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HERMITE–HADAMARD INTEGRAL INEQUALITY FOR HARMONICALLY CONVEX FUNCTIONS VIA RIEMANN–LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. The concept of convexity of functions is a useful instrument that is used to solve a wide range of pure and applied scientific issues. The Hermite-Hadamard inequality which is also used frequently in many other parts of practical mathematics notably in optimization and probability is one of the most important mathematical inequalities relevant to convex maps. The fractional calculus, a calculus of non-integer order has applications in diverse fields of physical sciences. In this paper, we have established Hermite-Hadamard's inequalities via Riemann-Liouville fractional integral for the case of harmonically convex function as well as the products of two harmonically convex functions via Riemann-Liouville fractional integrals.

1. Introduction

The notion of non-integer order calculus, the generalization of traditional integer order calculus, called fractional calculus, was introduced by Leibnitz and L'Hopital in 1695 but it was popularized in the end of nineteenth century by Riemann and Liouville. The rapid growth of the fractional calculus is because of its applications in diverse fields ranging from physical sciences to engineering to biological sciences and economics. Due to the wide application of fractional integrals and importance of Hermite-Hadamard type inequalities, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes.

The concept of convexity of functions is a useful instrument that is used to solve a wide range of pure and applied scientific issues. Many researchers have recently comitted themselves to investigate the attributes and inequalities of convexity in various directions. The Hermite-Hadamard inequality which is also used frequently in many other parts of practical mathematics notably in optimization and in probability is one of the most important mathematical inequalities relevant to convex maps. This famous inequality gives error bounds for the mean value of a continuous convex mapping. The inequalities discovered independently by Ch. Hermite and J. Hadamard for convex functions are very essential in the literature of mathematical analysis. These

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inequalities state that if $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval $\mathbb{I}$ of real numbers and $a, b \in \mathbb{I}$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}.$$

Both the inequalities hold in reversed direction if f is concave. We note that Hermite-Hadamard inequality (hereafter it is called as H-H inequality in this paper) may be regarded as a refinement of the concept of convexity and it easily follows from Jension's inequality. H-H inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been obtained. The H-H inequality provides estimates of the mean value of a continuous convex functions. B. G. Pachpatte established new Hermite-Hadamard type inequalities for products of classical convex functions as follows:

THEOREM 1. [3] *Let f and g be real-valued, non-negative and convex functions on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \leqslant \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b)$$

and,

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)g(x)\,dx + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b).$$

Some new integral inequalities involving two non-negative and integrable functions that are related to Hermite-Hadamard type are also obtained by many authors. B. G. Pachpatte proposed some Hermite-Hadamard type inequalities involving two log-convex functions. Similar results for s-convex functions are established by Kirmaci et al. M. Z. Sarikaya presented some integral inequalities for two h-convex functions. It is remarkable that M. Z. Sarikaya proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals. For more detail see [1, 2, 4] and the references therein.

THEOREM 2. [1] *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is convex function on $[a,b]$ then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[\mathbb{J}_{a^+}^{\alpha} f(b) + \mathbb{J}_{b^-}^{\alpha} f(a)\right] \leqslant \frac{f(a)+f(b)}{\alpha}.$$

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus which are used further in this paper.

DEFINITION 1. [1] Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $\mathbb{J}_{a^+}^{\alpha} f$ and $\mathbb{J}_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geqslant 0$ are defined by

$$\mathbb{J}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a$$

$$\mathbb{J}_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \quad x < b$$

respectively.

The symbols $\mathbb{J}_{a^+}^{\alpha} f(x)$ and $\mathbb{J}_{b^-}^{\alpha} f(x)$ are left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$, with $a \geqslant 0$. Here $\Gamma(\alpha)$ is gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1}\, dt.$$

DEFINITION 2. [2] A function $f : \mathbf{I} \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonically convex if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant t f(y) + (1-t) f(x)$$

holds for all $x, y \in \mathbb{I}$ and $t \in [0,1]$.

The aim of this paper is to establish Hermite-Hadamard's inequalities via Riemann-Liouville fractional integral for the case of harmonically convex function as well as to present the results on the products of two harmonically convex functions via Riemann-Liouville fractional integrals.

2. Main results

Hermite-Hadamard's inequalities for harmonically convex functions via Riemann-Liouville fractional integral can be represented as follows:

THEOREM 3. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leqslant a \leqslant b$ and $f \in L[a,b]$. If f is harmonically convex function on $[a,b]$, then the following inequalities for fractional inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right) \leqslant \frac{(a^{\alpha}-b^{\alpha})\Gamma(\alpha)}{2(b-a)^{\alpha}} \left[\frac{1}{a^{\alpha-1}} \mathbb{J}_{a^+}^{\alpha} f(b) + \frac{1}{b^{\alpha-1}} \mathbb{J}_{b^-}^{\alpha} f(a)\right] \leqslant \frac{f(a)+f(b)}{2\alpha}.$$

Proof. Since $f : \mathbb{I} \subset \mathbb{R}\backslash\{0\}$ is a harmonically convex, for all $x, y \in \mathbb{I}$ with $t = \frac{1}{2}$, we have

$$f\left(\frac{2xy}{x+y}\right) \leqslant \frac{f(x)+f(y)}{2}.$$

Choosing $x = \frac{ab}{ta+(1-t)b}$, $y = \frac{ab}{tb+(1-t)a}$, we get

$$f\left(\frac{2\frac{ab}{ta+(1-t)b}\frac{ab}{tb+(1-t)a}}{\frac{ab}{ta+(1-t)b}+\frac{ab}{tb+(1-t)a}}\right) \leqslant \frac{f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)}{2}$$

$$2f\left(\frac{2ab}{a+b}\right) \leqslant f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{tb+(1-t)a}\right).$$

Multiplying both sides by $t^{\alpha-1}$, then integrating with respect to t over $[0,1]$, we obtain

$$2\int_0^1 t^{\alpha-1} f\left(\frac{2ab}{a+b}\right) dt \leqslant \int_0^1 t^{\alpha-1} f\left(\frac{ab}{ta+(1-t)b}\right) dt + \int_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)a}\right) dt$$

$$\frac{2}{\alpha} f\left(\frac{2ab}{a+b}\right) \leqslant I_1 + I_2 \tag{1}$$

where $I_1 = \int_0^1 t^{\alpha-1}\left(\frac{ab}{ta+(1-t)b}\right) dt$. Put $u = \frac{ab}{ta+(1-t)b}$. Then $dt = \frac{ab}{b-a}\frac{du}{u^2}$ when $t \to 0$, $u \to a$; $t \to 1$, $u \to b$ and, $t = \frac{b(u-a)}{(a-b)u}$.

On substituting these values, we obtain

$$\begin{aligned} I_1 &= \int_a^b \left(\frac{b(u-a)}{(b-a)u}\right)^{\alpha_1-1} f(u)\frac{ab}{b-a}\frac{du}{u^2} \\ &= \frac{ab}{b-a}\int_a^b \frac{b^{\alpha-1}(u-a)^{\alpha-1}}{(b-a)^{\alpha-1}u^{\alpha-1}}\frac{f(u)du}{u^2} \\ &= \frac{ab^\alpha}{(b-a)^\alpha}\int_a^b \frac{(u-a)^{\alpha-1}f(u)du}{u^{\alpha+1}} \\ &= \frac{b^\alpha-a^\alpha}{\alpha(b-a)^\alpha a^{\alpha-1}}\Gamma(\alpha)\mathbb{J}^\alpha_{a^+}f(b). \end{aligned}$$

And, $I_2 = \int_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)a}\right) dt$. Put $v = \frac{ab}{tb+(1-t)a}$, then $dt = \frac{ab}{b-a}\frac{dv}{v^2}$.

When $t \to 0$, $v \to b$; $t \to 1$, $v \to a$ and, $t = \frac{a(b-v)}{(b-a)v}$.

On substituting these values, we obtain

$$I_2 = \frac{b^\alpha-a^\alpha}{\alpha(b-a)^\alpha b^{\alpha-1}}\Gamma(\alpha)\mathbb{J}^\alpha_{b^-}f(a).$$

Now, we substitute the values of I_1 and I_2 in (1), we obtain

$$\frac{2}{\alpha} f\left(\frac{2ab}{a+b}\right) \leqslant \frac{b^\alpha-a^\alpha}{\alpha(b-a)^\alpha a^{\alpha-1}}\Gamma(\alpha)\mathbb{J}^\alpha_{a^+}f(b) + \frac{b^\alpha-a^\alpha}{\alpha(b-a)^\alpha b^{\alpha-1}}\Gamma(\alpha)\mathbb{J}^\alpha_{b^-}f(a)$$

$$2f\left(\frac{2ab}{a+b}\right) \leqslant \frac{(b^\alpha-a^\alpha)\Gamma(\alpha)}{(b-a)^\alpha}\left(\frac{1}{a^{\alpha-1}}\mathbb{J}^\alpha_{a^+}f(b) + \frac{1}{b^{\alpha-1}}\mathbb{J}^\alpha_{b^-}f(a)\right). \tag{2}$$

Also, as f is harmonically convex function, then for $t \in [0,1]$, it yields

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leqslant tf(a) + (1-t)f(b)$$

and

$$f\left(\frac{ab}{tb+(1-t)a}\right) \leqslant (1-t)f(a) + tf(b).$$

Adding these two inequalities,

$$f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)\leqslant f(a)(t+1-t)+f(b)(1-t+t)$$
$$= f(a)+f(b).$$

Multiplying both sides by $t^{\alpha-1}$ and integrating with respect to t over $[0,1]$, we have

$$\int_0^1 t^{\alpha-1} f\left(\frac{ab}{ta+(1-t)b}\right)dt+\int_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)a}\right)dt$$
$$\leqslant f(a)\int_0^1 t^{\alpha-1}\,dt+f(b)\int_0^1 t^{\alpha-1}\,dt$$

$$\frac{(b^\alpha-a^\alpha)\Gamma(\alpha)}{(b-a)^\alpha}\left(\frac{1}{a^{\alpha-1}}\mathbb{J}^\alpha_{a^+}f(b)+\frac{1}{b^{\alpha-1}}\mathbb{J}^\alpha_{b^-}f(a)\right)\leqslant\frac{f(a)+f(b)}{\alpha}. \tag{3}$$

From (1), (2), and (3), we have

$$f\left(\frac{2ab}{a+b}\right)\leqslant\frac{(a^\alpha-b^\alpha)\Gamma(\alpha)}{2(b-a)^\alpha}\left[\frac{1}{a^{\alpha-1}}\mathbb{J}^\alpha_{a^+}f(b)+\frac{1}{b^{\alpha-1}}\mathbb{J}^\alpha_{b^-}f(a)\right]\leqslant\frac{f(a)+f(b)}{2\alpha}.$$

This completes the proof. □

THEOREM 4. *Let f and g be two real-valued, non-negative and harmonically convex functions on $[a,b]$. Then the following inequalities hold:*

$$\frac{(b^\alpha-a^\alpha)\Gamma(\alpha)}{\alpha(b-a)^\alpha}\left[\frac{1}{a^{\alpha-1}}\mathbb{J}^\alpha_{a^+}f(b)g(b)+\frac{1}{b^{\alpha-1}}\mathbb{J}^\alpha_{b^-}f(a)g(a)\right]$$
$$\leqslant\left(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\right)M(a,b)+\frac{2}{(\alpha+1)(\alpha+2)}N(a,b)$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$; $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Let f and g be harmonically convex functions on $[a,b]$. Then, for $t\in[0,1]$, we have

$$f\left(\frac{ab}{ta+(1-t)b}\right)\leqslant tf(b)+(1-t)f(a)$$

and

$$g\left(\frac{ab}{ta+(1-t)b}\right)\leqslant tg(b)+(1-t)g(a).$$

Then their product is given by

$$f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)$$
$$\leqslant t^2f(b)g(b)+(1-t)^2f(a)g(a)+t(1-t)(f(a)b(b)+f(b)g(a)).$$

Similarly,

$$f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{(1-t)a+tb}\right)$$
$$\leqslant t^2 f(a)g(a)+(1-t)^2 f(b)g(b)+t(1-t)(f(a)b(b)+f(b)g(a)).$$

On adding the above two inequalities, we have

$$f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{(1-t)a+tb}\right)$$
$$\leqslant (2t^2-2t+1)(f(a)g(a)+f(b)g(b))+2t(1-t)(f(a)g(b)+f(b)g(a)).$$

Multiplying the above inequality by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1} f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)dt$$
$$+\int_0^1 t^{\alpha-1} f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{(1-t)a+tb}\right)dt$$
$$\leqslant (f(a)g(a)+f(b)g(b))\int_0^1 t^{\alpha-1}(2t^2-2t+1)\,dt$$
$$+2(f(a)g(b)+f(b)g(a))\int_0^1 t^{\alpha-1}t(1-t)\,dt.$$

Here,

$$\int_0^1 t^{\alpha-1}(2t^2-2t+1)\,dt=\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}$$
$$\int_0^1 t^{\alpha-1}t(1-t)\,dt=\frac{1}{(\alpha+1)(\alpha+2)}.$$

Put $\frac{ab}{ta+(1-t)b}=u$. Then $dt=\frac{ab}{b-a}\frac{du}{u^2}$. And, when $t\to 0$, $u\to a$ and $t\to 1$, $u\to b$ and, $t=\frac{b(a-u)}{u(a-b)}$. Again, put $\frac{ab}{(1-t)a+tb}=v$. Then $dt=\frac{ab}{b-a}\frac{dv}{v^2}$. Also, when $t\to 0$, $v\to b$, and when $t\to 1$, $v\to a$. On substituting these values, we obtain

$$\int_a^b\left(\frac{b(a-u)}{u(a-b)}\right)^{\alpha-1} f(u)g(u)\frac{ab}{(b-a)u^2}\,du+\int_b^a\left(\frac{a(v-a)}{v(b-a)}\right)^{\alpha-1} f(v)g(v)\frac{ab}{(b-a)v^2}\,du$$
$$\leqslant\left(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\right)M(a,b)+\frac{2}{(\alpha+1)(\alpha+2)}N(a,b)$$

$$\frac{ab^\alpha}{(a-b)^\alpha}\int_b^a\frac{(a-u)^{\alpha-1}f(u)g(u)}{u^{\alpha+1}}\,du+\frac{a^\alpha b}{(b-a)^\alpha}\int_b^a\frac{(v-a)^{\alpha-1}f(v)g(v)}{u^{\alpha+1}}\,dv$$
$$\leqslant\left(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\right)M(a,b)+\frac{2}{(\alpha+1)(\alpha+2)}N(a,b)$$

$$\frac{(b^\alpha - a^\alpha)\Gamma(\alpha)}{\alpha(b-a)^\alpha}\left[\frac{1}{a^{\alpha-1}}\mathbb{J}^\alpha_{a^+}f(b)g(b)+\frac{1}{b^{\alpha-1}}\mathbb{J}^\alpha_{b^-}f(a)g(a)\right]$$
$$\leqslant\left(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\right)M(a,b)+\frac{2}{(\alpha+1)(\alpha+2)}N(a,b).$$

This completes the proof. □

3. Conclusion

In this paper, we have established some new Hermite-Hadamard type inequalities for harmonically convex function and the products of two harmonically convex functions via Riemann-Liouville fractional integrals. An interesting concern is that whether we can further use it to establish Hermite-Hadamard inequality for other kinds of convex functions?

REFERENCES

[1] F. CHEN, *A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals*, Ital. J. Pure Appl. Math., **33**, (2014), 299–306.

[2] I. ISCAN, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacetteppe, J. Math. Stats., **43** (6), (2014), 935–942.

[3] B. G. PACHPATTE, *A note on integral inequalities involving two log convex functions*, Math. Inequal. Appl., **7** (4) (2004), 511–515.

[4] M. Z. SARIKAYA, E. SET, H. YALDIZ, AND N. BASAK, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model., **57** (2013), 2403–2407.

[5] E. SET, M. F. OZDEMIR, AND S. S. DRAGOMIR, *On the Hermite-Hadamard inequality and other integral inequalities involving two functions*, J. Inequal. Appl., (2010), 2010:148102.

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