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CORRECTED SIMPSON'S SECOND FORMULA INEQUALITIES ON FRACTAL SET

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Abstract. The aim of this research is to investigate the corrected Simpson's second formula within the context of local fractional calculus. Firstly, we present a new integral identity that is related to the formula, which enables us to derive several integral inequalities for functions whose local fractional derivatives are generalized (s,P)-convex functions. Lastly, we discuss potential practical applications.

1. Introduction

A function $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is referred to as a convex function on I, if the following inequality holds

$$\phi\left(\tau z_1 + (1-\tau) z_2\right) \leqslant \tau \phi\left(z_1\right) + (1-\tau)\phi\left(z_2\right),$$

for any $z_1, z_2 \in I$ and $\tau \in [0,1]$.

Convex functions have significant importance across various fields, including biology, economics, and optimization [21]. These functions have been the subject of numerous integral inequalities, among which the Hermite-Hadamard inequality stands out as a notable example. The Hermite-Hadamard inequality states that for any convex function ϕ defined on the interval $[a,b]$, the following inequality holds:

$$\phi\left(\tfrac{a+b}{2}\right) \leqslant \tfrac{1}{b-a}\int_a^b \phi(z)dz \leqslant \tfrac{\phi(a)+\phi(b)}{2}. \tag{1}$$

The Hermite-Hadamard inequality has a wide range of applications and has been extensively studied in the field of mathematical analysis [21].

Fractal sets and local fractional calculus are closely related to the concept of non-differentiability. Traditional calculus assumes that functions are differentiable everywhere, which is often not the case in complex systems with fractal properties. Local fractional calculus provides a framework for dealing with non-differentiable functions and has been used to model many real-world phenomena that exhibit fractal

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characteristics. In this context, fractal sets provide a geometric representation of non-differentiability, and local fractional calculus provides a mathematical tool for analyzing and modeling such systems. Recently, local fractional calculus has emerged as a new tool for studying fractal sets, providing insights into the intricate behavior of these complex systems exhibiting multi-scale behavior [7, 30].

The concept of generalized convexity on fractal sets was introduced by Yang in 2012 [28], aiming to expand the conventional notion of convexity to the realm of fractal geometry.

DEFINITION 1. (Generalized convex function) The function $\phi : I \subseteq \mathbb{R} \to \mathbb{R}^{\delta}$ is referred to as a generalized convex function on I, if the inequality

$$\phi\left(\tau z_1 + (1-\tau) z_2\right) \leqslant \tau^{\delta} \phi\left(z_1\right) + (1-\tau)^{\delta} \phi\left(z_2\right)$$

holds for any $z_1, z_2 \in I$ and $\tau \in [0,1]$.

The analogue of inequality (1) for generalized convex functions was provided in [16] as follows:

$$\phi\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(1+\delta)}{(b-a)^{\delta}}\, {}_aI_b^{\delta} \phi(z) \leqslant \frac{\phi(a)+\phi(b)}{2^{\delta}}, \tag{2}$$

where ϕ is a generalized convex function on $I = [a,b]$.

Since its introduction, the concept of generalized convexity on fractal sets has been subject to further extensions by various researchers aiming to encompass a wider range of functions.

The class of generalized s-convex functions was introduced by Mo et al. in [17].

DEFINITION 2. (Generalized s-convex function) The function $\phi : I \subseteq \mathbb{R} \to \mathbb{R}^{\delta}$ is referred to as a generalized s-convex function in the second sense on I, if the inequality

$$\phi\left(\tau z_1 + (1-\tau) z_2\right) \leqslant \tau^{s\delta} \phi\left(z_1\right) + (1-\tau)^{s\delta} \phi\left(z_2\right)$$

holds for any $z_1, z_2 \in I$ and $\tau \in [0,1]$.

Moreover, the authors of [18] gave the Hermite-Hadamard inequality via generalized s-convexity on fractal set as follows:

$$\frac{2^{(s-1)\delta}}{\Gamma(1+\delta)} \phi\left(\frac{a+b}{2}\right) \leqslant \frac{{}_aI_b^{\delta} \phi(z)}{(b-a)^{\delta}} \leqslant \frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)} \left(\phi(a)+\phi(b)\right), \; 0 < s \leqslant 1,$$

where ϕ is a generalized s-convex function in the second sense on $I = [a,b]$.

The article [33] introduces and discusses the definition and properties of generalized (s,P)-convex functions.

DEFINITION 3. (Generalized (s,P)-convex function [33]) The function $\phi : I \subseteq \mathbb{R} \to \mathbb{R}^{\delta}$ is referred as a generalized (s,P)-convex function on I, if the inequality

$$\phi\left(\tau z_1 + (1-\tau) z_2\right) \leqslant \left(\tau^{s\delta} + (1-\tau)^{s\delta}\right) \left(\phi\left(z_1\right) + \phi\left(z_2\right)\right)$$

holds for any $z_1, z_2 \in I$ and $\tau \in [0,1]$.

In the same paper, the authors gave the corresponding Hermite-Hadamard inequality for this class of functions as follows:

$$\frac{2^{(s-2)\delta}}{\Gamma(1+\delta)}\phi\left(\frac{a+b}{2}\right)\leqslant\frac{{}_aI_b^{\delta}\phi(z)}{(b-a)^{\delta}}\leqslant\frac{2^{\delta}\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\left(\phi(a)+\phi(b)\right),\ 0<s\leqslant 1.$$

PROPOSITION 1. [33] *If a function $\phi : I\subseteq\mathbb{R}\to\mathbb{R}$ is non-negative and generalized s-convex, then ϕ is a generalized (s,P)-convex function.*

Based on these definitions and results, several studies have been carried out to establish error estimates of different quadrature formulas using various types of generalized convexity, see [1, 4, 5, 9, 10, 11, 13, 14, 22, 23, 27, 31].

In [8], Iftikhar et al. studied the Simpson second formula and were able to establish the following results for generalized convex local fractional derivatives:

THEOREM 1. *Let $\phi : [a,b]\to\mathbb{R}^{\delta}$ be a differentiable function on (a,b) such that $\phi\in D_{\delta}[a,b]$ and $\phi^{(\delta)}\in C_{\delta}[a,b]$ with $0\leqslant a<b$. If $\left|\phi^{(\delta)}\right|$ is generalized convex on $[a,b]$, then the following inequality holds*

$$\begin{aligned}&\left|\left(\tfrac{1}{8}\right)^{\delta}\left(\phi(a)+3^{\delta}\phi\left(\frac{2a+b}{3}\right)+3^{\delta}\phi\left(\frac{a+2b}{3}\right)+\phi(b)\right)-\tfrac{\Gamma(\delta+1)}{(b-a)^{\delta}}\,{}_aI_b^{\delta}\phi(z)\right|\\&\leqslant(b-a)^{\delta}\left[\left(\tfrac{528}{13824}\right)^{\delta}\frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)}+\left(\tfrac{1008}{13824}\right)^{\delta}\frac{\Gamma(1+2\delta)}{\Gamma(1+3\delta)}\right]\left(\left|\phi^{(\delta)}(a)\right|+\left|\phi^{(\delta)}(b)\right|\right).\end{aligned}$$

THEOREM 2. *Assume that all the assumptions of Theorem 1 are satisfied. If $\left|\phi^{(\delta)}\right|^q$ is generalized (s,P)-convex. Then for all $\tau\in[0,1]$ the following inequality holds*

$$\begin{aligned}&\left|\left(\tfrac{1}{8}\right)^{\delta}\left(\phi(a)+3^{\delta}\phi\left(\frac{2a+b}{3}\right)+3^{\delta}\phi\left(\frac{a+2b}{3}\right)+\phi(b)\right)-\tfrac{\Gamma(\delta+1)}{(b-a)^{\delta}}\,{}_aI_b^{\delta}\phi(z)\right|\\&\leqslant(b-a)^{\delta}\left(\tfrac{\Gamma(1+p\delta)}{\Gamma(1+(p+1)\delta)}\right)^{\frac{1}{p}}\left(\tfrac{\Gamma(1+\delta)}{\Gamma(1+2\delta)}\right)^{\frac{1}{q}}\left[\left(\tfrac{3^{p+1}+5^{p+1}}{24^{p+1}}\right)^{\frac{\delta}{p}}\left(\frac{\left|\phi^{(\delta)}(a)\right|^q+\left|\phi^{(\delta)}\left(\frac{2a+b}{3}\right)\right|^q}{3^{\delta}}\right)^{\frac{1}{q}}\right.\\&+\left(\tfrac{2}{6^{p+1}}\right)^{\frac{\delta}{p}}\left(\frac{\left|\phi^{(\delta)}\left(\frac{2a+b}{3}\right)\right|^q+\left|\phi^{(\delta)}\left(\frac{a+2b}{3}\right)\right|^q}{3^{\delta}}\right)^{\frac{1}{q}}\\&\left.+\left(\tfrac{3^{p+1}+5^{p+1}}{24^{p+1}}\right)^{\frac{\delta}{p}}\left(\frac{\left|\phi^{(\delta)}\left(\frac{a+2b}{3}\right)\right|^q+\left|\phi^{(\delta)}(b)\right|^q}{3^{\delta}}\right)^{\frac{1}{q}}\right].\end{aligned}$$

In this article, we study another 4-point formula known as the corrected Simpson second formula, which can be formulated as follows.

$$\frac{\Gamma(\delta+1)}{(b-a)^{\delta}}\, {}_aI_b^{\delta}\phi(z)$$
$$=\left(\tfrac{1}{80}\right)^{\delta}\left(13^{\delta}\phi(a)+27^{\delta}\phi\left(\frac{2a+b}{3}\right)+27^{\delta}\phi\left(\frac{a+2b}{3}\right)+13^{\delta}\phi(b)\right)+\mathscr{R}(\phi),$$

where $\mathscr{R}$ represents the approximation error.

By introducing a new identity related to the formula in question, we establish several new inequalities for generalized (s,P)-convex local fractional derivatives. Finally, we provide some practical applications that demonstrate the significance of our obtained results.

2. Preliminaries

In this section, we recall some definitions and tools related to fractal sets that are essential for our study, [28].

The following statements are true for all z_1^{δ}, z_2^{δ}, and z_3^{δ} from the set $\mathbb{R}^{\delta}$, where $0<\delta\leqslant 1$.

1. $z_1^{\delta}+z_2^{\delta}$ and $z_1^{\delta}z_2^{\delta}$ belong to $\mathbb{R}^{\delta}$,
2. $z_1^{\delta}+z_2^{\delta}=z_2^{\delta}+z_1^{\delta}=(z_1+z_2)^{\delta}=(z_2+z_1)^{\delta}$,
3. $z_1^{\delta}+\left(z_2^{\delta}+z_3^{\delta}\right)=(z_1+z_2)^{\delta}+z_3^{\delta}$,
4. $z_1^{\delta}z_2^{\delta}=z_2^{\delta}z_1^{\delta}=(z_1z_2)^{\delta}=(z_2z_1)^{\delta}$,
5. $z_1^{\delta}\left(z_2^{\delta}z_3^{\delta}\right)=\left(z_1^{\delta}z_2^{\delta}\right)z_3^{\delta}$,
6. $z_1^{\delta}\left(z_2^{\delta}+z_3^{\delta}\right)=z_1^{\delta}z_2^{\delta}+z_1^{\delta}z_3^{\delta}$,
7. $z_1^{\delta}+0^{\delta}=0^{\delta}+z_1^{\delta}=z_1^{\delta}$ and $z_1^{\delta}1^{\delta}=1^{\delta}z_1^{\delta}=z_1^{\delta}$.

The concept of local fractional derivative and local fractional integral was originally proposed by Gao-Yang-Kang, as described in [28, 29].

DEFINITION 4. ([28]) We define a function $\phi:[a,b]\rightarrow\mathbb{R}^{\delta}$ to be local fractional continuous at $z=z_0$, if for any $\eta>0$ there exists $\varepsilon>0$ satisfying

$$|\phi(z)-\phi(z_0)|<\varepsilon^{\delta}$$

for $|z-z_0|<\eta$.

We denote the set of all functions that are local fractional continuous on $[a,b]$ by $C_{\delta}[a,b]$.

DEFINITION 5. ([28]) The local fractional derivative of $\phi(z)$ of order δ at $z=z_0$ is defined by:

$$\phi^{(\delta)}(z_0)=\left.\frac{d^{\delta}\phi(z)}{dz^{\delta}}\right|_{z=z_0}=\lim_{z\to z_0}\frac{\Delta^{\delta}(\phi(z)-\phi(z_0))}{(z-z_0)^{\delta}},$$

where $\Delta^{\delta}(\phi(z)-\phi(z_0))\cong\Gamma(\delta+1)(\phi(z)-\phi(z_0))$.

We say that $\phi\in D_{(m+1)\delta}(I)$, if there exists $\phi^{(m+1)\delta}(z)=\overbrace{D^{\delta}D^{\delta}\ldots D^{\delta}}^{(m+1)\text{ times}}\phi(z)$ for any $z\in I\subseteq\mathbb{R}$, where $m\in\mathbb{N}$.

DEFINITION 6. ([28]) Let $\phi(z)\in C_{\delta}[a,b]$. Then the local fractional integral is defined by,

$$_aI_b^{\delta}\phi(z)=\frac{1}{\Gamma(\delta+1)}\int_a^b\phi(u)(du)^{\delta}=\frac{1}{\Gamma(\delta+1)}\lim_{\Delta u\to 0}\sum_{i=0}^{M-1}\phi(u_i)(\Delta u_i)^{\delta}$$

with $\Delta u_i=u_{i+1}-u_i$ and $\Delta u=\max\{\Delta u_1,\Delta u_2,\ldots,\Delta u_{M-1}\}$, where $[u_i,u_{i+1}]$, $i=0,1,\ldots,M-1$ and $a=u_0<u_1<\ldots<u_M=b$ is partition of interval $[a,b]$.

It can be inferred that, $_aI_b^{\delta}\phi(z)=0$ for $a=b$ and $_aI_b^{\delta}\phi(z)=-{}_bI_a^{\delta}\phi(z)$ for $a<b$. If for any $z\in[a,b]$, $_aI_b^{\delta}\phi(z)$ exists, then we denoted by $\phi(z)\in I_z^{\delta}[a,b]$.

LEMMA 1. ([28])

1. *Suppose that* $\phi(z)=\psi^{(\delta)}(z)\in C_{\delta}[a,b]$, *then we have*

$$_aI_b^{\delta}\phi(z)=\psi(b)-\psi(a).$$

2. *Suppose that* $\phi,\psi\in D_{\delta}[a,b]$ *and* $\phi^{(\delta)}(z),\psi^{(\delta)}(z)\in C_{\delta}[a,b]$, *then we have*

$$_aI_b^{\delta}\phi(z)\psi^{(\delta)}(z)=\phi(z)\psi(z)|_a^b-{}_aI_b^{\delta}\phi^{(\delta)}(z)\psi(z).$$

LEMMA 2. ([28]) *For* $\phi(z)=z^{m\delta}$, *we have following equations*

$$\frac{d^{\delta}z^{m\delta}}{dz^{\delta}}=\frac{\Gamma(1+m\delta)}{\Gamma(1+(m-1)\delta)}z^{(m-1)\delta},$$

$$\frac{1}{\Gamma(1+\delta)}\int_a^b z^{m\delta}(dz)^{\delta}=\frac{\Gamma(1+m\delta)}{\Gamma(1+(m+1)\delta)}\left(b^{(m+1)\delta}-a^{(m+1)\delta}\right),\quad m\in\mathbb{R}.$$

LEMMA 3. (Generalized Hölder's inequality [2]) *Let* $\phi,\psi\in C_{\delta}[a,b]$, $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\frac{1}{\Gamma(1+\delta)}\int_a^b|\phi(z)\psi(z)|(dz)^{\delta}\leqslant\left(\frac{1}{\Gamma(1+\delta)}\int_a^b|\phi(z)|^p(dz)^{\delta}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\delta)}\int_a^b|\psi(z)|^q(dz)^{\delta}\right)^{\frac{1}{q}}.$$

3. Main results

In order to prove our results, we need the following lemma.

LEMMA 4. *Let $\phi : I \to \mathbb{R}^{\delta}$ be a differentiable function on I°, $a, b \in I^{\circ}$ with $a < b$, and $\phi^{(\delta)} \in C_{\delta}[a,b]$, then the following equality holds*

$$\frac{13^{\delta}\phi(a)+27^{\delta}\phi\left(\frac{2a+b}{3}\right)+27^{\delta}\phi\left(\frac{2a+b}{3}\right)+13^{\delta}\phi(b)}{80^{\delta}}-\frac{\Gamma(\delta+1)}{(b-a)^{\delta}}\,{}_{a}I_{b}^{\delta}\phi(z)$$
$$=\frac{(b-a)^{\delta}}{36^{\delta}}\left(\frac{4^{\delta}}{\Gamma(\delta+1)}\int_{0}^{1}\left(\tau-\frac{39}{80}\right)^{\delta}\phi^{(\delta)}\left((1-\tau)a+\tau\frac{2a+b}{3}\right)(d\tau)^{\delta}\right.$$
$$+\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}(\tau-1)^{\delta}\phi^{(\delta)}\left((1-\tau)\frac{2a+b}{3}+\tau\frac{a+b}{2}\right)(d\tau)^{\delta}$$
$$+\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\tau^{\delta}\phi^{(\delta)}\left((1-\tau)\frac{a+b}{2}+\tau\left(\frac{a+2b}{3}\right)\right)(d\tau)^{\delta}$$
$$\left.+\frac{4^{\delta}}{\Gamma(\delta+1)}\int_{0}^{1}\left(\tau-\frac{41}{80}\right)^{\delta}\phi^{(\delta)}\left((1-\tau)\left(\frac{a+2b}{3}\right)+\tau b\right)(d\tau)^{\delta}\right).$$

Proof. Let

$$I_1=\frac{4^{\delta}}{\Gamma(\delta+1)}\int_{0}^{1}\left(\tau-\frac{39}{80}\right)^{\delta}\phi^{(\delta)}\left((1-\tau)a+\tau\frac{2a+b}{3}\right)(d\tau)^{\delta},$$

$$I_2=\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}(\tau-1)^{\delta}\phi^{(\delta)}\left((1-\tau)\frac{2a+b}{3}+\tau\frac{a+b}{2}\right)(d\tau)^{\delta},$$

$$I_3=\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\tau^{\delta}\phi^{(\delta)}\left((1-\tau)\frac{a+b}{2}+\tau\frac{a+2b}{3}\right)(d\tau)^{\delta}$$

and

$$I_4=\frac{4^{\delta}}{\Gamma(\delta+1)}\int_{0}^{1}\left(\tau-\frac{41}{80}\right)^{\delta}\phi^{(\delta)}\left((1-\tau)\frac{a+2b}{3}+\tau b\right)(d\tau)^{\delta}.$$

Using the local fractional integration by parts, I_1 gives

$$\begin{aligned}
I_1 =& \frac{4^{\delta}}{\Gamma(\delta+1)}\int_0^1 \left(\tau-\frac{39}{80}\right)^{\delta}\phi^{(\delta)}\left((1-\tau)a+\tau\frac{2a+b}{3}\right)(d\tau)^{\delta}\\
=& \frac{12^{\delta}}{(b-a)^{\delta}}\left(\tau-\frac{39}{80}\right)^{\delta}\phi\left((1-\tau)a+\tau\frac{2a+b}{3}\right)\Big|_0^1\\
&-\frac{12^{\delta}}{(b-a)^{\delta}\Gamma(\delta+1)}\int_0^1 \Gamma(\delta+1)\phi\left((1-\tau)a+\tau\frac{2a+b}{3}\right)(d\tau)^{\delta}\\
=& \frac{492^{\delta}}{80^{\delta}(b-a)^{\delta}}\phi\left(\frac{2a+b}{3}\right)+\frac{468^{\delta}}{80^{\delta}(b-a)^{\delta}}\phi(a)-\frac{36^{\delta}}{(b-a)^{2\delta}}\int_a^{\frac{2a+b}{3}}\phi(z)(dz)^{\delta}. \qquad (3)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
I_2 =& \frac{1}{\Gamma(\delta+1)}\int_0^1 (\tau-1)^{\delta}\phi^{(\delta)}\left((1-\tau)\frac{2a+b}{3}+\tau\frac{a+b}{2}\right)(d\tau)^{\delta}\\
=& \frac{6^{\delta}}{(b-a)^{\delta}}(\tau-1)^{\delta}\phi\left((1-\tau)\frac{2a+b}{3}+\tau\frac{a+b}{2}\right)\Big|_0^1\\
&-\frac{6^{\delta}}{(b-a)^{\delta}\Gamma(\delta+1)}\int_0^1 \Gamma(\delta+1)\phi\left((1-\tau)\frac{2a+b}{3}+\tau\frac{a+b}{2}\right)(d\tau)^{\delta}\\
=& \frac{6^{\delta}}{(b-a)^{\delta}}\phi\left(\frac{2a+b}{3}\right)-\frac{36^{\delta}}{(b-a)^{2\delta}}\int_{\frac{2a+b}{3}}^{\frac{a+b}{2}}\phi(z)(dz)^{\delta}, \qquad (4)
\end{aligned}$$

$$\begin{aligned}
I_3 =& \frac{1}{\Gamma(\delta+1)}\int_0^1 \tau^{\delta}\phi^{(\delta)}\left((1-\tau)\frac{a+b}{2}+\tau\frac{a+2b}{3}\right)(d\tau)^{\delta}\\
=& \frac{6^{\delta}}{(b-a)^{\delta}}\tau^{\delta}\phi\left((1-\tau)\frac{a+b}{2}+\tau\frac{a+2b}{3}\right)\Big|_0^1\\
&-\frac{6^{\delta}}{(b-a)^{\delta}\Gamma(\delta+1)}\int_0^1 \Gamma(\delta+1)\phi\left((1-\tau)\frac{a+b}{2}+\tau\frac{a+2b}{3}\right)(d\tau)^{\delta}\\
=& \frac{6^{\delta}}{(b-a)^{\delta}}\phi\left(\frac{a+2b}{3}\right)-\frac{36^{\delta}}{(b-a)^{2\delta}}\int_{\frac{a+b}{2}}^{\frac{a+2b}{3}}\phi(z)(dz)^{\delta} \qquad (5)
\end{aligned}$$

and

$$
\begin{aligned}
I_4 &= \frac{4^\delta}{\Gamma(\delta+1)} \int_0^1 \left(\tau - \tfrac{41}{80}\right)^\delta \phi^{(\delta)}\left((1-\tau)\tfrac{a+2b}{3} + \tau b\right)(d\tau)^\delta \\
&= \frac{12^\delta}{(b-a)^\delta} \left(\tau - \tfrac{41}{80}\right)^\delta \phi\left((1-\tau)\tfrac{a+2b}{3} + \tau b\right)\Big|_0^1 \\
&\quad - \frac{12^\delta}{(b-a)^\delta \Gamma(\delta+1)} \int_0^1 \Gamma(\delta+1)\,\phi\left((1-\tau)\tfrac{a+2b}{3} + \tau b\right)(d\tau)^\delta \\
&= \frac{468^\delta}{80^\delta (b-a)^\delta}\phi(b) + \frac{492^\delta}{(b-a)^\delta}\phi\tfrac{a+2b}{3} - \frac{36^\delta}{(b-a)^{2\delta}} \int_{\frac{a+2b}{3}}^{b} \phi(z)\,(dz)^\delta . \qquad (6)
\end{aligned}
$$

Summing (3)–(6), then multiplying the resulting equality by $\frac{(b-a)^\delta}{36^\delta}$, we get the desired result. $\square$

THEOREM 3. *Let $\phi : [a,b] \to \mathbb{R}^\delta$ be a differentiable function on (a,b) such that $\phi \in D_\delta[a,b]$ and $\phi^{(\delta)} \in C_\delta[a,b]$ with $0 \leqslant a < b$. If $\left|\phi^{(\delta)}\right|$ is generalized (s,P)-convex on $[a,b]$, then the following inequality holds*

$$
\begin{aligned}
&\left| \frac{13^\delta \phi(a) + 27^\delta \phi\left(\frac{2a+b}{3}\right) + 27^\delta \phi\left(\frac{a+2b}{3}\right) + 13^\delta \phi(b)}{80^\delta} - \frac{\Gamma(\delta+1)}{(b-a)^\delta}\, {}_aI_b^\delta \phi(z) \right| \\
&\leqslant \frac{(b-a)^\delta}{36^\delta} \left(\left(\left(\frac{39^{s+2} + 41^{s+2}}{10 \times 80^{(s+1)}} \right)^\delta \left(\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)} - \frac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)} \right) + \frac{8^\delta \Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)} - \frac{4^\delta \Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)} \right) \right. \\
&\quad \times \left(\left|\phi^{(\delta)}(a)\right| + \left|\phi^{(\delta)}\left(\tfrac{2a+b}{3}\right)\right| + \left|\phi^{(\delta)}\left(\tfrac{a+2b}{3}\right)\right| + \left|\phi^{(\delta)}(b)\right| \right) \\
&\quad \left. + \frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)} \left(\left|\phi^{(\delta)}\left(\tfrac{2a+b}{3}\right)\right| + 2^\delta \left|\phi^{(\delta)}\left(\tfrac{a+b}{2}\right)\right| + \left|\phi^{(\delta)}\left(\tfrac{a+2b}{3}\right)\right| \right) \right).
\end{aligned}
$$

Proof. From Lemma 4, properties of modulus, and the generalized (s,P)-convexity of $\left|\phi^{(\delta)}\right|$, we have

$$
\begin{aligned}
&\left| \frac{13^\delta \phi(a) + 27^\delta \phi\left(\frac{2a+b}{3}\right) + 27^\delta \phi\left(\frac{2a+b}{3}\right) + 13^\delta \phi(b)}{80^\delta} - \frac{\Gamma(\delta+1)}{(b-a)^\delta}\, {}_aI_b^\delta \phi(z) \right| \\
&\leqslant \frac{(b-a)^\delta}{36^\delta} \left(\frac{4^\delta}{\Gamma(\delta+1)} \int_0^1 \left|\tau - \tfrac{39}{80}\right|^\delta \left|\phi^{(\delta)}\left((1-\tau)a + t\tfrac{2a+b}{3}\right)\right| (d\tau)^\delta \right. \\
&\quad + \frac{1}{\Gamma(\delta+1)} \int_0^1 (1-\tau)^\delta \left|\phi^{(\delta)}\left((1-\tau)\tfrac{2a+b}{3} + \tau\tfrac{a+b}{2}\right)\right| (d\tau)^\delta
\end{aligned}
$$

$$+\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\tau^{\delta}\left|\phi^{(\delta)}\left((1-\tau)\frac{a+b}{2}+\tau\left(\frac{a+2b}{3}\right)\right)\right|(d\tau)^{\delta}$$

$$\left.+\frac{4^{\delta}}{\Gamma(\delta+1)}\int_{0}^{1}\left|\tau-\frac{41}{80}\right|^{\delta}\left|\phi^{(\delta)}\left((1-\tau)\left(\frac{a+2b}{3}\right)+\tau b\right)\right|(d\tau)^{\delta}\right)$$

$$\leqslant\frac{(b-a)^{\delta}}{36^{\delta}}\left(\frac{4^{\delta}}{\Gamma(\delta+1)}\int_{0}^{1}\left|\tau-\frac{39}{80}\right|^{\delta}\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)\left(\left|\phi^{(\delta)}(a)\right|+\left|\phi^{(\delta)}\left(\frac{2a+b}{3}\right)\right|\right)(d\tau)^{\delta}\right.$$

$$+\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}(1-\tau)^{\delta}\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)\left(\left|\phi^{(\delta)}\left(\frac{2a+b}{3}\right)\right|+\left|\phi^{(\delta)}\left(\frac{a+b}{2}\right)\right|\right)(d\tau)^{\delta}$$

$$+\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\tau^{\delta}\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)\left(\left|\phi^{(\delta)}\left(\frac{a+b}{2}\right)\right|+\left|\phi^{(\delta)}\left(\frac{a+2b}{3}\right)\right|\right)(d\tau)^{\delta}$$

$$\left.+\frac{4^{\delta}}{\Gamma(\delta+1)}\int_{0}^{1}\left|\tau-\frac{41}{80}\right|^{\delta}\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)\left(\left|\phi^{(\delta)}\left(\frac{a+2b}{3}\right)\right|+\left|\phi^{(\delta)}(b)\right|\right)(d\tau)^{\delta}\right)$$

$$=\frac{(b-a)^{\delta}}{36^{\delta}}\left(\left(\left(\frac{39^{s+2}+41^{s+2}}{10\times 80^{(s+1)}}\right)^{\delta}\left(\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}-\frac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)}\right)\right.\right.$$

$$\left.+\frac{8^{\delta}\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)}-\frac{4^{\delta}\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\right)$$

$$\times\left(\left|\phi^{(\delta)}(a)\right|+\left|\phi^{(\delta)}\left(\frac{2a+b}{3}\right)\right|+\left|\phi^{(\delta)}\left(\frac{a+2b}{3}\right)\right|+\left|\phi^{(\delta)}(b)\right|\right)$$

$$\left.+\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\left(\left|\phi^{(\delta)}\left(\frac{2a+b}{3}\right)\right|+2^{\delta}\left|\phi^{(\delta)}\left(\frac{a+b}{2}\right)\right|+\left|\phi^{(\delta)}\left(\frac{a+2b}{3}\right)\right|\right)\right),$$

where we have used the facts that

$$\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\left|\tau-\frac{41}{80}\right|^{\delta}\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)(d\tau)^{\delta}$$

$$=\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\left|\tau-\frac{39}{80}\right|^{\delta}\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)(d\tau)^{\delta}$$

$$=\frac{1}{\Gamma(\delta+1)}\int_{0}^{\frac{39}{80}}\left(\frac{39}{80}-\tau\right)^{\delta}\tau^{s\delta}(d\tau)^{\delta}+\frac{1}{\Gamma(\delta+1)}\int_{0}^{\frac{39}{80}}\left(\frac{39}{80}-\tau\right)^{\delta}(1-\tau)^{s\delta}(d\tau)^{\delta}$$

$$+\frac{1}{\Gamma(\delta+1)}\int_{\frac{39}{80}}^{1}\left(\tau-\frac{39}{80}\right)^{\delta}\tau^{s\delta}(d\tau)^{\delta}+\frac{1}{\Gamma(\delta+1)}\int_{\frac{39}{80}}^{1}\left(\tau-\frac{39}{80}\right)^{\delta}(1-\tau)^{s\delta}(d\tau)^{\delta}$$

$$=\frac{1}{\Gamma(\delta+1)}\int_{0}^{\frac{39}{80}}\left(\left(\frac{39}{80}\right)^{\delta}\tau^{s\delta}-\tau^{(s+1)\delta}\right)(d\tau)^{\delta}+\frac{1}{\Gamma(\delta+1)}\int_{\frac{41}{80}}^{1}\left(\tau^{(s+1)\delta}-\left(\frac{41}{80}\right)^{\delta}\tau^{s\delta}\right)(d\tau)^{\delta}$$

$$+\frac{1}{\Gamma(\delta+1)}\int_{\frac{39}{80}}^{1}\left(\tau^{(s+1)\delta}-\left(\frac{39}{80}\right)^{\delta}\tau^{s\delta}\right)(d\tau)^{\delta}+\frac{1}{\Gamma(\delta+1)}\int_{0}^{\frac{41}{80}}\left(\left(\frac{41}{80}\right)^{\delta}\tau^{s\delta}-\tau^{(s+1)\delta}\right)(d\tau)^{\delta}$$

$$=\left(\frac{39^{s+2}+41^{s+2}}{40\times 80^{(s+1)}}\right)^{\delta}\left(\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}-\frac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)}\right)+2^{\delta}\frac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)}-\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)} \tag{7}$$

and

$$\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}(1-\tau)^{\delta}\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)(d\tau)^{\delta}$$

$$=\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\tau^{\delta}\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)(d\tau)^{\delta}$$

$$=\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\tau^{(s+1)\delta}(d\tau)^{\delta}+\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\tau^{\delta}(1-\tau)^{s\delta}(d\tau)^{\delta}$$

$$=\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\tau^{(s+1)\delta}(d\tau)^{\delta}+\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}(1-\tau)^{\delta}\tau^{s\delta}(d\tau)^{\delta}$$

$$=\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\tau^{(s+1)\delta}(d\tau)^{\delta}+\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\left(\tau^{s\delta}-\tau^{(s+1)\delta}\right)(d\tau)^{\delta}$$

$$=\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}. \tag{8}$$

The proof is completed. □

THEOREM 4. *Assume that all the assumptions of Theorem 3 are satisfied. If* $\left|\phi^{(\delta)}\right|^{q}$ *is generalized* (s,P)*-convex. Then for all* $\tau\in[0,1]$ *the following inequality holds*

$$\left|\frac{13^{\delta}\phi(a)+27^{\delta}\phi\left(\frac{2a+b}{3}\right)+27^{\delta}\phi\left(\frac{2a+b}{3}\right)+13^{\delta}\phi(b)}{80^{\delta}}-\frac{\Gamma(\delta+1)}{(b-a)^{\delta}}\,{}_{a}I_{b}^{\delta}\phi(z)\right|$$

$$\leqslant\frac{(b-a)^{\delta}}{36^{\delta}}\left(4^{\delta}\left(\frac{(41)^{p+1}+(39)^{p+1}}{(80)^{p+1}}\right)^{\frac{1}{p}\delta}\left(\frac{\Gamma(1+p\delta)}{\Gamma(1+(p+1)\delta)}\right)^{\frac{1}{p}}\left(2^{\delta}\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\right)^{\frac{1}{q}}\right.$$

$$\times\left(\left(\left|\phi^{(\delta)}(a)\right|^{q}+\left|\phi^{(\delta)}\left(\frac{2a+b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|\phi^{(\delta)}\left(\frac{a+2b}{3}\right)\right|^{q}+\left|\phi^{(\delta)}(b)\right|^{q}\right)^{\frac{1}{q}}\right)$$

$$+\left(\frac{\Gamma(1+p\delta)}{\Gamma(1+(p+1)\delta)}\right)^{\frac{1}{p}}\left(2^{\delta}\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\right)^{\frac{1}{q}}$$
$$\times\left(\left(\left|\phi^{(\delta)}\left(\frac{2a+b}{3}\right)\right|^{q}+\left|\phi^{(\delta)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|\phi^{(\delta)}\left(\frac{a+b}{2}\right)\right|^{q}+\left|\phi^{(\delta)}\left(\frac{a+2b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right)\Bigg).$$

Proof. From Lemma 4, properties of modulus, generalized Hölder inequality and generalized (s,P)-convexity of $\left|\phi^{(\delta)}\right|^{q}$, we have

$$\left|\frac{13^{\delta}\phi(a)+27^{\delta}\phi\left(\frac{2a+b}{3}\right)+27^{\delta}\phi\left(\frac{2a+b}{3}\right)+13^{\delta}\phi(b)}{80^{\delta}}-\frac{\Gamma(\delta+1)}{(b-a)^{\delta}}\,{}_aI_b^{\delta}\phi(z)\right|$$
$$\leqslant\frac{(b-a)^{\delta}}{36^{\delta}}\left(4^{\delta}\left(\frac{1}{\Gamma(\delta+1)}\int_0^1\left|\tau-\frac{39}{80}\right|^{p\delta}(d\tau)^{\delta}\right)^{\frac{1}{p}}\right.$$
$$\times\left(\frac{1}{\Gamma(\delta+1)}\int_0^1\left|\phi^{(\delta)}\left((1-\tau)a+\tau\frac{2a+b}{3}\right)\right|^{q}(d\tau)^{\delta}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\delta+1)}\int_0^1(1-\tau)^{p\delta}(d\tau)^{\delta}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\delta+1)}\int_0^1\left|\phi^{(\delta)}\left((1-\tau)\frac{2a+b}{3}+\tau\frac{a+b}{2}\right)\right|^{q}(d\tau)^{\delta}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\delta+1)}\int_0^1\tau^{p\delta}(d\tau)^{\delta}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\delta+1)}\int_0^1\left|\phi^{(\delta)}\left((1-\tau)\frac{a+b}{2}+\tau\frac{a+2b}{3}\right)\right|^{q}(d\tau)^{\delta}\right)^{\frac{1}{q}}$$
$$+4^{\delta}\left(\frac{1}{\Gamma(\delta+1)}\int_0^1\left|\tau-\frac{41}{80}\right|^{p\delta}(d\tau)^{\delta}\right)^{\frac{1}{p}}$$
$$\left.\times\left(\frac{1}{\Gamma(\delta+1)}\int_0^1\left|\phi^{(\delta)}\left((1-\tau)\frac{a+2b}{3}+\tau b\right)\right|^{q}(d\tau)^{\delta}\right)^{\frac{1}{q}}\right)$$
$$\leqslant\frac{(b-a)^{\delta}}{36^{\delta}}\left(4^{\delta}\left(\frac{1}{\Gamma(\delta+1)}\int_0^1\left|\tau-\frac{39}{80}\right|^{p\delta}(d\tau)^{\delta}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\delta+1)}\int_0^1\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)(d\tau)^{\delta}\right)^{\frac{1}{q}}\right.$$
$$\times\left(\left(\left|\phi^{(\delta)}(a)\right|^{q}+\left|\phi^{(\delta)}\left(\frac{2a+b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|\phi^{(\delta)}\left(\frac{a+2b}{3}\right)\right|^{q}+\left|\phi^{(\delta)}(b)\right|^{q}\right)^{\frac{1}{q}}\right)$$
$$+\left(\frac{1}{\Gamma(\delta+1)}\int_0^1\tau^{p\delta}(d\tau)^{\delta}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\delta+1)}\int_0^1\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)(d\tau)^{\delta}\right)^{\frac{1}{q}}$$

$$\times\left(\left(\left|\phi^{(\delta)}\left(\tfrac{2a+b}{3}\right)\right|^q+\left|\phi^{(\delta)}\left(\tfrac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\phi^{(\delta)}\left(\tfrac{a+b}{2}\right)\right|^q+\left|\phi^{(\delta)}\left(\tfrac{a+2b}{3}\right)\right|^q\right)^{\frac{1}{q}}\right)\Bigg)$$

$$=\tfrac{(b-a)^\delta}{36^\delta}\left(4^\delta\left(\left(\left(\tfrac{41}{80}\right)^{p+1}+\left(\tfrac{39}{80}\right)^{p+1}\right)^{\frac{1}{p}}\right)^\delta\left(\tfrac{\Gamma(1+p\delta)}{\Gamma(1+(p+1)\delta)}\right)^{\frac{1}{p}}\left(2^\delta\tfrac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\right)^{\frac{1}{q}}\right.$$

$$\times\left(\left(\left|\phi^{(\delta)}(a)\right|^q+\left|\phi^{(\delta)}\left(\tfrac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\phi^{(\delta)}\left(\tfrac{a+2b}{3}\right)\right|^q+\left|\phi^{(\delta)}(b)\right|^q\right)^{\frac{1}{q}}\right)$$

$$+\left(\tfrac{\Gamma(1+p\delta)}{\Gamma(1+(p+1)\delta)}\right)^{\frac{1}{p}}\left(2^\delta\tfrac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\right)^{\frac{1}{q}}$$

$$\left.\times\left(\left(\left|\phi^{(\delta)}\left(\tfrac{2a+b}{3}\right)\right|^q+\left|\phi^{(\delta)}\left(\tfrac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\phi^{(\delta)}\left(\tfrac{a+b}{2}\right)\right|^q+\left|\phi^{(\delta)}\left(\tfrac{a+2b}{3}\right)\right|^q\right)^{\frac{1}{q}}\right)\right),$$

where we have used the facts that

$$\tfrac{1}{\Gamma(\delta+1)}\int_0^1\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)(d\tau)^\delta=2^\delta\tfrac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}, \tag{9}$$

$$\tfrac{1}{\Gamma(\delta+1)}\int_0^1\tau^{p\delta}(d\tau)^\delta=\tfrac{\Gamma(1+p\delta)}{\Gamma(1+(p+1)\delta)} \tag{10}$$

and

$$\tfrac{1}{\Gamma(\delta+1)}\int_0^1\left|\tau-\tfrac{39}{80}\right|^{p\delta}(d\tau)^\delta=\left(\left(\tfrac{41}{80}\right)^{p+1}+\left(\tfrac{39}{80}\right)^{p+1}\right)^\delta\tfrac{\Gamma(1+p\delta)}{\Gamma(1+(p+1)\delta)}. \tag{11}$$

The proof is completed. □

THEOREM 5. *Assume that all the assumptions of Theorem 3 are satisfied. If* $\left|\phi^{(\delta)}\right|^q$ *is generalized* (s,P)*-convex. Then the following inequality holds*

$$\left|\tfrac{13^\delta\phi(a)+27^\delta\phi\left(\frac{2a+b}{3}\right)+27^\delta\phi\left(\frac{2a+b}{3}\right)+13^\delta\phi(b)}{80^\delta}-\tfrac{\Gamma(\delta+1)}{(b-a)^\delta}\,{}_aI_b^\delta\phi(z)\right|$$

$$\leqslant\tfrac{(b-a)^\delta}{36^\delta}\left(\tfrac{\Gamma(1+\delta)}{\Gamma(1+2\delta)}\right)^{1-\frac{1}{q}}\left(4^\delta\left(\left(\tfrac{1601}{3200}\right)^\delta\right)^{1-\frac{1}{q}}\left(2^\delta\tfrac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)}-\tfrac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\right.\right.$$

$$\left.+\left(\tfrac{39^{(s+2)}+41^{(s+2)}}{40\times80^{(s+1)}}\right)^\delta\left(\tfrac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}-\tfrac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)}\right)\right)^{\frac{1}{q}}$$

$$\times\left(\left(\left|\phi^{(\delta)}(a)\right|^q+\left|\phi^{(\delta)}\left(\tfrac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\phi^{(\delta)}\left(\tfrac{a+2b}{3}\right)\right|^q+\left|\phi^{(\delta)}(b)\right|^q\right)^{\frac{1}{q}}\right)$$

$$+\left(\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\right)^{\frac{1}{q}}\left(\left(\left|\phi^{(\delta)}\left(\frac{2a+b}{3}\right)\right|^{q}+\left|\phi^{(\delta)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|\phi^{(\delta)}\left(\frac{a+b}{2}\right)\right|^{q}+\left|\phi^{(\delta)}\left(\frac{a+2b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right)\right).$$

Proof. From Lemma 4, properties of modulus, generalized power mean inequality and generalized (s,P)-convexity of $\left|\phi^{(\delta)}\right|^{q}$, we have

$$\left|\frac{13^{\delta}\phi(a)+27^{\delta}\phi\left(\frac{2a+b}{3}\right)+27^{\delta}\phi\left(\frac{2a+b}{3}\right)+13^{\delta}\phi(b)}{80^{\delta}}-\frac{\Gamma(\delta+1)}{(b-a)^{\delta}}\,{}_{a}I_{b}^{\delta}\phi(z)\right|$$
$$\leqslant\frac{(b-a)^{\delta}}{36^{\delta}}\left(4^{\delta}\left(\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\left|\tau-\frac{39}{80}\right|^{\delta}(d\tau)^{\delta}\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\left|\tau-\frac{39}{80}\right|^{\delta}\left|\phi^{(\delta)}\left((1-\tau)a+t\frac{2a+b}{3}\right)\right|^{q}(d\tau)^{\delta}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}(1-\tau)^{\delta}(d\tau)^{\delta}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}(1-\tau)^{\delta}\left|\phi^{(\delta)}\left((1-\tau)\frac{2a+b}{3}+\tau\frac{a+b}{2}\right)\right|^{q}(d\tau)^{\delta}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\tau^{\delta}(d\tau)^{\delta}\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\tau^{\delta}\left|\phi^{(\delta)}\left((1-\tau)\frac{a+b}{2}+\tau\frac{a+2b}{3}\right)\right|^{q}(d\tau)^{\delta}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\left|\tau-\frac{49}{80}\right|^{\delta}(d\tau)^{\delta}\right)^{1-\frac{1}{q}}$$
$$\left.\times\left(\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\left|\tau-\frac{49}{80}\right|^{\delta}\left|\phi^{(\delta)}\left((1-\tau)\frac{a+2b}{3}+\tau b\right)\right|^{q}(d\tau)^{\delta}\right)^{\frac{1}{q}}\right)$$
$$\leqslant\frac{(b-a)^{\delta}}{36^{\delta}}\left(4^{\delta}\left(\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\left|\tau-\frac{39}{80}\right|^{\delta}(d\tau)^{\delta}\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\left(\frac{1}{\Gamma(\delta+1)}\int_{0}^{1}\left|\tau-\frac{39}{80}\right|^{\delta}\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)\left(\left|\phi^{(\delta)}(a)\right|^{q}+\left|\phi^{(\delta)}\left(\frac{2a+b}{3}\right)\right|^{q}\right)(d\tau)^{\delta}\right)^{\frac{1}{q}}\right.$$

$$
\begin{aligned}
&+\left(\frac{1}{\Gamma(\delta+1)}\int\limits_0^1\left|\tau-\tfrac{41}{80}\right|^{\delta}\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)\left(\left|\phi^{(\delta)}\left(\tfrac{a+2b}{3}\right)\right|^q+\left|\phi^{(\delta)}(b)\right|^q\right)(d\tau)^{\delta}\right)^{\frac{1}{q}}\Bigg)\\
&+\left(\frac{1}{\Gamma(\delta+1)}\int\limits_0^1\tau^{\delta}(d\tau)^{\delta}\right)^{1-\frac{1}{q}}\\
&\times\Bigg(\left(\frac{1}{\Gamma(\delta+1)}\int\limits_0^1(1-\tau)^{\delta}\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)\left(\left|\phi^{(\delta)}\left(\tfrac{2a+b}{3}\right)\right|^q+\left|\phi^{(\delta)}\left(\tfrac{a+b}{2}\right)\right|^q\right)(d\tau)^{\delta}\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{\Gamma(\delta+1)}\int\limits_0^1\tau^{\delta}\left(\tau^{s\delta}+(1-\tau)^{s\delta}\right)\left(\left|\phi^{(\delta)}\left(\tfrac{a+b}{2}\right)\right|^q+\left|\phi^{(\delta)}\left(\tfrac{a+2b}{3}\right)\right|^q\right)(d\tau)^{\delta}\right)^{\frac{1}{q}}\Bigg)\Bigg)\\
&=\frac{(b-a)^{\delta}}{36^{\delta}}\left(\frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)}\right)^{1-\frac{1}{q}}\Bigg(4^{\delta}\left(\left(\tfrac{1601}{3200}\right)^{\delta}\right)^{1-\frac{1}{q}}\Bigg(2^{\delta}\frac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)}-\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\\
&+\left(\frac{39^{(s+2)}+41^{(s+2)}}{40\times 80^{(s+1)}}\right)^{\delta}\left(\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}-\frac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)}\right)\Bigg)^{\frac{1}{q}}\\
&\times\left(\left(\left|\phi^{(\delta)}(a)\right|^q+\left|\phi^{(\delta)}\left(\tfrac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\phi^{(\delta)}\left(\tfrac{a+2b}{3}\right)\right|^q+\left|\phi^{(\delta)}(b)\right|^q\right)^{\frac{1}{q}}\right)\\
&+\left(\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\right)^{\frac{1}{q}}\Bigg(\left(\left|\phi^{(\delta)}\left(\tfrac{2a+b}{3}\right)\right|^q+\left|\phi^{(\delta)}\left(\tfrac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\left|\phi^{(\delta)}\left(\tfrac{a+b}{2}\right)\right|^q+\left|\phi^{(\delta)}\left(\tfrac{a+2b}{3}\right)\right|^q\right)^{\frac{1}{q}}\Bigg)\Bigg),
\end{aligned}
$$

where we have used (9), (10) and the facts that

$$
\frac{1}{\Gamma(\delta+1)}\int\limits_0^1\tau^{\delta}(d\tau)^{\delta}=\frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)}
$$

and

$$
\frac{1}{\Gamma(\delta+1)}\int\limits_0^1\left|\tau-\tfrac{39}{80}\right|^{\delta}(d\tau)^{\delta}=\left(\tfrac{1601}{3200}\right)^{\delta}\frac{\Gamma(1+\delta)}{\Gamma(1+2\delta)}.
$$

The proof is completed. □

4. Applications

Quadrature formula

Let Θ be the partition of the points $a = x_0 < x_1 < \ldots < x_n = b$ of the interval $[a,b]$, and consider the quadrature formula

$$\frac{1}{\Gamma(\delta+1)}\int_a^b \phi(u)(du)^{\delta} = \Omega(\phi,\Theta) + R(\phi,\Theta),$$

where

$$\Omega(\phi,\Theta) = \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^{\delta}}{80^{\delta}\Gamma(\delta+1)}\left(13^{\delta}\left(\phi(x_i)+\phi(x_{i+1})\right)+27^{\delta}\left(\phi\left(\frac{2x_i+x_{i+1}}{3}\right)+\phi\left(\frac{x_i+2x_{i+1}}{3}\right)\right)\right),$$

and $R(\phi,\Theta)$ denotes the associated approximation error.

PROPOSITION 2. *Let $n \in \mathbb{N}$, and $\phi : [a,b] \to \mathbb{R}^{\delta}$ be a differentiable function on (a,b) with $0 \leqslant a < b$ and $\phi^{(\delta)} \in C_{\delta}[a,b]$. If $\left|\phi^{(\delta)}\right|$ is genaralized (s,P)-convex function, we have*

$$\begin{aligned}
&|R(\phi,\Theta)| \\
&\leqslant \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^{2\delta}}{36^{\delta}\Gamma(1+\delta)}\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\left(\left(\frac{39^{(s+2)}+41^{(s+2)}}{10\times 80^{(s+1)}}\right)^{\delta}\left(\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}-\frac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)}\right)\right. \\
&\quad + \frac{8^{\delta}\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)} - \frac{4^{\delta}\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\Bigg) \\
&\quad \times\left(\left|\phi^{(\delta)}(x_i)\right|+\left|\phi^{(\delta)}\left(\frac{2x_i+x_{i+1}}{3}\right)\right|+\left|\phi^{(\delta)}\left(\frac{x_i+2x_{i+1}}{3}\right)\right|+\left|\phi^{(\delta)}(x_{i+1})\right|\right) \\
&\quad \left.+\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\left(\left|\phi^{(\delta)}\left(\frac{2x_i+x_{i+1}}{3}\right)\right|+2^{\delta}\left|\phi^{(\delta)}\left(\frac{x_i+x_{i+1}}{2}\right)\right|+\left|\phi^{(\delta)}\left(\frac{x_i+2x_{i+1}}{3}\right)\right|\right)\right).
\end{aligned}$$

Proof. Applying Theorem 3 on the subintervals $[x_i, x_{i+1}]$ $(i = 0,1,\ldots,n-1)$ of the partition Θ, we get

$$\begin{aligned}
&\left|\frac{13^{\delta}\phi(x_i)+27^{\delta}\phi\left(\frac{2x_i+x_{i+1}}{3}\right)+27^{\delta}\phi\left(\frac{x_i+2x_{i+1}}{3}\right)+13^{\delta}\phi(x_{i+1})}{80^{\delta}} - \frac{\Gamma(\delta+1)}{(x_{i+1}-x_i)^{\delta}}\, {}_{x_i}I^{\delta}_{x_{i+1}}\phi(t)\right| \\
&\leqslant \frac{(x_{i+1}-x_i)^{\delta}}{36^{\delta}}\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\left(\left(\frac{39^{(s+2)}+41^{(s+2)}}{10\times 80^{(s+1)}}\right)^{\delta}\left(\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}-\frac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)}\right)+8^{\delta}\frac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)}\right. \\
&\quad -4^{\delta}\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\Bigg)\left(\left|\phi^{(\delta)}(x_i)\right|+\left|\phi^{(\delta)}\left(\frac{2x_i+x_{i+1}}{3}\right)\right|+\left|\phi^{(\delta)}\left(\frac{x_i+2x_{i+1}}{3}\right)\right|+\left|\phi^{(\delta)}(x_{i+1})\right|\right) \\
&\quad \left.+\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\left(\left|\phi^{(\delta)}\left(\frac{2x_i+x_{i+1}}{3}\right)\right|+2^{\delta}\left|\phi^{(\delta)}\left(\frac{x_i+x_{i+1}}{2}\right)\right|+\left|\phi^{(\delta)}\left(\frac{x_i+2x_{i+1}}{3}\right)\right|\right)\right).
\end{aligned}$$

Multiplying both sides of above inequality by $\frac{1}{\Gamma(1+\delta)}\left(x_{i+1}-x_i\right)^{\delta}$, and then summing the obtained inequalities for all $i=0,1,\ldots,n-1$ and using the triangular inequality, we get the desired result. □

PROPOSITION 3. *Let $n\in\mathbb{N}$, and $\wp:[a,b]\to\mathbb{R}^{\delta}$ be a differentiable function on (a,b) with $0\leqslant a<b$ and $\wp^{(\delta)}\in C_{\delta}[a,b]$. If $\left|\wp^{(\delta)}\right|^{q}$ is genaralized (s,P)-convex function, we have*

$$\begin{aligned}
&|R(\phi,\Theta)|\\
&\leqslant\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^{2\delta}}{9^{\delta}\Gamma(1+\delta)}\left(\frac{\Gamma(1+p\delta)}{\Gamma(1+(p+1)\delta)}\right)^{\frac{1}{p}}\left(2^{\delta}\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\right)^{\frac{1}{q}}\left(4^{\delta}\left(\frac{(41)^{p+1}+(39)^{p+1}}{(80)^{p+1}}\right)^{\frac{1}{p}\delta}\right.\\
&\quad\times\left(\left(\left|\phi^{(\delta)}(x_i)\right|^{q}+\left|\phi^{(\delta)}\left(\frac{2x_i+x_{i+1}}{3}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|\phi^{(\delta)}\left(\frac{x_i+2x_{i+1}}{3}\right)\right|^{q}+\left|\phi^{(\delta)}(x_{i+1})\right|^{q}\right)^{\frac{1}{q}}\right)\\
&\quad+\left(\left|\phi^{(\delta)}\left(\frac{2x_i+x_{i+1}}{3}\right)\right|^{q}+\left|\phi^{(\delta)}\left(\frac{x_i+x_{i+1}}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\left|\phi^{(\delta)}\left(\frac{x_i+x_{i+1}}{2}\right)\right|^{q}+\left|\phi^{(\delta)}\left(\frac{x_i+2x_{i+1}}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Proof. Applying Theorem 4 on the subintervals $[x_i,x_{i+1}]$ $(i=0,1,\ldots,n-1)$ of the partition Θ, we get

$$\begin{aligned}
&\left|\frac{13^{\delta}\phi(x_i)+27^{\delta}\phi\left(\frac{2x_i+x_{i+1}}{3}\right)+27^{\delta}\phi\left(\frac{x_i+2x_{i+1}}{3}\right)+13^{\delta}\phi(x_{i+1})}{80^{\delta}}-\frac{\Gamma(\delta+1)}{(x_{i+1}-x_i)^{\delta}}\,{}_{x_i}I^{\delta}_{x_{i+1}}\phi(t)\right|\\
&\leqslant\frac{(x_{i+1}-x_i)^{\delta}}{36^{\delta}}\left(\frac{\Gamma(1+p\delta)}{\Gamma(1+(p+1)\delta)}\right)^{\frac{1}{p}}\left(2^{\delta}\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\right)^{\frac{1}{q}}\left(4^{\delta}\left(\frac{(41)^{p+1}+(39)^{p+1}}{80^{p+1}}\right)^{\frac{1}{p}\delta}\right.\\
&\quad\times\left(\left(\left|\phi^{(\delta)}(x_i)\right|^{q}+\left|\phi^{(\delta)}\left(\frac{2x_i+x_{i+1}}{3}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|\phi^{(\delta)}\left(\frac{x_i+2x_{i+1}}{3}\right)\right|^{q}+\left|\phi^{(\delta)}(x_{i+1})\right|^{q}\right)^{\frac{1}{q}}\right)\\
&\quad+\left(\left|\phi^{(\delta)}\left(\frac{2x_i+x_{i+1}}{3}\right)\right|^{q}+\left|\phi^{(\delta)}\left(\frac{x_i+x_{i+1}}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\left|\phi^{(\delta)}\left(\frac{x_i+x_{i+1}}{2}\right)\right|^{q}+\left|\phi^{(\delta)}\left(\frac{x_i+2x_{i+1}}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Multiplying both sides of above inequality by $\frac{1}{\Gamma(1+\delta)}\left(x_{i+1}-x_i\right)^{\delta}$, and then summing the obtained inequalities for all $i=0,1,\ldots,n-1$ and using the triangular inequality, we get the desired result. □

Application to special means

For arbitrary real numbers a,b we have:

- the generalized Arithmetic mean:

$$A(a,b)=\frac{a^{\delta}+b^{\delta}}{2^{\delta}},$$

- the generalized p-Logarithmic mean:

$$L_p(a,b)=\left[\frac{\Gamma(1+p\delta)}{\Gamma(1+(p+1)\delta)}\left(\frac{b^{(p+1)\delta}-a^{(p+1)\delta}}{(b-a)^{\delta}}\right)\right]^{\frac{1}{p}},$$

$a,b\in\mathbb{R}, a\neq b$ and $p\in\mathbb{Z}\setminus\{-1,0\}$.

PROPOSITION 4. *Let $a,b\in\mathbb{R}$ with $0<a<b$, then we have*

$$\left|\frac{13^{\delta}A^{(s+1)\delta}(a,b)+27^{\delta}A\left(A^{(s+1)\delta}(a,a,b),A^{(s+1)\delta}(a,b,b)\right)}{\Gamma(\delta+1)}-40^{\delta}L_{s+1}^{s+1}(a,b)\right|$$
$$\leqslant\frac{10^{\delta}(b-a)^{\delta}}{9^{\delta}\Gamma(\delta+1)}\frac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s)\delta)}\left(\left(\left(\frac{39^{(s+2)}+41^{(s+2)}}{10\times 80^{(s+1)}}\right)^{\delta}\left(\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}-\frac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)}\right)\right.\right.$$
$$\left.+8^{\delta}\frac{\Gamma(1+(s+1)\delta)}{\Gamma(1+(s+2)\delta)}-4^{\delta}\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\right)\left(a^{s\delta}+\left(\frac{2a+b}{3}\right)^{s\delta}+\left(\frac{a+2b}{3}\right)^{s\delta}+b^{s\delta}\right)$$
$$\left.+\frac{\Gamma(1+s\delta)}{\Gamma(1+(s+1)\delta)}\left(\left|\left(\frac{2a+b}{3}\right)^{s\delta}\right|+2^{\delta}\left(\frac{a+b}{2}\right)^{s\delta}+\left(\frac{a+2b}{3}\right)^{s\delta}\right)\right).$$

Proof. This follows from Theorem 3, applied to the function $\phi(z)=z^{(s+1)\delta}$ where $\phi:(0,+\infty)\to\mathbb{R}^{\delta}$. □

5. Conclusion

In conclusion, this study examines the corrected Simpson $3/8$-type inequalities on fractal set. The introduction of a novel integral identity has allowed for the derivation of integral inequalities related to the formula in question for local fractional differentiable generalized (s,P)-convex functions. Finally, applications to quadrature formulas and special means are presented, yielding specific results. Overall, this study expands the scope of understanding and paves the way for future research in the development of integral inequalities in the context of fractional local calculus.

Conflict of interest. The authors declare that they have no conflict of interest.

Availability of data and materials. Not applicable.

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