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NEW COMPARISON METHOD FOR NONAUTONOMOUS CAPUTO–TYPE TIME–DELAY SYSTEMS

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Abstract. This paper exploits non-autonomous Caputo-type systems associated with unbounded delayed arguments. We formulate a new theorem and provide proof that gives limiting behaviour of nontrivial solutions to zero vector. Two examples are provided to illustrate the proposed theorem.

1. Introduction

Real order systems (well known as fractional order systems) describe evolution of state variables in a mathematical space that containing long-term memory. Applications of real order systems are inevitable in diverse areas of science and engineering [15, 21, 20, 14, 3, 11, 13, 2].

Time-delays are inherent and the effect of time-delays has been well established for integer time-delay differential systems [9, 6]. It adds an extra degree of freedom to such systems by the inclusion of section-wise memory of state variables before the current state of time. Such investigations demonstrate that the presence of time-delay in such systems might affect the performance of such systems and thus, it cannot be neglected in real order systems. Real order time-delay systems are very difficult in contrast to integer order time-delay systems. Recently, the stability analysis of real order systems that involve time-delay or no time-delay has received significant attention from many researchers (see, [5,17,18,26,8,23,16,4,19,22,10,7,24]). Different methods, such as the Lyapunov method, comparison method, Laplace transform method, has been proposed in these researches that provide useful tools for obtaining the conditions for the asymptotic stability analysis of such systems. Moreover, Tuan and Trinh [24] developed an asymptotic stability theory for autonomous nonlinear bounded or unbounded delay systems for the case when the fractional orders are equal. Further, in [25], Tuan et al. developed criteria for autonomous linear unbounded delay systems for the case when the fractional orders were different. He et al. [10] developed a theory for unbounded delay systems that involve equal fractional order. These works established

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asymptotic stability criteria of such systems that are dependent on time-delays, dependent on fractional orders, independent of time-delays, independent of fractional orders, and a combination of these arguments. In [7], Gallegos et al. have developed a fractional vector Lyapunov-like method for the asymptotic stability analysis of real order systems that involve bounded delays.

The question of asymptotic stability analysis of many real order time-delay systems is quite challenging and difficult. We ask the following question: Can we discover some relative simple real order system if equations are associated with unbounded delays? Motivated by the mentioned question, this paper investigates nonautonomous Caputo time-delay system

$$ {}^{C}D_{0,t}^{\widehat{\alpha}}x(t) = f(t,x,x(t-\tau_1(t)),\cdots,x(t-\tau_m(t))) \tag{1} $$

subject to the initial continuous function $x(t)=\phi(t)=(\phi_1(t),\cdots,\phi_n(t))^T$ on $[-\tau,\,0]$, where $\tau_j(t)$ is nonnegative, continuous and satisfies $t-\tau_j(t)\geqslant -\tau_j$ with $t-\tau_j(t)\to\infty$ as $t\to\infty$, $\tau=\max\{\tau_1,\cdots,\tau_m\}$, $x(t)\in\mathbb{R}^n$, ${}^{C}D_{0,t}^{\widehat{\alpha}}x(t)=\left({}^{C}D_{0,t}^{\alpha_1}x_1(t),\cdots,{}^{C}D_{0,t}^{\alpha_n}x_n(t)\right)^T\in\mathbb{R}^n$, ${}^{C}D_{0,t}^{\star}x_i(t)$ stands for Caputo operator, $\widehat{\alpha}=(\alpha_1,\cdots,\alpha_n)$, $\alpha_1,\cdots,\alpha_n\in(0,1]$, and the function $f:[0,\infty)\times\mathbb{R}^{(m+1)n}\to\mathbb{R}^n$ is continuously differentiable with $f(t,0,0,\cdots,0)=0$, $\forall t\geqslant 0$.

2. Background

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N}$ the set of natural numbers, $\mathbb{R}^n$ the Euclidean space and $\|\cdot\|$ the Euclidean norm. The inequality $x\leqslant y$ means its components x_i and y_i satisfy $x_i\leqslant y_i$ for $i=1,2,\cdots,n$, where $x,y\in\mathbb{R}^n$. The symbol $i=1(1)n$ means $i=1,2,\cdots,n$.

Here, we recall the popular fractional Riemann-Liouville (RL) and Caputo (C) operators [21,20]. The α-order Riemann-Liouville operator of $x:[0,\infty)\to\mathbb{R}$ is given by

$$ {}^{RL}D_{0,t}^{-\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}x(\tau)d\tau,\ t>0, \tag{2} $$

where $0<\alpha\in\mathbb{R}$ and $\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt$.

The α-order Caputo operator of $x:[0,\infty)\to\mathbb{R}$ is given by

$$ {}^{C}D_{0,t}^{\alpha}x(t)=\begin{cases} {}^{RL}D_{0,t}^{-(n-\alpha)}\left(\frac{d^nx(t)}{dt^n}\right), & \text{if } \alpha\in(n-1,n),\\ \frac{d^nx(t)}{dt^n}, & \text{if } \alpha=n, \end{cases} \tag{3} $$

where $n\in\mathbb{N}$ and $0<\alpha\in\mathbb{R}$.

A square matrix $A\in\mathbb{R}^{n\times n}$ is called a Metzler matrix if its off-diagonal elements are non-negative [12].

Next, we introduce the following definition.

DEFINITION 1. Throughout this work, a function $g = (g_1, \cdots, g_m)^T : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ is called a quasi-monotone non-decreasing (QN) function of *class* W^* if, for every fixed $t \in \mathbb{R}$ and $y \in \mathbb{R}^p$, there exists a $h = (h_1, \cdots, h_m)^T : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ with $g_i(t,u,y) \leqslant h_i(t,\widehat{u},y)$, $i = 1(1)m$, for all u, $\widehat{u} \in \mathbb{R}^n$ such that $u_j \leqslant \widehat{u_j}$, $u_i = \widehat{u_i}$, $j = 1(1)m$, $i \neq j$, where u_i denotes the i-th component of u.

When the function g is given by $g = h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$, then it becomes quasi-monotone nondecreasing function (see [7]).

DEFINITION 2. We say the null solution to system (1) is globally asymptotically stable (GAS) if, for any initial continuous function $\phi(t)$ on $[-\tau, 0]$, the limit $\lim_{t\to\infty} \|x(t)\| = 0$ holds.

LEMMA 1. (Comparison principle) *Consider the fractional differential inequality*

$$ {}^{C}D_{0,t}^{\alpha_i} x_i(t) \leqslant {}^{C}D_{0,t}^{\alpha_i} y_i(t) \tag{4}$$

with $y_i(t) \geqslant x_i(t) = \phi_i(t) \geqslant 0$ *on* $[-\tau, 0]$, *for* $i = 1(1)n$, *where* $\phi_1(t), \cdots, \phi_n(t)$ *are continuous,* $\tau > 0$ *and* $\alpha_1, \cdots, \alpha_n \in (0,1]$. *Then, the inequality*

$$0 \leqslant x_i(t) \leqslant y_i(t), \ \forall t \geqslant -\tau, \ i = 1(1)n \tag{5}$$

holds.

Proof. Let $z_i(t) = {}^{C}D_{0,t}^{\alpha_i} x_i(t) - {}^{C}D_{0,t}^{\alpha_i} y_i(t)$, for $i = 1(1)n$. Applying the RL operator, one gets

$$x_i(t) - y_i(t) = x_i(0) - y_i(0) + \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} z_i(\tau) d\tau, \ i = 1(1)n. \tag{6}$$

Since $z_i(t) \leqslant 0$ and $y_i(0) \geqslant x_i(0)$, one obtains $x_i(t) \leqslant y_i(t)$, $\forall t \geqslant -\tau$ for $i = 1(1)n$. Set $w_i(t) = y_i(t) - x_i(t)$ for $i = 1(1)n$. Then, one can obtain $w_i(t) \geqslant 0$ for all $t \geqslant -\tau$. Note that if $x_i(t) \leqslant 0$ for all $t \geqslant -\tau$, then $w_i(t) \geqslant y_i(t)$ for all $t \geqslant -\tau$. Consequently, one gets $x_i(t) \leqslant 0$ on $[-\tau, 0]$. It violets the priori assumption $x_i(t) = \phi_i(t) \geqslant 0$ on $[-\tau, 0]$. Thus, one must have $x_i(t) \geqslant 0$ for all $t \geqslant -\tau$, for $i = 1(1)n$. This closes the proof. □

REMARK 1. Finding some new different proofs to Lemma 1 often fascinates to many students and researches. This problem remains an open exercise problem in the comparison principle theory.

3. Main results

Let $V(t,x) : [0,\infty) \times D \subseteq \mathbb{R}^n \to \mathbb{R}^d_{\geqslant 0}$ be a continuously differentiable vector function with $V(t,0) = 0 \ \forall t \geqslant 0$ for some $d \in \mathbb{N}$. Define the vector function $V(t,x) = (v_1(t,x), v_2(t,x), \cdots, v_d(t,x))^T$, where $v_i(t,x) : [0,\infty) \times D \subseteq \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ are continuously differentiable with $v_i(t,0) = 0 \ \forall t \geqslant 0$, for $i = 1(1)d$.

DEFINITION 3. Whenever $V(t,x) \in \mathbb{R}^d_{\geqslant 0}$, we mean Caputo memory derivative along the nontrivial solution to (1) as

$$ {}^{C}D^{\widehat{\delta}}_{0,t}V(t,x(t)) = \left({}^{C}D^{\delta_1}_{0,t}v_1(t,x(t)), {}^{C}D^{\delta_2}_{0,t}v_2(t,x(t)), \cdots, {}^{C}D^{\delta_d}_{0,t}v_d(t,x(t))\right)^T \tag{7} $$

where $\widehat{\delta} = (\delta_1, \cdots, \delta_d)$, and $\delta_1, \delta_2, \cdots, \delta_d \in (0,1]$.

The below mentioned Theorem 1 introduces a new concept for the understanding of system (1) and provides a new insight toward the analysis of such a system.

THEOREM 1. (Nonnegative comparison theorem) *Let $x = 0$ be the equilibrium point of the system (1). Suppose there exists a continuously differentiable function $V(t,x) : [0,\infty) \times D \subseteq \mathbb{R}^n \to [0,\infty)$ such that*

A_1) $k_1\|x\|^{p_1} \leqslant \sum_{i=1}^{d} v_i(t,x) \leqslant k_2\|x\|^{p_2}$, $\forall x \in D$, $\forall t \geqslant 0$, *where the constants k_1, k_2, p_1 and p_2 are positive, and satisfy $k_1 \leqslant k_2$ with $k_2 \geqslant 1$ and $p_1 \leqslant p_2$ with $p_2 \geqslant 1$.*

A_2) *Along the nontrivial solution $x(t)$ of (1), the inequality*

$$ {}^{C}D^{\widehat{\delta}}_{0,t}V(t,x(t)) \leqslant g(t, V(t,x(t)), V(t,x(t-\tau_1(t))), \cdots, V(t,x(t-\tau_m(t)))), \tag{8} $$

$\forall x \in D - \{0\}$, $\forall t > 0$ *holds, where the function $g : [0,\infty) \times \mathbb{R}^d_{\geqslant 0} \times \mathbb{R}^{md}_{\geqslant 0} \to \mathbb{R}^d$ is continuously differentiable and satisfies $g(t,0,\cdots,0) = 0$ for all $t \geqslant 0$.*

A_3) *There exists a continuously differentiable quasi-monotone nondecreasing function $h : [0,\infty) \times \mathbb{R}^d_{\geqslant 0} \times \mathbb{R}^{md}_{\geqslant 0} \to \mathbb{R}^d$ with $h(t,0,\cdots,0) = 0$ for all $t \geqslant 0$ and satisfies*

$$ {}^{C}D^{\widehat{\delta}}_{0,t}U(t,z(t)) = h(t, U(t,z(t)), U(t,z(t-\tau_1(t))), \cdots, U(t,z(t-\tau_m(t)))), \tag{9} $$

$\forall z \in D - \{0\}$, $\forall t > 0$, *subject to $U(t,z(t)) \geqslant V(t,x(t)) \geqslant 0$ on $[-\tau, 0]$.*

If the function g is of QN class W^*, then the equilibrium point $x = 0$ to the system (1) is asymptotically stable provided the zero solution to (9) is asymptotically stable. If the result holds globally, then the zero solution is globally asymptotically stable.

Proof. Assume inequality (8) with equality (9). Since g is of QN *class* W^*, it follows from Lemma 1 that $0 \leqslant V(t,x(t)) \leqslant U(t,z(t))$ for all $t \geqslant -\tau$. Moreover, if the zero solution to (9) is asymptotically stable, then one has $\lim_{t\to\infty} V(t,x(t)) = 0$. Then, from the assumption A_1), one gets $\lim_{t\to\infty} \|x(t)\| = 0$. □

REMARK 2. The idea of the proposed comparison theory is quite new in the current state of the art of qualitative asymptotic stability theory. In short, it says that in order to conclude the asymptotic behaviour of the original system one must discover/construct/find at least a relative non-negative asymptotic stable system. The continuously differentiable function $V(t,x)$ enables a pathway to figure out the possibility of the existence of many such relative systems. It has been formulated in Theorem 1 to show the existence of many new results.

REMARK 3. In Theorem 1, the notation ${}^{C}D_{0,t}^{\widehat{\delta}}$ gives abstract measurement of fractional derivative of order $\widehat{\delta} = (\delta_1, \cdots, \delta_d)$ for a given system of order $\widehat{\alpha} = (\alpha_1, \cdots, \alpha_n)$. Some special cases could occur depending on the matching of orders for a smooth application process. The introduction of ${}^{C}D_{0,t}^{\widehat{\delta}}$ in Theorem 1 is absolutely necessary since the right measurement of the memory derivative of order $\widehat{\delta}$ to the given system remains an open problem to date.

Next, the following corollaries are introduced that provides a useful tool for the construction of some suitable comparison systems.

COROLLARY 1. *Let* $x = 0$ *be the equilibrium point of the system* (1)*. Suppose there exists a continuously differentiable function* $V(t,x) : [0,\infty) \times D \subseteq \mathbb{R}^n \to [0,\infty)$ *such that*

$A_1)$ $k_1\|x\|^{p_1} \leqslant \sum_{i=1}^{d} v_i(t,x) \leqslant k_2\|x\|^{p_2}$, $\forall x \in D$, $\forall t \geqslant 0$, *where the constants* k_1, k_2, p_1 *and* p_2 *are positive, and satisfy* $k_1 \leqslant k_2$ *with* $k_2 \geqslant 1$ *and* $p_1 \leqslant p_2$ *with* $p_2 \geqslant 1$.

$A_2)$ *Along the nontrivial solution* $x(t)$ *of* (1)*, the inequality*

$$ {}^{C}D_{0,t}^{\widehat{\delta}} V(t,x(t)) \leqslant AV(t,x(t)) + \sum_{j=1}^{m} B_j V(t,x(t-\tau_j(t))), \tag{10} $$

$\forall x \in D - \{0\}$, $\forall t > 0$ *holds, where* $A \in \mathbb{R}^{n\times n}$ *is a Metzler matrix and* $B_j \in \mathbb{R}^{n\times n}$ *are non-negative matrices for* $j = 1,2,\cdots,m$.

Then, the equilibrium point $x = 0$ *to the system* (1) *is asymptotically stable if the zero solution to the system*

$$ {}^{C}D_{0,t}^{\widehat{\delta}} U(t,x(t)) = AU(t,x(t)) + \sum_{j=1}^{m} B_j U(t,x(t-\tau_j(t))), \tag{11} $$

with $U(t,x(t)) = V(t,x(t))$ *on* $[-\tau,0]$*, is asymptotically stable. If the result holds globally, then the zero solution is globally asymptotically stable.*

Proof. Consider (10) with (11). Take $g(t,u,y_1,\cdots,y_m) = h(t,u,y_1,\cdots,y_m) = Au + \sum_{j=1}^{m} B_j y_j$. Then, it follows from Theorem 1 that the equilibrium point $x = 0$ to the system (1) is asymptotically stable since the zero solution to the system (11) is asymptotically stable. $\square$

COROLLARY 2. *Let* $x = 0$ *be the equilibrium point of the system* (1)*. Suppose there exists a continuously differentiable function* $V(t,x) : [0,\infty) \times D \subseteq \mathbb{R}^n \to [0,\infty)$ *such that*

$A_1)$ $k_1\|x\|^{p_1} \leqslant \sum_{i=1}^{d} v_i(t,x) \leqslant k_2\|x\|^{p_2}$, $\forall x \in D$, $\forall t \geqslant 0$, *where the constants* k_1, k_2, p_1 *and* p_2 *are positive, and satisfy* $k_1 \leqslant k_2$ *with* $k_2 \geqslant 1$ *and* $p_1 \leqslant p_2$ *with* $p_2 \geqslant 1$.

A_2) *Along the nontrivial solution $x(t)$ of (1), the inequality*

$$
\begin{aligned}
{}^{C}D_{0,t}^{\widehat{\delta}} V(t,x(t)) &\leqslant AV(t,x(t)) \\
&\quad + g(t,V(t,x(t)),V(t,x(t-\tau_1(t))),\cdots,V(t,x(t-\tau_m(t)))), \quad (12)
\end{aligned}
$$

$\forall x \in D-\{0\}$, $\forall t>0$ holds, where $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix and $g:[0,\infty)\times \mathbb{R}^{d}_{\geqslant 0}\times\mathbb{R}^{md}_{\geqslant 0}\to\mathbb{R}^{d}$ is a continuously differentiable quasi-monotone nondecreasing function and satisfies $g(t,0,\cdots,0)=0$ for all $t\geqslant 0$.

Then, the equilibrium point $x=0$ to the system (1) is asymptotically stable if the zero solution to the system

$$
\begin{aligned}
{}^{C}D_{0,t}^{\widehat{\delta}} U(t,x(t)) &= AU(t,x(t)) \\
&\quad + g(t,U(t,x(t)),U(t,x(t-\tau_1(t))),\cdots,U(t,x(t-\tau_m(t)))), \qquad (13)
\end{aligned}
$$

with $U(t,x(t))=V(t,x(t))$ on $[-\tau,0]$, is asymptotically stable. If the result holds globally, then the zero solution is globally asymptotically stable.

Proof. Consider (12) with its corresponding system (13). Then, it follows from Theorem 1 that the equilibrium point $x=0$ to the system (1) is asymptotically stable since the zero solution to the system (13) is asymptotically stable. □

REMARK 4. The new emerging Corollary 1 and Corollary 2 are some simplest kind of discoverable results of Theorem 1. Deriving many conditions for asymptotic stability remains some exercise problems in applicable results.

4. Examples

This section introduces two examples to illustrate the importance of some proposed mathematical results.

EXAMPLE 1. Consider the scalar real order time-delay system

$$ {}^{C}D_{0,t}^{\alpha} x(t) = -(5+e^{-t})x(t)+2x(t-\tau_1(t)), \qquad (14) $$

with $x(t)=\phi(t)$ on $[-3,0]$, where $\alpha\in(0,1]$ and $\tau_1(t)=\frac{t}{2}+3$.

Take $V(t,x)=x^2$. Then, by using Lemma 1 of [1], one gets the Caputo derivative of $V(t,x)$ along the solution $x(t)$ of (1) as

$$ {}^{C}D_{0,t}^{\alpha} V(t,x(t)) \leqslant -8V(t,x(t))+2V(t,x(t-\tau_1(t))),\ \forall t>0. $$

Construct an equation

$$ {}^{C}D_{0,t}^{\alpha} U(t,x(t)) = -8U(t,x(t))+2U(t,x(t-\tau_1(t))), \qquad (15) $$

where $U(t,x(t))=\phi^2(t)$ on $[-3,0]$. Set $d=1$ and $\widehat{\delta}=\delta_1=\alpha$. Clearly, all the assumptions of Theorem 1 are satisfied. Since the zero solution to (15) is asymptotically stable by Corollary 3 of [10], it follows from Theorem 1 that the zero solution to (1) should be globally asymptotically stable.

EXAMPLE 2. Consider the vector real order time-delay system

$$\begin{aligned} {}^{C}D_{0,t}^{\alpha_1}x_1(t) &= -\frac{11}{2}x_1(t)+\cos^2(t)x_1(t)-x_1^3(t)+\sin(x_2(t-\tau_1(t))),\\ {}^{C}D_{0,t}^{\alpha_2}x_2(t) &= -\frac{11}{2}x_2(t)+\sin^2(t)x_2(t)+\sin(x_1(t-\tau_2(t))), \end{aligned} \tag{16}$$

with $x_i(t)=\phi_i(t)$ on $[-5,0]$, where $\alpha_1,\alpha_2\in(0,1]$, $\tau_1(t)=\frac{t}{3}+1$ and $\tau_2(t)=\frac{t}{3}+\frac{t\sin^2(t)}{3}+5$.

Let $V(t,x)=(v_1(t,x),v_2(t,x))^T=\left(x_1^2,x_2^2\right)^T$. Applying Lemma 1 of [1], one computes the Caputo derivative of $V(t,x)$ along the solution $x(t)$ of (2) as

$$\begin{aligned} {}^{C}D_{0,t}^{\alpha_1}v_1(t,x(t)) &\leqslant -8v_1(t,x(t))+v_2(t,x(t-\tau_1(t))),\\ {}^{C}D_{0,t}^{\alpha_2}v_2(t,x(t)) &\leqslant -8v_2(t,x(t))+v_1(t,x(t-\tau_2(t))). \end{aligned} \tag{17}$$

Construct a system

$$\begin{aligned} {}^{C}D_{0,t}^{\alpha_1}u_1(t,x(t)) &= -8u_1(t,x(t))+u_2(t,x(t-\tau_1(t))),\\ {}^{C}D_{0,t}^{\alpha_2}u_2(t,x(t)) &= -8u_2(t,x(t))+u_1(t,x(t-\tau_2(t))), \end{aligned} \tag{18}$$

where $u_i(t,x(t))=\phi_i^2(t)$ on $[-5,0]$ for $i=1,2$. Set $d=2$, $\widehat{\delta}=(\delta_1,\delta_2)$, $\delta_1=\alpha_1$ and $\delta_2=\alpha_2$. Clearly, all the assumptions of Theorem 1 are satisfied. Since the zero solution to (18) is asymptotically stable by Theorem 1 of [25], it follows from Theorem 1 that the zero solution to (17) should be globally asymptotically stable.

5. Conclusions

In short, determining the asymptotic stability of nonautonomous real order time-delay systems to the incommensurate order is challenging in the stability theory. This paper proposes some new simple comparison theories that provide asymptotic stability of nonautonomous Caputo-type real order time-delay systems. The proposed asymptotic theory is applied successively to two examples that involve unbounded delayed arguments. The theory may bring new insight into the understanding and analysis of many advanced systems.

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