leqslant 1$. We further establish an existence and uniqueness result for optimal (ω,c)-asymptotically periodic mild solution if $\mathbb{X}$ is a uniformly convex Banach space.

1. Introduction

Let A be the generator of a strongly continuous semigroup $\{T(\theta)\}_{\theta\geqslant 0}$ on a reflexive Banach space $\mathbb{X}$ equipped with a norm $\|\cdot\|_{\mathbb{X}}$ and such that there exist positive constants $M,b \geqslant 0$ with

$$\|T(\theta)\|_{\mathbb{X}} \leqslant Me^{-b\theta}, \quad \theta \geqslant 0; \tag{1}$$

Let $f:\mathbb{R} \longrightarrow \mathbb{X}$ be a strongly continuous function. We consider the following fractional evolution equation:

$${}^{C}D_t^{\alpha}u(t) = Au(t) + f(t), \quad t \in \mathbb{R} \tag{2}$$

where ${}^{C}D_t^{\alpha}$ denotes the Caputo fractional derivative of order $0 < \alpha \leqslant 1$.

In 2018, E. Alvarez et al. [4] introduced the concept of (ω,c)-periodicity. It all started with the fact that, according to the Floquet's theorem, the Mathieu's equation $y'' + ay - 2qy\cos(2t) = 0$ admits solutions of the form $y(t) = e^{\mu t}p(t)$, $t \in \mathbb{R}$ where $\mu \in \mathbb{C}$ and $p:\mathbb{R} \longrightarrow \mathbb{C}$ satisfies $p(t+\omega) = p(t)$ for $\omega > 0$. It follows that $y(t+\omega) = cy(t)$, $t \in \mathbb{R}$ where $c = e^{\mu\omega}$. From this observation, the authors defined for the first time an (ω,c)-periodic function as any function f for which, for some strictly positive number ω, there exists a non-zero complex number c such that $f(t+\omega) = cf(t)$ for all $t \in \mathbb{R}$. This appears to be a generalization of ω-periodic functions (for $c = 1$). When $c = -1$, we have the so-called ω-anti-periodic functions.

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One year later in 2019, E. Alvarez et al. [3] extended the latter concept to a new class of functions called (ω,c)-asymptotically periodic functions, including asymptotically periodic, asymptotically anti-periodic, asymptotically Bloch-periodic, and unbounded functions. The definition of an (ω,c)-asymptotically periodic function is provided in the preliminary section of this work. The same year, E. Alvarez et al. [2] also introduced the so-called (ω,c)-pseudo periodic functions including pseudo periodic, pseudo anti-periodic, and pseudo Bloch-periodic; with some applications on the first order abstract Cauchy problem and the Lasota-Wazewska equation. Since then, several results have been published regarding the (ω,c)-periodicity (see [1, 12, 14, 16, 17, 18, 24, 32]).

More than two decades earlier, S. Zaidman [35] introduced in 1994 the concept of optimal mild solutions dealing with the so-called min-max Amerio method in some problems concerning bounded (on the real line) solutions of the differential equation

$$u'(t) = Au(t) + f(t), \quad t \in \mathbb{R} \tag{3}$$

in Banach spaces $(X, \|\cdot\|)$, where A is a bounded linear operator. The author defined an optimal mild solution as follows: A mild solution $\tilde{u}(t)$ of (3) is said to be optimal if it minimizes the functional $\mu(u) = \sup\limits_{t\in\mathbb{R}} \|u(t)\|_{\mathbb{X}}$ over the collection Ω_f of such solutions which are bounded on $\mathbb{R}$; that is,

$$\mu(\tilde{u}) \equiv \mu^* = \inf_{u\in\Omega_f} \mu(u).$$

In 2005, G. M. N'Guérékata [29] studied the existence and uniqueness of the optimal mild solution $u(t)$ of the equation (3) and proved that $u(t)$ is weakly almost periodic, where $(X, \|\cdot\|)$ is a uniformly convex Banach space, $A : D(A) \subseteq X \longrightarrow X$ a linear operator that generates a C_0-semigroup of uniformly bounded linear operator, and $f : \mathbb{R} \longrightarrow X$ is a nontrivial strongly continuous function.

A few years later in 2009, A. Debbouche et al. [11] proved the existence and uniqueness of the optimal mild solution for the linear fractional evolution equation

$$D_t^\alpha u(t) + (A - B(t))u(t) = f(t), \;\; t > t_0 \;\; (\text{with } t_0 \in \mathbb{R})$$

in a uniformly convex Banach space $(X, \|\cdot\|)$, where $0 < \alpha \leqslant 1$, f is a given abstract function with values in X, $-A$ is a linear closed operator which generates an analytic semigroup, $\{B(t) : \; t \in \mathbb{R}^+\}$ is a family of linear bounded operators defined on X into X. The authors used the Gelfand-Shilov principle to prove the existence, and then the Bochner almost periodicity condition to show that solutions are weakly almost periodic. More results on optimal mild solutions of differential equations can be found in [7, 26, 28].

Recently in 2020, R. G. Foko Tiomela et al. [13] used the Banach contraction principle to prove the existence and uniqueness of a mild solution to the semilinear fractional differential equation

$${}^C D_t^\alpha x(t) = Ax(t) + f(t, x(t)), \;\; t \in [0,b] \;\; \text{with } \; x(0) = x_0,$$

in an infinite dimensional Banach space $(X,\|\cdot\|)$, where A is the infinitesimal generator of a semigroup of uniformly bounded operators. The authors also established the stability and stabilization of the system. Numerous researchers have been interested on the existence of solutions for evolution equations, and we refer for instance to [6, 8, 9, 10, 15, 19, 21, 22, 23, 25, 27].

The main purpose of this paper is to establish the existence and uniqueness of optimal (ω,c)-asymptotically periodic mild solution to the fractional differential equation (2). The rest of this paper is organized as follows. Section 2 is devoted to some preliminaries very useful in the sequel, including new properties of the Mittag-Leffler function. In Section 3, we prove the existence of an (ω,c)-asymptotically periodic mild solution to the equation (2) when the Banach space $(\mathbb{X},\|\cdot\|_{\mathbb{X}})$ is reflexive and $\Omega_f \neq \emptyset$. Afterwards, we prove in Section 4, the uniqueness of the optimal (ω,c)-asymptotically periodic mild solution assuming that the Banach space $(\mathbb{X},\|\cdot\|_{\mathbb{X}})$ is uniformly convex. Finally, Section 5 concludes this work.

2. Preliminaries

Throughout this article, $t_0 \in \mathbb{R}$ and we denote by $\mathscr{C}([t_0,\infty),\mathbb{X})$ the collection of all functions $h:[t_0,\infty) \longrightarrow \mathbb{X}$ which are continuous; $\mathscr{BC}([t_0,\infty),\mathbb{X})$ denotes the collection of all functions $h:[t_0,\infty) \longrightarrow \mathbb{X}$ which are bounded and continuous, and we set

$$C_0(\mathbb{X}) := \{h \in \mathscr{BC}([t_0,\infty),\mathbb{X}) \text{ such that } \lim_{t \longrightarrow \infty} h(t) = 0\}.$$

A mild solution of the evolution equation (2) is defined as follows.

DEFINITION 2.1. [13] A function $u:\mathbb{R} \longrightarrow \mathbb{X}$ is said to be a mild solution of (2) if the function $s \longmapsto (t-s)^{\alpha-1}R(t-s)f(s)$ is integrable on (t_0,t) for each $t \geqslant t_0$, $t_0 \in \mathbb{R}$ and

$$u(t) = Q(t-t_0)u(t_0) + \int_{t_0}^{t} (t-s)^{\alpha-1}R(t-s)f(s)ds, \quad t \geqslant t_0, \tag{4}$$

where:

$$Q(t) = \int_0^\infty \zeta_\alpha(\theta)T(t^\alpha\theta)d\theta \tag{5}$$

and

$$R(t) = \int_0^\infty \alpha\theta\zeta_\alpha(\theta)T(t^\alpha\theta)d\theta, \tag{6}$$

with

$$\zeta_\alpha(\theta) = \frac{1}{\alpha}\theta^{-\frac{1}{\alpha}-1}\rho_\alpha\left(\theta^{-\frac{1}{\alpha}}\right) \tag{7}$$

denoting the probability density function defined on $(0,\infty)$ (that is, $\zeta_\alpha(\theta) \geqslant 0$ for $\theta \in (0,\infty)$ and $\int_0^\infty \zeta_\alpha(\theta)d\theta = 1$) and $\rho_\alpha(\theta)$ is the one-sided stable probability density defined by

$$\rho_\alpha(\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\theta^{-\alpha k-1}\frac{\Gamma(\alpha k+1)}{k!}\sin(k\pi\alpha), \quad \theta \in (0,\infty).$$

We will need the following definitions and properties of the Mittag-Leffler function.

DEFINITION 2.2. [30] The two-parameter Mittag-Leffler function is defined by the series expansion:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0,\ \beta > 0,\ z \in \mathbb{C}$$

where $\Gamma(x)$ is the gamma function.

PROPOSITION 2.1. [34] *Let $t \in \mathbb{R}$ and $0 < \alpha \leqslant 1$. Then the following holds:*

$$E_{\alpha,1}(t) > 0, \quad E_{\alpha,\alpha}(t) > 0, \quad \text{and} \quad E'_{\alpha,\alpha}(t) > 0.$$

PROPOSITION 2.2. [33] *Let $t, t_1, t_2 \in \mathbb{R}$ and $0 < \alpha \leqslant 1$. The following results hold:*

$$0 < E_{\alpha,\alpha}(t_1) < E_{\alpha,\alpha}(0) < E_{\alpha,\alpha}(t_2) \ \text{ for } \ t_1 < 0 < t_2.$$

Moreover,

$$\lim_{t \longrightarrow \infty} E_{\alpha,\alpha}(t) = \infty \quad \text{and} \quad \lim_{t \longrightarrow -\infty} E_{\alpha,\alpha}(t) = 0.$$

Now, we have the following very important result, obtained immediately from the asymptotic expansion formula for the Mittag-Leffler function (see [5, Page 12]).

PROPOSITION 2.3. *Let $t \in \mathbb{R}$ and $0 < \alpha \leqslant 1$. Then the following holds:*

$$\lim_{t \longrightarrow \infty} E_{\alpha,1}(t) = \infty \quad \text{and} \quad \lim_{t \longrightarrow -\infty} E_{\alpha,1}(t) = 0.$$

PROPOSITION 2.4. [31] *Let $0 < \alpha \leqslant 1$. Considering the probability density function $\zeta_\alpha(\theta)$ defined by (7), the following results hold:*

(i) $\displaystyle\int_0^\infty \zeta_\alpha(\theta) e^{-z\theta} d\theta = E_{\alpha,1}(-z), \ \ z \in \mathbb{C}$

(ii) $\displaystyle\int_0^\infty \alpha\theta\zeta_\alpha(\theta) e^{-z\theta} d\theta = E_{\alpha,\alpha}(-z), \ \ z \in \mathbb{C}.$

Inspired by Proposition 2.4, we established and proved the proposition below.

PROPOSITION 2.5. *The following results hold:*

$$\|Q(\eta)\|_{\mathbb{X}} \leqslant M E_{\alpha,1}(-b\eta^\alpha), \quad \eta \geqslant 0 \tag{8}$$

and

$$\|R(\eta)\|_{\mathbb{X}} \leqslant M E_{\alpha,\alpha}(-b\eta^\alpha), \quad \eta \geqslant 0 \tag{9}$$

where $Q(\eta)$ and $R(\eta)$ are respectively defined by (5) and (6), M, b two positive numbers satisfying (1).

Proof. Let $\eta \geqslant 0$; we have:

$$\|Q(\eta)\|_{\mathbb{X}} = \left\| \int_0^{\infty} \zeta_\alpha(\theta) T(\eta^\alpha \theta) d\theta \right\|_{\mathbb{X}} \leqslant \int_0^{\infty} \zeta_\alpha(\theta) \|T(\eta^\alpha \theta)\|_{\mathbb{X}} d\theta.$$

Since $\eta^\alpha \theta \geqslant 0$, then according to (1), there exist two positive numbers M and b such that $\|T(\eta^\alpha \theta)\|_{\mathbb{X}} \leqslant M e^{-b\eta^\alpha \theta}$. So, from the equality (i) in Proposition 2.4 we obtain:

$$\|Q(\eta)\|_{\mathbb{X}} \leqslant M \int_0^{\infty} \zeta_\alpha(\theta) e^{-b\eta^\alpha \theta} d\theta = M E_{\alpha,1}(-b\eta^\alpha).$$

Similarly, using the equality (ii) in Proposition 2.4 we have:

$$\begin{aligned} \|R(\eta)\|_{\mathbb{X}} &= \left\| \int_0^{\infty} \alpha\theta \zeta_\alpha(\theta) T(\eta^\alpha \theta) d\theta \right\|_{\mathbb{X}} \\ &\leqslant \int_0^{\infty} \alpha\theta \zeta_\alpha(\theta) \|T(\eta^\alpha \theta)\|_{\mathbb{X}} d\theta \\ &\leqslant M \int_0^{\infty} \alpha\theta \zeta_\alpha(\theta) e^{-b\eta^\alpha \theta} d\theta \\ &= M E_{\alpha,\alpha}(-b\eta^\alpha). \end{aligned}$$

This ends the proof of the proposition. □

The following definitions on the (ω,c)-periodicity were introduced by E. Alvarez et *al.* [3, 4] and are essential in the characterization of an (ω,c)-asymptotically periodic function.

DEFINITION 2.3. [4] A function $h \in \mathscr{C}([t_0,\infty),\mathbb{X})$ is said to be (ω,c)-periodic if for some $\omega > 0$, there exists $c \in \mathbb{C}\backslash\{0\}$ such that

$$h(t+\omega) \;=\; ch(t) \quad \forall\, t \geqslant t_0. \tag{10}$$

We denote by $P_{\omega c}(\mathbb{X})$, the collection of all functions $h \in \mathscr{C}([t_0,\infty),\mathbb{X})$ which are (ω,c)-periodic.

DEFINITION 2.4. [3] Let $c \in \mathbb{C}\backslash\{0\}$ and $\omega > 0$. A function $h \in \mathscr{C}([t_0,\infty),\mathbb{X})$ is said to be c-asymptotic if $c^\wedge(-t)h(t) \in C_0(\mathbb{X})$; that is,

$$\lim_{t \longrightarrow \infty} c^\wedge(-t)h(t) = 0,$$

where $c^\wedge(-t) = c^{-t/\omega}$. The collection of those functions are denoted by $C_{0,c}(\mathbb{X})$.

DEFINITION 2.5. [3] Let $c \in \mathbb{C}\backslash\{0\}$ and $\omega > 0$. A function $h \in \mathscr{C}([t_0,\infty),\mathbb{X})$ is said to be (ω,c)-asymptotically periodic if $h = h_1 + h_2$ where $h_1 \in P_{\omega c}(\mathbb{X})$ and $h_2 \in C_{0,c}(\mathbb{X})$. The collection of those functions (with the same period ω for the first component) are denoted by $AP_{\omega c}(\mathbb{X})$.

Endowed with the norm

$$\|h\|_{a\omega c} := \sup_{t \geqslant t_0} \||c|^{\wedge}(-t)h(t)\|_{\mathbb{X}}, \tag{11}$$

$AP_{\omega c}(\mathbb{X})$ is a Banach space (see [3]).

The following Hahn-Banach theorem is very useful in establishing the existence result.

THEOREM 2.1. (Hahn-Banach theorem) *Let X be a normed vector space and X^* the topological dual of X. Let Y be a subspace of X and Y^* the topological dual of Y. Then any continuous linear functional $\lambda \in Y^*$ on Y can be extended to a continuous linear functional $\tilde{\lambda} \in X^*$ on X with the same operator norm; thus $\tilde{\lambda}$ agrees with λ on Y and $\|\tilde{\lambda}\|_{X^*} = \|\lambda\|_{Y^*}$. (We note that the extension $\tilde{\lambda}$ is, in general, not unique).*

For the uniqueness result, we will need the following definition of uniform convexity.

DEFINITION 2.6. A uniformly convex space is a normed vector space such that, for every $0 < \varepsilon \leqslant 2$ there is some $\delta > 0$ such that for any two vectors x, y with $\|x\| \leqslant 1$ and $\|y\| \leqslant 1$, the condition $\|x-y\| \geqslant \varepsilon$ implies that $\left\|\frac{x+y}{2}\right\| \leqslant 1-\delta$.

3. Existence result

In this section, we establish an existence result for optimal (ω, c)-asymptotically periodic mild solution to the fractional differential equation (2).

The following intermediate results are very useful for clarity of our demonstrations.

PROPOSITION 3.1. *Let $f \in P_{\omega c}(\mathbb{X})$, and define v by*

$$v(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1} R(t-s) f(s) ds.$$

Then v belongs to $P_{\omega c}(\mathbb{X})$.

Proof. Let $\omega > 0$. We have:

$$v(t+\omega) = \int_{-\infty}^{t+\omega} (t+\omega-s)^{\alpha-1} R(t+\omega-s) f(s) ds.$$

By the change of variable $z = s - \omega$ we have:

$$v(t+\omega) = \int_{-\infty}^{t} (t-z)^{\alpha-1} R(t-z) f(z+\omega) dz. \tag{12}$$

Since f is an (ω, c)-periodic function, then there exists $c \in \mathbb{C} \setminus \{0\}$ such that $f(z + \omega) = cf(z)$. Hence, (12) becomes:

$$\begin{aligned} v(t+\omega) &= c \int_{-\infty}^{t} (t-z)^{\alpha-1} R(t-z) f(z) dz \\ &= cv(t). \quad \square \end{aligned}$$

PROPOSITION 3.2. *Let $f \in AP_{\omega c}(\mathbb{X})$, and define v by*

$$v(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1} R(t-s) f(s) ds.$$

Then v belongs to $AP_{\omega c}(\mathbb{X})$ if $|c| \geqslant 1$.

Proof. Since $f \in AP_{\omega c}(\mathbb{X})$, then there exist $f_1 \in P_{\omega c}(\mathbb{X})$ and $f_2 \in C_{0,c}(\mathbb{X})$ such that $f = f_1 + f_2$. We have:

$$\begin{aligned} v(t) &= \int_{-\infty}^{t} (t-s)^{\alpha-1} R(t-s)(f_1(s) + f_2(s)) ds \\ &= v_1(t) + v_2(t), \end{aligned}$$

where

$$\begin{aligned} v_1(t) &= \int_{-\infty}^{t} (t-s)^{\alpha-1} R(t-s) f_1(s) ds, \\ v_2(t) &= \int_{-\infty}^{t} (t-s)^{\alpha-1} R(t-s) f_2(s) ds. \end{aligned}$$

Based on Proposition 3.1, it is clear that $v_1 \in P_{\omega c}(\mathbb{X})$. It remains to prove that $v_2 \in C_{0,c}(\mathbb{X})$.

We have:

$$\|c^\wedge(-t) v_2(t)\|_{\mathbb{X}} = \left\| \int_{-\infty}^{t} c^{-(t-s)/\omega} (t-s)^{\alpha-1} R(t-s) c^{-s/\omega} f_2(s) ds \right\|_{\mathbb{X}}.$$

Making the change of variable $\tau = t - s$, we obtain:

$$\begin{aligned} \|c^\wedge(-t) v_2(t)\|_{\mathbb{X}} &= \left\| \int_{0}^{\infty} c^{-\tau/\omega} \tau^{\alpha-1} R(\tau) c^{-(t-\tau)/\omega} f_2(t-\tau) d\tau \right\|_{\mathbb{X}} \\ &\leqslant \int_{0}^{\infty} |c|^{-\tau/\omega} \tau^{\alpha-1} \|R(\tau)\|_{\mathbb{X}} \left\| c^{-(t-\tau)/\omega} f_2(t-\tau) \right\|_{\mathbb{X}} d\tau. \end{aligned} \tag{13}$$

We note that $\tau = t - s > 0$; if $|c| \geqslant 1$, then we have $|c|^{-\tau/\omega} \leqslant 1$. Based on Proposition 2.5, we have also

$$\|R(\tau)\|_{\mathbb{X}} \leqslant M E_{\alpha,\alpha}(-b\tau^{\alpha}).$$

Since $t - \tau = s$ and $s \in (-\infty, t]$, the inequality (13) becomes:

$$\|c^\wedge(-t) v_2(t)\|_{\mathbb{X}} \leqslant M \left(\sup_{s \in (-\infty, t]} \left\{ \|c^{-s/\omega} f_2(s)\|_{\mathbb{X}} \right\} \right) \int_{0}^{\infty} \tau^{\alpha-1} E_{\alpha,\alpha}(-b\tau^{\alpha}) d\tau.$$

We have previously established that:

$$E'_{\alpha,1}(t) = \frac{1}{\alpha}E_{\alpha,\alpha}(t) \quad \forall t \in \mathbb{R} \ \ (\text{this is also true in } \mathbb{C}). \tag{14}$$

Hence, we obtain

$$\|c^{\wedge}(-t)v_2(t)\|_{\mathbb{X}} \leqslant M\left(\sup_{s\in(-\infty,t]}\left\{\|c^{-s/\omega}f_2(s)\|_{\mathbb{X}}\right\}\right)\int_0^{\infty}\alpha\tau^{\alpha-1}E'_{\alpha,1}(-b\tau^{\alpha})d\tau. \tag{15}$$

Let us consider the integral

$$I = \int_0^{\infty}\alpha\tau^{\alpha-1}E'_{\alpha,1}(-b\tau^{\alpha})d\tau.$$

Considering the change of variable $\zeta = -b\tau^{\alpha}$, we have:

$$\tau^{\alpha} = -\frac{1}{b}\zeta \quad \text{and} \quad \tau = -\frac{1}{b^{\frac{1}{\alpha}}}\cdot\zeta^{\frac{1}{\alpha}}.$$

So,

$$d\tau = -\frac{1}{\alpha}\cdot\frac{1}{b^{\frac{1}{\alpha}}}\cdot\zeta^{\frac{1}{\alpha}-1}.$$

We have also:

$$\tau^{\alpha-1} = \frac{-\frac{1}{b}\zeta}{-\frac{1}{b^{\frac{1}{\alpha}}}\cdot\zeta^{\frac{1}{\alpha}}} = \frac{1}{b^{1-\frac{1}{\alpha}}}\cdot\zeta^{1-\frac{1}{\alpha}}.$$

In addition, $\zeta \longrightarrow 0$ as $\tau \longrightarrow 0$ and $\zeta \longrightarrow -\infty$ as $\tau \longrightarrow \infty$.

Hence,

$$\begin{aligned} I &= -\int_0^{-\infty}\alpha\frac{1}{b^{1-\frac{1}{\alpha}}}\cdot\zeta^{1-\frac{1}{\alpha}}E'_{\alpha,1}(\zeta)\frac{1}{\alpha}\cdot\frac{1}{b^{\frac{1}{\alpha}}}\cdot\zeta^{\frac{1}{\alpha}-1}d\zeta \\ &= \int_{-\infty}^{0}\frac{1}{b}E'_{\alpha,1}(\zeta)d\zeta \\ &= \frac{1}{b}\left[E_{\alpha,1}(\zeta)\right]_{\zeta=-\infty}^{0}. \end{aligned}$$

According to Proposition 2.3, we know that $\lim_{\zeta\longrightarrow-\infty}E_{\alpha,1}(\zeta) = 0$. It is also clear that $E_{\alpha,1}(0) = 1$. Hence, we have:

$$I = \frac{1}{b}(1-0) = \frac{1}{b}$$

and the inequality (15) becomes:

$$\|c^{\wedge}(-t)v_2(t)\|_{\mathbb{X}} \leqslant \frac{M}{b}\left(\sup_{s\in(-\infty,t]}\left\{\|c^{-s/\omega}f_2(s)\|_{\mathbb{X}}\right\}\right).$$

Now, since $f_2 \in C_{0,c}(\mathbb{X})$, then

$$\lim_{t\longrightarrow\infty}\left(\sup_{s\in(-\infty,t]}\left\{\|c^{-s/\omega}f_2(s)\|_{\mathbb{X}}\right\}\right)=0$$

and therefore,

$$\lim_{t\longrightarrow\infty} c^{\wedge}(-t)v_2(t)=0,$$

which means that $v_2 \in C_{0,c}(\mathbb{X})$ and therefore $v \in AP_{\omega c}(\mathbb{X})$. □

PROPOSITION 3.3. *Let $f \in AP_{\omega c}(\mathbb{X})$, and define v by*

$$v(t)=\int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)ds.$$

Then v belongs to $AP_{\omega c}(\mathbb{X})$ if $|c|\geqslant 1$.

Proof. Since $f \in AP_{\omega c}(\mathbb{X})$, then there exist $f_1 \in P_{\omega c}(\mathbb{X})$ and $f_2 \in C_{0,c}(\mathbb{X})$ such that $f=f_1+f_2$. We have:

$$\begin{aligned}
v(t)&=\int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)(f_1(s)+f_2(s))ds\\
&=\int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f_1(s)ds+\int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f_2(s)ds\\
&=\int_{-\infty}^{t}(t-s)^{\alpha-1}R(t-s)f_1(s)ds-\int_{-\infty}^{t_0}(t-s)^{\alpha-1}R(t-s)f_1(s)ds\\
&+\int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f_2(s)ds\\
&=v_1(t)+v_2(t),
\end{aligned}$$

where

$$\begin{aligned}
v_1(t)&=\int_{-\infty}^{t}(t-s)^{\alpha-1}R(t-s)f_1(s)ds, \qquad \text{and}\\
v_2(t)&=\int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f_2(s)ds-\int_{-\infty}^{t_0}(t-s)^{\alpha-1}R(t-s)f_1(s)ds.
\end{aligned}$$

Based on Proposition 3.1, it is clear that $v_1 \in P_{\omega c}(\mathbb{X})$. It remains to prove that $v_2 \in C_{0,c}(\mathbb{X})$.

Let $\varepsilon>0$. Since $f_2 \in C_{0,c}(\mathbb{X})$, then there exists $T>0$ such that for all $s>T$,

$$\|c^{-s/\omega}f_2(s)\|_{\mathbb{X}}<\varepsilon. \tag{16}$$

In order to prove that $v_2 \in C_{0,c}(\mathbb{X})$, we must consider two cases: $t>T$ and $t<-T$.

Without any loss of generality, let us assume $t > T$; we have:

$$v_2(t) = \int_{t_0}^{T} (t-s)^{\alpha-1} R(t-s) f_2(s)ds + \int_{T}^{t} (t-s)^{\alpha-1} R(t-s) f_2(s)ds \\ - \int_{-\infty}^{t_0} (t-s)^{\alpha-1} R(t-s) f_1(s)ds$$

and

$$\|c^{\wedge}(-t)v_2(s)\|_{\mathbb{X}} \leqslant \sum_{i=1}^{3} I_i(t),$$

where

$$I_1(t) = \left\| c^{-t/\omega} \int_{t_0}^{T} (t-s)^{\alpha-1} R(t-s) f_2(s)ds \right\|_{\mathbb{X}},$$
$$I_2(t) = \left\| c^{-t/\omega} \int_{T}^{t} (t-s)^{\alpha-1} R(t-s) f_2(s)ds \right\|_{\mathbb{X}},$$
$$I_3(t) = \left\| c^{-t/\omega} \int_{-\infty}^{t_0} (t-s)^{\alpha-1} R(t-s) f_1(s) \right\|_{\mathbb{X}}.$$

Now, we have:

$$I_1(t) = \left\| \int_{t_0}^{T} c^{-(t-s)/\omega} (t-s)^{\alpha-1} R(t-s) c^{-s/\omega} f_2(s)ds \right\|_{\mathbb{X}} \\ \leqslant \int_{t_0}^{T} |c|^{-(t-s)/\omega} (t-s)^{\alpha-1} \|R(t-s)\|_{\mathbb{X}} \|c^{-s/\omega} f_2(s)\|_{\mathbb{X}} ds.$$

We note that $t > T$ and $t_0 \leqslant s \leqslant T$ implies that $t-s > 0$. Then, if $|c| \geqslant 1$ we have $|c|^{-(t-s)/\omega} \leqslant 1$. Based on Proposition 2.5, we have also

$$\|R(t-s)\|_{\mathbb{X}} \leqslant M E_{\alpha,\alpha}(-b(t-s)^{\alpha}). \tag{17}$$

So, we obtain:

$$I_1(t) \leqslant M \left(\sup_{t_0 \leqslant s \leqslant T} \left\{ \|c^{-s/\omega} f_2(s)\|_{\mathbb{X}} \right\} \right) \int_{t_0}^{T} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-b(t-s)^{\alpha}) ds.$$

According to (14) and the chain rule, this inequality becomes:

$$I_1(t) \leqslant M \left(\sup_{t_0 \leqslant s \leqslant T} \left\{ \|c^{-s/\omega} f_2(s)\|_{\mathbb{X}} \right\} \right) \left[\frac{1}{b} E_{\alpha,1}(-b(t-s)^{\alpha}) \right]_{s=t_0}^{T} \\ = \frac{M}{b} \left(\sup_{t_0 \leqslant s \leqslant T} \left\{ \|c^{-s/\omega} f_2(s)\|_{\mathbb{X}} \right\} \right) \left(E_{\alpha,1}(-b(t-T)^{\alpha}) - E_{\alpha,1}(-b(t-t_0)^{\alpha}) \right).$$

Since $\lim_{t \longrightarrow -\infty} E_{\alpha,1}(t) = 0$, we deduce that $E_{\alpha,1}(-b(t-T)^{\alpha}) \longrightarrow 0$ as $t \longrightarrow \infty$ and $E_{\alpha,1}(-b(t-t_0)^{\alpha}) \longrightarrow 0$ as $t \longrightarrow \infty$ because $t > T$ and $t > t_0$.

In addition, since $f_2 \in C_{0,c}(\mathbb{X})$, then $\sup_{t_0 \leqslant s \leqslant T} \left\{ \|c^{-s/\omega} f_2(s)\|_{\mathbb{X}} \right\} < \infty$ and therefore,

$$\lim_{t \longrightarrow \infty} I_1(t) = 0. \tag{18}$$

We have also:

$$\begin{aligned} I_2(t) &= \left\| \int_T^t c^{-(t-s)/\omega}(t-s)^{\alpha-1} R(t-s) c^{-s/\omega} f_2(s) ds \right\|_{\mathbb{X}} \\ &\leqslant \int_T^t |c|^{-(t-s)/\omega}(t-s)^{\alpha-1} \|R(t-s)\|_{\mathbb{X}} \|c^{-s/\omega} f_2(s)\|_{\mathbb{X}} ds. \end{aligned}$$

We note that $T \leqslant s \leqslant t$ implies that $t-s \geqslant 0$. Then, if $|c| \geqslant 1$ we have $|c|^{-(t-s)/\omega} \leqslant 1$. According to (16) and (17), the above inequality becomes:

$$\begin{aligned} I_2(t) &\leqslant \varepsilon M \int_T^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-b(t-s)^{\alpha}) ds \\ &= \varepsilon M \left[\frac{1}{b} E_{\alpha,1}(-b(t-s)^{\alpha}) \right]_{s=T}^{t} \\ &= \frac{\varepsilon M}{b} \left(E_{\alpha,1}(0) - E_{\alpha,1}(-b(t-T)^{\alpha}) \right). \end{aligned}$$

But we know that $E_{\alpha,1}(0) = 1$ and $E_{\alpha,1}(-b(t-T)^{\alpha}) \longrightarrow 0$ as $t \longrightarrow \infty$. So, for any $\varepsilon > 0$, we have $I_2(t) \leqslant \dfrac{M}{b}\varepsilon$ and consequently,

$$\lim_{t \longrightarrow \infty} I_2(t) = 0. \tag{19}$$

Finally, making the change of variable $\tau = t - s$, we have:

$$\begin{aligned} I_3(t) &= \left\| \int_{-\infty}^{t_0} c^{-(t-s)/\omega}(t-s)^{\alpha-1} R(t-s) c^{-s/\omega} f_1(s) ds \right\|_{\mathbb{X}} \\ &\leqslant \int_{t-t_0}^{\infty} |c|^{-\tau/\omega} \tau^{\alpha-1} \|R(\tau)\|_{\mathbb{X}} \|c^{-(t-\tau)/\omega} f_1(t-\tau)\|_{\mathbb{X}} d\tau. \end{aligned}$$

We note that $\tau = t - s \geqslant 0$; Then, if $|c| \geqslant 1$ we have $|c|^{-\tau/\omega} \leqslant 1$. According to (17), the above inequality becomes:

$$\begin{aligned} I_3(t) &\leqslant M \left(\sup_{\tau \in [t-t_0,\infty)} \left\{ \|c^{-(t-\tau)/\omega} f_1(t-\tau)\|_{\mathbb{X}} \right\} \right) \int_{t-t_0}^{\infty} \tau^{\alpha-1} E_{\alpha,\alpha}(-b\tau^{\alpha}) d\tau \\ &= M \left(\sup_{\tau \in [t-t_0,\infty)} \left\{ \|c^{-(t-\tau)/\omega} f_1(t-\tau)\|_{\mathbb{X}} \right\} \right) \left[\frac{1}{-b} E_{\alpha,1}(-b\tau^{\alpha}) \right]_{\tau=t-t_0}^{\infty}. \end{aligned}$$

Since $\lim_{t \longrightarrow -\infty} E_{\alpha,1}(t) = 0$, we deduce that $E_{\alpha,1}(-b\tau^{\alpha}) \longrightarrow 0$ as $\tau \longrightarrow \infty$ and we have

$$I_3(t) \leqslant \frac{M}{b} \left(\sup_{\tau \in [t-t_0,\infty)} \left\{ \|c^{-(t-\tau)/\omega} f_1(t-\tau)\|_{\mathbb{X}} \right\} \right) \left(0 + E_{\alpha,1}(-b(t-t_0)^{\alpha}) \right).$$

Again, since $\lim_{t\longrightarrow-\infty} E_{\alpha,1}(t)=0$, we deduce that $E_{\alpha,1}(-b(t-t_0)^{\alpha})\longrightarrow 0$ as $\tau\longrightarrow\infty$ and therefore,

$$\lim_{t\longrightarrow\infty} I_3(t)=0. \tag{20}$$

We have just proved that for $t>T$, $\lim_{t\longrightarrow\infty} c^{\wedge}(-t)f_2(t)=0$. Similar arguments can be made if $t<-T$. Therefore, if $|c|\geqslant 1$ we have $f_2\in C_{0,c}(\mathbb{X})$ and therefore, $v\in AP_{\omega c}(\mathbb{X})$. $\square$

PROPOSITION 3.4. *Let $c\in\mathbb{C}\backslash\{0\}$ and $\omega>0$. For all $t\geqslant t_0$ and $u\in\mathbb{X}$, we have $Q(t-t_0)u(t_0)\in AP_{\omega c}(\mathbb{X})$ if $|c|\geqslant 1$.*

Proof. Let $t\geqslant t_0$ be fixed, $\omega>0$, $c\in\mathbb{C}\backslash\{0\}$ such that $|c|\geqslant 1$ and $u\in\mathbb{X}$. We have:

$$\begin{aligned}\|c^{\wedge}(-t)Q(t-t_0)u(t_0)\|_{\mathbb{X}} &\leqslant \|Q(t-t_0)\|_{\mathbb{X}}\frac{\|u(t_0)\|_{\mathbb{X}}}{|c|^{t/\omega}}\\ &\leqslant M\|u(t_0)\|_{\mathbb{X}}E_{\alpha,1}(-b(t-t_0)^{\alpha})\cdot\frac{1}{|c|^{t/\omega}}.\end{aligned}$$

Since $\lim_{t\longrightarrow-\infty} E_{\alpha,1}(t)=0$, we deduce that $E_{\alpha,1}(-b(t-t_0)^{\alpha})\longrightarrow 0$ as $t\longrightarrow\infty$. In addition, $\dfrac{1}{|c|^{t/\omega}}\longrightarrow 0$ as $t\longrightarrow\infty$ (because $|c|\geqslant 1$ and $\omega>0$). Then,

$$\lim_{t\longrightarrow\infty} c^{\wedge}(-t)Q(t-t_0)u(t_0)=0$$

which means that $Q(t-t_0)u(t_0)\in C_{0,c}(\mathbb{X})\subseteq AP_{\omega c}(\mathbb{X})$ and therefore, $Q(t-t_0)u(t_0)\in AP_{\omega c}(\mathbb{X})$ for any $u\in\mathbb{X}$. $\square$

As an immediate consequence of Propositions 3.3 and 3.4, we obtain the following very important theorem.

THEOREM 3.1. *Let us assume that the inequality (1) hold. Then, for each $f\in AP_{\omega c}(\mathbb{X})$, all mild solution u of (2) are (ω,c)-asymptotically periodic, where $\omega>0$ and $c\in\mathbb{C}\backslash\{0\}$ with $|c|\geqslant 1$.*

From now on, we denote by Ω_f the set of mild solutions u of (2) which are also bounded over the real line $\mathbb{R}$; that is, such that

$$\mu(u)=\sup_{t\in\mathbb{R}}\|u(t)\|_{\mathbb{X}}<\infty. \tag{21}$$

In what follows, let us assume that $\Omega_f\neq\emptyset$.

DEFINITION 3.1. [29] A bounded mild solution $\tilde{u}(t)$ of (2) is called optimal mild solution of (2) if

$$\mu(\tilde{u})\equiv\mu^{*}=\inf_{u\in\Omega_f}\mu(u). \tag{22}$$

THEOREM 3.2. *If the collection Ω_f is not empty and f is a strongly continuous function from $\mathbb{R}$ to a reflexive Banach space $\mathbb{X}$, then the fractional differential equation (2) has at least one optimal (ω,c)-asymptotically periodic mild solution.*

Proof. By the definition of μ^*, there exists for any $n\in\mathbb{N}$, a function $u_{n,0}(t)\in\Omega_f$ such that

$$\mu^* \leqslant \mu(u_{n,0}) < \mu^* + \frac{1}{n} \leqslant \mu^*+1, \quad n\in\mathbb{N}. \tag{23}$$

This means that the sequence $(u_{n,0}(0))_{n=1}^{\infty}$ is bounded in $\mathbb{X}$. Since $\mathbb{X}$ is a reflexive Banach space, then there exists a subsequence of $(u_{n,0}(0))_{n=1}^{\infty}$, denoted $(u_{n,1}(0))_{n=1}^{\infty}$ which converges weakly to $\hat{u}_0\in\mathbb{X}$.

Similarly, if we consider the sequence $(u_{n,1}(t))_{n=1}^{\infty}$, then according to the inequality (23) the sequence $(u_{n,1}(-1))_{n=1}^{\infty}$ is bounded in $\mathbb{X}$; so, there exists a subsequence of $(u_{n,1}(-1))_{n=1}^{\infty}$, denoted $(u_{n,2}(-1))_{n=1}^{\infty}$ which converges weakly to $\hat{u}_1\in\mathbb{X}$.

Proceeding like this, if we consider the sequence $(u_{n,q}(t))_{n=1}^{\infty}$ for all integer $q\geqslant 0$, then according to the inequality (23) the sequence $(u_{n,q}(-q))_{n=1}^{\infty}$ is bounded in $\mathbb{X}$; so, there exists a subsequence of $(u_{n,q}(-q))_{n=1}^{\infty}$, denoted $(u_{n,q+1}(-q))_{n=1}^{\infty}$ which converges weakly to $\hat{u}_q\in\mathbb{X}$.

Now, let us consider the sequence $(u_{q,q}(t))_{q=1}^{\infty}$. It is clear that $(u_{q,q}(-k))_{q=1}^{\infty}$ converges weakly to $\hat{u}_k$ for all integer $k\geqslant 0$. We note also that the sequence $(u_{q,q}(t))_{q=1}^{\infty}$ is bounded over the real line $\mathbb{R}$ and admits the representation formula (4). This means that $u_{q,q}(t)\in\Omega_f$ satisfies:

$$u_{q,q}(t) = Q(t-t_0)u_{q,q}(t_0) + \int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)ds, \quad t\geqslant t_0 \;\; (\forall\, t_0\in\mathbb{R}).$$

In particular, for $t_0=-k$ we have:

$$u_{q,q}(t) = Q(t+k)u_{q,q}(-k) + \int_{-k}^{t}(t-s)^{\alpha-1}R(t-s)f(s)ds, \;\; t\geqslant -k, \;\; (\forall\, k=0,1,2,3,\cdots).$$

Now, let $\mathbb{X}^*$ and T^* be the topological duals of $\mathbb{X}$ and T respectively. Let $u^*\in\mathbb{X}^*$; we have:

$$\begin{aligned}\left\langle u^*, Q(t+k)u_{q,q}(-k)\right\rangle &= \left\langle u^*, \left(\int_0^{\infty}\zeta_\alpha(\theta)T((t+k)^{\alpha}\theta)d\theta\right)u_{q,q}(-k)\right\rangle \\ &= \left\langle \left(\int_0^{\infty}\zeta_\alpha(\theta)T^*((t+k)^{\alpha}\theta)d\theta\right)u^*, u_{q,q}(-k)\right\rangle.\end{aligned}$$

Taking the limit when $q\longrightarrow\infty$, we have:

$$\begin{aligned}\lim_{q\longrightarrow\infty}\left\langle u^*, Q(t+k)u_{q,q}(-k)\right\rangle &= \lim_{q\longrightarrow\infty}\left\langle \left(\int_0^{\infty}\zeta_\alpha(\theta)T^*((t+k)^{\alpha}\theta)d\theta\right)u^*, u_{q,q}(-k)\right\rangle \\ &= \left\langle \left(\int_0^{\infty}\zeta_\alpha(\theta)T^*((t+k)^{\alpha}\theta)d\theta\right)u^*, \hat{u}_k\right\rangle \\ &= \left\langle u^*, \left(\int_0^{\infty}\zeta_\alpha(\theta)T((t+k)^{\alpha}\theta)d\theta\right)\hat{u}_k\right\rangle \\ &= \langle u^*, Q(t+k)\hat{u}_k\rangle.\end{aligned}$$

Hence, we deduce that $Q(t+k)u_{q,q}(-k)$ converges weakly to $Q(t+k)\hat{u}_k$ and we have:

$$u_{q,q}(t) \xrightarrow{w} Q(t+k)\hat{u}_k + \int_{-k}^{t}(t-s)^{\alpha-1}R(t-s)f(s)ds, \quad t \geqslant -k, \tag{24}$$

where " $\xrightarrow{w}$ " denotes the weak convergence.

Let us denote by

$$\hat{u}(t) = Q(t+k)\hat{u}_k + \int_{-k}^{t}(t-s)^{\alpha-1}R(t-s)f(s)ds \tag{25}$$

the weak limit of the sequence $(u_{q,q}(t))_{q=1}^{\infty}$. Now, we need to prove that $\hat{u}(t)$ admits the representation formula (4).

For all $t_0 \in \mathbb{R}$ and $t \geqslant t_0$, we recall that

$$u_{q,q}(t) = Q(t-t_0)u_{q,q}(t_0) + \int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)ds, \quad t \geqslant t_0 \ \ (\forall\, t_0 \in \mathbb{R}).$$

We have already proved that $u_{q,q}(t)$ converges weakly to $\hat{u}(t)$ and in particular $u_{q,q}(t_0)$ converges weakly to $\hat{u}(t_0)$. So, let $u^* \in \mathbb{X}^*$. For $t \geqslant t_0$, we have:

$$\begin{aligned}
\langle u^*, u_{q,q}(t)\rangle &= \langle u^*, Q(t-t_0)u_{q,q}(t_0)\rangle + \left\langle u^*, \int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)\right\rangle \\
&= \left\langle u^*, \left(\int_0^\infty \zeta_\alpha(\theta)T((t-t_0)^\alpha\theta)d\theta\right)u_{q,q}(t_0)\right\rangle \\
&\quad + \left\langle u^*, \int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)\right\rangle \\
&= \left\langle \left(\int_0^\infty \zeta_\alpha(\theta)T^*((t-t_0)^\alpha\theta)d\theta\right)u^*, u_{q,q}(t_0)\right\rangle \\
&\quad + \left\langle u^*, \int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)\right\rangle.
\end{aligned}$$

Taking the limit when $q \longrightarrow \infty$, we obtain:

$$\begin{aligned}
\langle u^*, \hat{u}(t)\rangle &= \left\langle \left(\int_0^\infty \zeta_\alpha(\theta)T^*((t-t_0)^\alpha\theta)d\theta\right)u^*, \hat{u}(t_0)\right\rangle \\
&\quad + \left\langle u^*, \int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)\right\rangle \\
&= \left\langle u^*, \left(\int_0^\infty \zeta_\alpha(\theta)T((t-t_0)^\alpha\theta)d\theta\right)\hat{u}(t_0)\right\rangle \\
&\quad + \left\langle u^*, \int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)\right\rangle \\
&= \langle u^*, Q(t-t_0)\hat{u}(t_0)\rangle + \left\langle u^*, \int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)\right\rangle \\
&= \left\langle u^*, Q(t-t_0)\hat{u}(t_0) + \int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)\right\rangle.
\end{aligned}$$

According to the Hahn-Banach theorem, this implies that

$$\hat{u}(t) = Q(t-t_0)\hat{u}(t_0) + \int_{t_0}^{t} (t-s)^{\alpha-1} R(t-s) f(s) ds, \quad t \geqslant t_0 \ (\forall\, t_0 \in \mathbb{R}). \tag{26}$$

Therefore, $\hat{u}(t)$ is a mild solution of the fractional differential equation ${}^{C}D_t^{\alpha} u(t) = Au(t) + f(t)$. Now, we need to prove that $\hat{u}(t)$ is an optimal solution.

We note that:

$$u_{q,q}(t) \xrightarrow{w} \hat{u}(t) \quad \forall\, t \in \mathbb{R}. \tag{27}$$

Let $u^* \in \mathbb{X}^*$ such that $\|u^*\|_{\mathbb{X}} \leqslant 1$; we have:

$$\begin{aligned} |\langle u^*, u_{q,q}(t)\rangle| &\leqslant \|u^*\|_{\mathbb{X}} \|u_{q,q}(t)\|_{\mathbb{X}} \\ &\leqslant \|u_{q,q}(t)\|_{\mathbb{X}} \\ &\leqslant \mu(u_{q,q}). \end{aligned} \tag{28}$$

In addition, according to the inequality (23), we have $\mu(u_{q,q}) \longrightarrow \mu^*$ as $q \longrightarrow \infty$ and hence, from (28) we obtain

$$\lim_{q \longrightarrow \infty} |\langle u^*, u_{q,q}(t)\rangle| \leqslant \mu^*. \tag{29}$$

But it is also clear that

$$\lim_{q \longrightarrow \infty} |\langle u^*, u_{q,q}(t)\rangle| = |\langle u^*, \hat{u}(t)\rangle|. \tag{30}$$

According to (29) and (30) we have just proved that

$$|\langle u^*, \hat{u}(t)\rangle| \leqslant \mu^* \quad \text{for all } \ t \in \mathbb{R}.$$

This means that

$$\|\hat{u}(t)\|_{\mathbb{X}} \leqslant \mu^* \ \text{ for all } \ t \in \mathbb{R} \ \text{ and therefore, } \ \mu(\hat{u}) \leqslant \mu^* \tag{31}$$

because $\mu(\hat{u}) = \sup\limits_{t \in \mathbb{R}} \|\hat{u}(t)\|_{\mathbb{X}}$. So, the function $\hat{u}(t)$ is bounded over the real line $\mathbb{R}$ and consequently $\hat{u} \in \Omega_f$.

Moreover, since $\hat{u} \in \Omega_f$ then according to (31) we deduce that $\mu(\hat{u}) = \mu^*$ because $\mu^* = \inf\limits_{u \in \Omega_f} \mu(u)$. Whence $\hat{u}$ is an optimal mild solution of (2). We have previously established in Theorem 3.1 that all mild solutions of the evolution equation (2) are (ω, c)-asymptotically periodic. Therefore, the fractional differential equation (2) has at least one optimal (ω, c)-asymptotically periodic mild solution. $\square$

4. Uniqueness result

In this section, we prove the uniqueness of the optimal (ω, c)-asymptotically periodic mild solution to the equation (2). To achieve this, we consider $\mathbb{X}$ as a uniformly convex Banach space.

THEOREM 4.1. *Assume that the Banach space $\mathbb{X}$ is uniformly convex, and any non-trivial and bounded (ω,c)-asymptotically periodic mild solution $u(t)$ of the fractional differential equation ${}^{C}D_t^{\alpha}u(t)=Au(t)$, $t\in\mathbb{R}$ satisfies*

$$\inf_{t\in\mathbb{R}}\|u(t)\|_{\mathbb{X}}>0.$$

Let $f\in\mathscr{C}(\mathbb{R},\mathbb{X})$ and assume $\Omega_f\neq\emptyset$. Then, the optimal (ω,c)-asymptotically periodic mild solution of the fractional differential equation (2) is unique.

Proof. Let us assume that $\hat{u}_1(t)$ and $\hat{u}_2(t)$ are two optimal (ω,c)-asymptotically periodic mild solutions of the fractional differential equation

$${}^{C}D_t^{\alpha}u(t)=Au(t)+f(t)$$

such that $\hat{u}_1(t)\neq\hat{u}_2(t)$.

Based on the development in Theorem 3.2, we have clearly that $\mu(\hat{u}_1)=\mu^*$ and $\mu(\hat{u}_2)=\mu^*$. Now, let us consider $z(t)=\hat{u}_1(t)-\hat{u}_2(t)$. We have:

$$\hat{u}_1(t)=Q(t-t_0)\hat{u}_1(t_0)+\int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)ds,\quad t\geqslant t_0\ (\forall\, t_0\in\mathbb{R});$$
$$\hat{u}_2(t)=Q(t-t_0)\hat{u}_2(t_0)+\int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)ds,\quad t\geqslant t_0\ (\forall\, t_0\in\mathbb{R}).$$

Then,

$$\begin{aligned}z(t)&=Q(t-t_0)(\hat{u}_1(t_0)-\hat{u}_2(t_0))\\&=Q(t-t_0)z(t_0),\ \ t\geqslant t_0,\ (\forall\, t_0\in\mathbb{R}).\end{aligned}$$

If we set $\varpi=\inf\limits_{t\in\mathbb{R}}\|z(t)\|_{\mathbb{X}}$, then by hypothesis it follows that $\varpi>0$. In other words,

$$\|z(t)\|_{\mathbb{X}}\geqslant\varpi\ \text{ for all }\ t\in\mathbb{R}.$$

Now, let us set

$$v(t)=\frac{1}{2}(\hat{u}_1(t)+\hat{u}_2(t)).$$

It is clear that the function v is continuous from $\mathbb{R}$ into $\mathbb{X}$, and we have

$$\begin{aligned}\mu(v)&=\mu\left(\frac{1}{2}(\hat{u}_1+\hat{u}_2)\right)\\&=\sup_{t\in\mathbb{R}}\left\|\frac{1}{2}(\hat{u}_1(t)+\hat{u}_2(t))\right\|_{\mathbb{X}}.\end{aligned}$$

By Triangle inequality, we obtain:

$$\begin{aligned}\mu(v)&\leqslant\frac{1}{2}\left(\sup_{t\in\mathbb{R}}\|\hat{u}_1(t)\|_{\mathbb{X}}+\sup_{t\in\mathbb{R}}\|\hat{u}_2(t)\|_{\mathbb{X}}\right)\\&=\frac{1}{2}(\mu(\hat{u}_1)+\mu(\hat{u}_2))=\frac{1}{2}(\mu^*+\mu^*)=\mu^*.\end{aligned}$$

This means that $\mu(v) \leqslant \mu^*$. Moreover, we have:

$$\begin{aligned}
v(t) &= \frac{1}{2}(\hat{u}_1(t)+\hat{u}_2(t)) \\
&= \frac{1}{2}\left(Q(t-t_0)\hat{u}_1(t_0)+Q(t-t_0)\hat{u}_2(t_0)+2\int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)ds\right) \\
&= Q(t-t_0)\left(\frac{1}{2}(\hat{u}_1(t_0)+\hat{u}_2(t_0))\right)+\int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)ds \\
&= Q(t-t_0)v(t_0)+\int_{t_0}^{t}(t-s)^{\alpha-1}R(t-s)f(s)ds, \;\; t \geqslant t_0 \quad (\forall\, t_0 \in \mathbb{R}).
\end{aligned}$$

Therefore, $v(t)$ is a mild solution of the fractional differential equation ${}^{C}D_t^{\alpha}u(t) = Au(t)+f(t)$. By construction, it is clear that the function $v(t)$ is also bounded over the real line $\mathbb{R}$ (because $\hat{u}_1$ and $\hat{u}_2$ are bounded over $\mathbb{R}$) and consequently $v \in \Omega_f$.

Now, for all $t \in \mathbb{R}$, we note that $\|u(t)\|_{\mathbb{X}} \leqslant \sup_{t\in\mathbb{R}}\|u(t)\|_{\mathbb{X}} = \mu(u)$ for all $u \in \Omega_f$ and $\inf_{u\in\Omega_f}\mu(u) = \mu^*$. So, it follows that $\mu^* \geqslant \|\hat{u}_1(t)\|_{\mathbb{X}}$ and $\mu^* \geqslant \|\hat{u}_2(t)\|_{\mathbb{X}}$ for all $t \in \mathbb{R}$. If we set

$$\hat{u}_m(t) = \max\{\|\hat{u}_1(t)\|_{\mathbb{X}}, \|\hat{u}_2(t)\|_{\mathbb{X}}\},$$

this implies that

$$\mu^* \geqslant \hat{u}_m(t) \tag{32}$$

and then

$$\frac{1}{\mu^*}\hat{u}_m(t) \leqslant 1 \;\; \forall\, t \in \mathbb{R}. \tag{33}$$

Multiplying both sides of (33) by $\varpi = \inf_{t\in\mathbb{R}}\|z(t)\|_{\mathbb{X}} > 0$ we obtain:

$$\frac{\varpi}{\mu^*}\hat{u}_m(t) \leqslant \varpi \;\; \forall\, t \in \mathbb{R}. \tag{34}$$

But we note that $\varpi \leqslant \|z(t)\|_{\mathbb{X}}$ for all $t \in \mathbb{R}$; then (34) implies that

$$\frac{\varpi}{\mu^*}\hat{u}_m(t) \leqslant \varpi \leqslant \|z(t)\|_{\mathbb{X}} \;\; \forall\, t \in \mathbb{R}.$$

This means that

$$\frac{1}{\hat{u}_m(t)}\|\hat{u}_1(t)-\hat{u}_2(t)\|_{\mathbb{X}} = \left\|\frac{\hat{u}_1(t)}{\hat{u}_m(t)} - \frac{\hat{u}_2(t)}{\hat{u}_m(t)}\right\|_{\mathbb{X}} \geqslant \frac{\varpi}{\mu^*} \;\; \forall\, t \in \mathbb{R}. \tag{35}$$

We note that

$$\left\|\frac{\hat{u}_1(t)}{\hat{u}_m(t)}\right\|_{\mathbb{X}} \leqslant \left\|\frac{\hat{u}_1(t)}{\|\hat{u}_1(t)\|_{\mathbb{X}}}\right\|_{\mathbb{X}} = 1$$

and

$$\left\|\frac{\hat{u}_2(t)}{\hat{u}_m(t)}\right\|_{\mathbb{X}} \leqslant \left\|\frac{\hat{u}_2(t)}{\|\hat{u}_2(t)\|_{\mathbb{X}}}\right\|_{\mathbb{X}} = 1.$$

Moreover, $\|z(t)\|_{\mathbb{X}} = \|\hat{u}_1(t) - \hat{u}_2(t)\|_{\mathbb{X}} \leqslant \|\hat{u}_1(t)\|_{\mathbb{X}} + \|\hat{u}_2(t)\|_{\mathbb{X}} \leqslant 2\mu^*$ and $\varpi \leqslant \|z(t)\|_{\mathbb{X}}$ for all $t \in \mathbb{R}$ implies that $\dfrac{\varpi}{\mu^*} \leqslant 2$.

Choosing $\varepsilon = \dfrac{\varpi}{\mu^*}$, then by the uniform convexity of the space $\mathbb{X}$, there exists $\delta > 0$ such that:

$$\left\| \frac{\dfrac{\hat{u}_1(t)}{\hat{u}_m(t)} + \dfrac{\hat{u}_2(t)}{\hat{u}_m(t)}}{2} \right\|_{\mathbb{X}} \leqslant 1 - \delta \;\; \forall\, t \in \mathbb{R};$$

that is,

$$\frac{1}{\hat{u}_m(t)} \left\| \frac{\hat{u}_1(t) + \hat{u}_2(t)}{2} \right\|_{\mathbb{X}} \leqslant 1 - \delta \;\; \forall\, t \in \mathbb{R}.$$

So,

$$\begin{aligned} \|v(t)\|_{\mathbb{X}} &\leqslant (1-\delta)\hat{u}_m(t) \;\; \forall\, t \in \mathbb{R} \\ &\leqslant (1-\delta)\mu^* \;\; \forall t \in \mathbb{R} \quad \text{(according to (32))}. \end{aligned}$$

This implies that:

$$\sup_{t\in\mathbb{R}} \|v(t)\|_{\mathbb{X}} \leqslant (1-\delta)\mu^*;$$

and then,

$$\mu(v) \leqslant (1-\delta)\mu^*.$$

Since $1 - \delta < 1$, we deduce that $\mu(v) < \mu^*$ (with $v \in \Omega_f$); which is a contradiction to the fact that $\mu^* = \inf\limits_{u\in\Omega_f} \mu(u)$. Therefore, $\hat{u}_1(t) = \hat{u}_2(t)$ for all $t \in \mathbb{R}$ and consequently, the optimal (ω, c)-asymptotically periodic mild solution of the fractional differential equation (2) is unique. $\square$

5. Conclusion

In this work, we have established the existence of optimal (ω, c)-asymptotically periodic mild solutions to the fractional differential equation (2) in a reflexive Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ when the collection Ω_f of bounded mild solutions over the real line $\mathbb{R}$ is not empty. Furthermore, we have established the uniqueness of the solution when the Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is uniformly convex, provided that any non-trivial and bounded (ω, c)-asymptotically periodic mild solution $u(t)$ of the fractional differential equation ${}^{C}D_t^{\alpha} u(t) = Au(t)$, $t \in \mathbb{R}$ satisfies $\inf\limits_{t\in\mathbb{R}} \|u(t)\|_{\mathbb{X}} > 0$.

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