Fractional
Differential
Calculus
Volume 14, Number 1 (2024), 61–81
doi:10.7153/fdc-2024-14-04

EXISTENCE AND UNIQUENESS RESULTS FOR A CLASS OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

MOHAMMED DERHAB* AND FATIHA MEZIANE

(*Communicated by H. Lopushanska*)

Abstract. This study focuses on constructing solutions for a specific type of second order fractional differential equation involving nonlocal boundary conditions. We additionally provide exemplifications that demonstrate the application of our results.

1. Introduction

This work aims to construct both minimal and maximal solutions for a specific class of second order fractional differential equations with nonlocal boundary conditions. Specifically, we tackle the following nonlinear problem:

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0^+}^{\alpha}u\right)(x) = f(x,u), \ x \in [0,1], \\ u(0) - au'(0) = g_1(u), \\ u(1) + bu'(1) = g_2(u), \end{cases} \tag{1}$$

where ${}^{C}\mathfrak{D}_{0^+}^{\alpha}$ is the Caputo fractional derivative of order α with $1 < \alpha < 2$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ continuous, a and b are two positive real numbers and $g_i : C([0,1];\mathbb{R}) \to \mathbb{R}$ continuous and non-decreasing for $i = 1,2$.

Fractional differential equations are used in various scientific fields such as viscoelasticity, electrical circuits, electroanalytical chemistry, biology, control theory, electromagnetic theory, and biomedical problems (see [19], [21], [22], [24] and the references cited in [28]).

Various techniques, such as topological degree theory, the nonlinear alternative of Leray-Schauder, Mawhin's continuation theorem, Guo-Krasnoselskii's fixed point theorem, Banach's fixed point theorem, Krasnoselskii's fixed point theorem, and the Adomian decomposition method, have been employed by several authors to study fractional differential equations with nonlocal boundary conditions (see [1], [2], [3], [4], [10], [15], [16], [20], [29] and the references cited therein).

Mathematics subject classification (2020): 34A08, 34B10.
Keywords and phrases: Fractional differential equations, upper and lower solutions, monotone iterative technique, nonlocal boundary conditions.

* Corresponding author.

Paper FDC-14-04

It is well knowledge that the method of upper and lower solutions combined with the monotone iterative technique, has been utilized by several authors in order to prove the existence of solutions to fractional differential equations (see [5], [9], [11], [12], [13], [25], [27] and the cited references therein).

The purpose of this work is to demonstrate its successful application to problems of type (1). To the best of our knowledge, this is the first work which shows the existence of minimal and maximal solutions for boundary value problems of type (1) by combining the method of upper and lower solutions with the monotone iterative technique.

The plan of this paper is organized as follows: In Section 2, we give some definitions and preliminary results. The main results are presented and proved in Section 3 and finally in Section 4, we give some examples illustrating the application of our results.

2. Preliminaries

In this section, we give some definitions and preliminary results that will be used in the remainder of this paper.

DEFINITION 1. ([26, Chapter 1 page 33]) Let $q>0$ and $h\in L^1([0,1];\ \mathbb{R})$. The Riemann-Liouville integral of order q of h is defined by

$$(I_{0^+}^{q}h)(x)=\frac{1}{\Gamma(q)}\int_0^x (x-s)^{q-1}h(s)\,ds,\text{ for a.e. } x\in[0,1],$$

where Γ is the Gamma Euler function defined by

$$\Gamma(x)=\int_0^{+\infty} e^{-t}t^{x-1}dt,$$

where $x\in\mathbb{R}$ with $x>0$.

REMARK 1. If $q=0$, we put by definition $(I_{0^+}^{0}h)(x)=h(x)$.

DEFINITION 2. ([14, Definition 1.5] or [19, Chapter 1 page 2]) By $AC([0,1];\mathbb{R})$ we denote the set of functions u which are absolutely continuous on $[0,1]$.

DEFINITION 3. ([14, Definition 1.5] or [19, Chapter 1 page 2]) By $AC^1([0,1];\mathbb{R})$ we denote the set of functions u which have continuous derivative u' on $[0,1]$ such that $u'\in AC([0,1];\mathbb{R})$.

DEFINITION 4. ([14, Chapter 3 page 50] or [19, Chapter 2 page 91]) Let $1<\alpha<2$ and $h\in AC^1([0,1];\ \mathbb{R})$. The Caputo fractional derivative of order α of h is defined by

$$({}^{C}\mathfrak{D}_{0^+}^{\alpha}h)(x)=({}^{RL}\mathfrak{D}_{0^+}^{\alpha}\left(h(t)-h(0)-th'(0)\right))(x),\text{ for a.e. } x\in[0,1],$$

where ${}^{RL}\mathfrak{D}_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative defined by

$$\begin{aligned}\left({}^{RL}\mathfrak{D}_{0^+}^{\alpha}h\right)(x) &= \frac{d^2}{dx^2}\left(I_{0^+}^{2-\alpha}h\right)(x), \text{ for a.e. } x\in[0,1]\\ &= \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_0^x (x-s)^{1-\alpha}h(s)\,ds, \text{ for a.e. } x\in[0,1].\end{aligned}$$

DEFINITION 5. ([19, Chapter 1 page 42]) The Mittag–Leffler function with two parameters $E_{\delta,\mu}$ is defined by

$$E_{\delta,\mu}(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\delta n+\mu)}, \quad \delta>0, \ \ \mu\in\mathbb{R} \text{ and } x\in\mathbb{R}.$$

We have the following result.

LEMMA 1. [19, Lemma 2.21] *Let* $1<\alpha<2$ *and let* $u\in C([0,1];\mathbb{R})$, *then we have*

$$\left({}^{C}\mathfrak{D}_{0^+}^{\alpha}\circ I_{0^+}^{\alpha}u\right)(x) = u(x), \ \textit{for all} \ \ x\in[0,1].$$

NOTATION 1. *For all* $1<\alpha<2$, *we denote by* $C^{\alpha,1}([0,1];\mathbb{R})$ *the following space of functions*

$$C^{\alpha,1}([0,1];\mathbb{R}) = \left\{u\in C^1([0,1];\mathbb{R}) : {}^{C}\mathfrak{D}_0^{\alpha}u\in C([0,1];\mathbb{R})\right\}.$$

We have the following results.

LEMMA 2. *If* $u\in C^{\alpha,1}([0,1];\mathbb{R})$, *then we have*

$$\left(I_{0^+}^{\alpha}\circ {}^{C}\mathfrak{D}_{0^+}^{\alpha}u\right)(x) = u(x)-u(0)-xu'(0), \ \textit{for all} \ \ x\in[0,1].$$

Proof. The proof is similar to that of the sufficient condition of Theorem 1 in [17], so we omit it. □

THEOREM 2. ([11, Theorem 2.11 page 85]) *Let* u *such that* $u\in C^{\alpha,1}([0,1];\mathbb{R})$ *and assume that* u *attains its maximum at* $x_0\in(0,1)$, *then*

$$-\left({}^{C}\mathfrak{D}_{0^+}^{\alpha}u\right)(x_0) \geqslant \frac{x_0^{-\alpha}}{\Gamma(2-\alpha)}\left((1-\alpha)(u(0)-u(x_0))+x_0u'(0)\right).$$

REMARK 2. Theorem 2 improve Theorem 2.1 in [6] since we suppose only that $u\in C^{\alpha,1}([0,1];\mathbb{R})$.

PROPOSITION 3. *Let u such that $u \in C^{\alpha,1}([0,1];\mathbb{R})$ and assume that*

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0^+}^{\alpha}u\right)(x) + M_1 u(x) \leqslant 0, & x \in [0,1], \\ u(0) - au'(0) \leqslant 0, \\ u(1) + bu'(1) \leqslant 0, \end{cases}$$

where M_1 is a positive real number and $a \geqslant \dfrac{1}{\alpha - 1}$.

Then $u(x) \leqslant 0$, for all $x \in [0,1]$.

Proof. Assume that there exists $x_0 \in [0,1]$ such that

$$u(x_0) = \max_{x \in [0,1]} u(x) = \varepsilon > 0.$$

Since $u \in C^1([0,1];\mathbb{R})$, we have

$$u'(x_0) = 0.$$

Case 1: If $x_0 = 0$, we obtain the contradiction

$$0 < u(0) \leqslant 0.$$

Case 2: If $x_0 = 1$, we obtain the contradiction

$$0 < u(1) \leqslant 0.$$

Case 3: If $x_0 \in (0,1)$, we have

$$-\left({}^{C}\mathfrak{D}_{0^+}^{\alpha}u\right)(x_0) \leqslant -M_1 u(x_0) \leqslant 0.$$

That is

$$-\left({}^{C}\mathfrak{D}_{0^+}^{\alpha}u\right)(x_0) \leqslant 0. \tag{2}$$

On the other hand from Theorem 2, we have

$$\begin{aligned} -\left({}^{C}\mathfrak{D}_{0^+}^{\alpha}u\right)(x_0) &\geqslant \frac{x_0^{-\alpha}}{\Gamma(2-\alpha)}\left((1-\alpha)(u(0) - u(x_0)) + x_0 u'(0)\right) \\ &\geqslant \frac{x_0^{-\alpha}}{a\Gamma(2-\alpha)}\left(u(x_0) - u(0) + a x_0 u'(0)\right) \\ &\geqslant \frac{x_0^{-\alpha}}{a\Gamma(2-\alpha)}\left(u(x_0) - u(0) + x_0 u(0)\right) \\ &= \frac{x_0^{-\alpha}}{a\Gamma(2-\alpha)}\left(u(x_0) - (1-x_0)u(0)\right) \\ &\geqslant \frac{x_0^{-\alpha}(1-x_0)}{a\Gamma(2-\alpha)}\left(u(x_0) - u(0)\right) > 0. \end{aligned}$$

Which is a contradiction with (2). □

Now, we consider the following problem

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0+}^{\alpha}u\right)(x)+M_2u(x)=F(x), & x\in[0,1], \\ u(0)-au'(0)=a_1, \\ u(1)+bu'(1)=a_2, \end{cases} \tag{3}$$

where $F:[0,1]\to\mathbb{R}$ continuous, $M_2\geqslant 0$, and a_1 and a_2 are real numbers.

THEOREM 4. ([14, Theorem 7.19]) *Let u_1 and u_2 be two linearly independent solutions of the homogeneous differential equation*

$$-\left({}^{C}\mathfrak{D}_{0+}^{\alpha}u\right)(x)+M_2u(x)=0,$$

and let M_{hom} the matrix defined by

$$M_{\text{hom}}=\begin{pmatrix} u_1(0)-au_1'(0) & u_2(0)-au_2'(0) \\ u_1(1)+bu_1'(1) & u_2(1)+bu_2'(1) \end{pmatrix}.$$

(i) If $\det M_{\text{hom}}\neq 0$, then the boundary value problem

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0+}^{\alpha}u\right)(x)+M_2u(x)=0, & x\in[0,1], \\ u(0)-au'(0)=0, \\ u(1)+bu'(1)=0, \end{cases} \tag{4}$$

has only the trivial solution, and the problem (3) admits a unique solution.

(ii) If $\det M_{\text{hom}}=0$, then the boundary value problem (4) has nontrivial solutions.

PROPOSITION 5. *The problem (3) admits a unique solution $u\in C^{\alpha,1}([0,1];\mathbb{R})$.*

Proof. From Theorem 1 in [18] the functions $u_1(x)=E_{\alpha,1}(M_2x^{\alpha})$ and $u_2(x)=xE_{\alpha,2}(M_2x^{\alpha})$ are two linearly independent solutions of the homogeneous fractional differential equation

$$-\left({}^{C}\mathfrak{D}_{0+}^{\alpha}u\right)(x)+M_2u(x)=0.$$

We have

$$M_{\text{hom}}=\begin{pmatrix} u_1(0)-au_1'(0) & u_2(0)-au_2'(0) \\ u_1(1)+bu_1'(1) & u_2(1)+bu_2'(1) \end{pmatrix}.$$

Since $u_1(0)=u_2'(0)=1$, $u_1'(0)=0$, $u_1'(1)=M_2E_{\alpha,\alpha}(M_2)$ and $u_2'(1)=E_{\alpha,1}(M_2)$, we obtain

$$M_{\text{hom}}=\begin{pmatrix} 1 & -a \\ E_{\alpha,1}(M_2)+bM_2E_{\alpha,\alpha}(M_2) & E_{\alpha,2}(M_2)+bE_{\alpha,1}(M_2) \end{pmatrix}.$$

Which implies that

$$\det M_{\text{hom}} = E_{\alpha,2}(M_2) + bE_{\alpha,1}(M_2) + a(E_{\alpha,1}(M_2) + bM_2E_{\alpha,\alpha}(M_2))$$
$$= \begin{cases} 1+b+a \text{ if } M_2 = 0, \\ \sum\limits_{n=0}^{+\infty}\left(\dfrac{1}{\Gamma(\alpha n+2)} + \dfrac{a+b}{\Gamma(\alpha n+1)} + \dfrac{abM_2}{\Gamma(\alpha n+\alpha)}\right) M_2^n \text{ if } M_2 > 0. \end{cases}$$

Now taking into account that M_2, a and b are all positive real numbers, we deduce that $\det M_{\text{hom}} > 0$ and consequently according to Theorem 4, it follows that the problem (3) admits a unique solution $u \in C^{\alpha,1}([0,1];\mathbb{R})$. □

PROPOSITION 6. *Let u such that $u \in C^{\alpha,1}([0,1];\mathbb{R})$ and assume that*

$$\begin{cases} -\left({}^C\mathfrak{D}_{0+}^{\alpha}u\right)(x) \leqslant 0, \ x \in [0,1], \\ u(0) - au'(0) = \int\limits_0^1 \mathfrak{h}_1(x)\, u(x)\, dx, \\ u(1) + bu'(1) = \int\limits_0^1 \mathfrak{h}_2(x)\, u(x)\, dx, \end{cases}$$

where $\mathfrak{h}_i : [0,1] \to \mathbb{R}^+$ are continuous with $\int\limits_0^1 \mathfrak{h}_i(x)\, dx < 1$ for $i = 1,2$ and $a \geqslant \dfrac{1}{\alpha - 1}$.

Then $u(x) \leqslant 0$, for all $x \in [0,1]$.

Proof. Assume that there exists $x_0 \in [0,1]$ such that

$$u(x_0) = \max_{x\in[0,1]} u(x) = \varepsilon > 0.$$

Since $u \in C^1([0,1];\mathbb{R})$, we have

$$u'(x_0) = 0.$$

Case 1: If $x_0 = 0$, we have

$$\begin{aligned} u(0) &= \int\limits_0^1 \mathfrak{h}_1(x)\, u(x)\, dx \\ &\leqslant u(0) \int\limits_0^1 \mathfrak{h}_1(x)\, dx \\ &< u(0). \end{aligned}$$

That is

$$u(0) < u(0).$$

Which is a contradiction.

Case 2: If $x_0 = 1$, we have

$$\begin{aligned} u(1) &= \int_0^1 \mathfrak{h}_2(x)\, u(x)\, dx \\ &\leqslant u(1) \int_0^1 \mathfrak{h}_2(x)\, dx \\ &< u(1). \end{aligned}$$

That is

$$u(1) < u(1).$$

Which is a contradiction.

Case 3: $x_0 \in (0,1)$.

In this case, we have

$$-\left({}^{C}\mathfrak{D}_{0+}^{\alpha} u\right)(x_0) \leqslant 0. \tag{5}$$

On the other hand from Theorem 2, we have

$$\begin{aligned} -\left({}^{C}\mathfrak{D}_{0+}^{\alpha} u\right)(x_0) &\geqslant \frac{x_0^{-\alpha}}{\Gamma(2-\alpha)} \left((1-\alpha)\left(u(0) - u(x_0)\right) + x_0 u'(0)\right) \\ &\geqslant \frac{x_0^{-\alpha}}{a\Gamma(2-\alpha)} \left(u(x_0) - u(0) + a x_0 u'(0)\right) \\ &> \frac{x_0^{-\alpha}}{a\Gamma(2-\alpha)} \left(u(x_0) - u(0) + x_0 (u(0) - u(x_0)\right) \\ &= \frac{x_0^{-\alpha}(1-x_0)}{a\Gamma(2-\alpha)} \left(u(x_0) - u(0)\right) > 0. \end{aligned}$$

Which is a contradiction with (5). □

3. Main results

In this section, we give some definitions, state and prove our main results.

DEFINITION 6. We say that u is a solution of (1) if $u \in C^{\alpha,1}([0,1];\mathbb{R})$ and

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0+}^{\alpha} u\right)(x) = f(x,u), \ \ x \in [0,1], \\ u(0) - a u'(0) = g_1(u), \\ u(1) + b u'(1) = g_2(u). \end{cases}$$

DEFINITION 7. We say that $\underline{u}$ is a lower solution of (1) if $\underline{u} \in C^{\alpha,1}([0,1];\mathbb{R})$ and

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0+}^{\alpha} \underline{u}\right)(x) \leqslant f(x,\underline{u}), \ \ x \in [0,1], \\ \underline{u}(0) - a \underline{u}'(0) \leqslant g_1(\underline{u}), \\ \underline{u}(1) + b \underline{u}'(1) \leqslant g_2(\underline{u}). \end{cases}$$

DEFINITION 8. We say that $\overline{u}$ is an upper solution of (1) if $\overline{u} \in C^{\alpha,1}([0,1];\mathbb{R})$ and

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0+}^{\alpha}\overline{u}\right)(x) \geqslant f\left(x,\overline{u}\right), \ x \in [0,1], \\ \overline{u}(0) - au'(0) \geqslant g_1(\overline{u}), \\ \overline{u}(1) + b\overline{u}'(1) \geqslant g_2(\overline{u}). \end{cases}$$

Assume the existence of an ordered pair of lower and upper solutions $\underline{u}$ and $\overline{u}$ satisfying

$$\underline{u}(x) \leqslant \overline{u}(x), \text{ for all } x \in [0,1].$$

Also, on the nonlinearity f and the real number a, we shall impose the following conditions.

(H1) There exists a positive real number M such that $u \mapsto f(x,u) + Mu$ is non-decreasing for all $x \in [0,1]$ and $\underline{u} \leqslant u \leqslant \overline{u}$.

(H2) $a \geqslant \dfrac{1}{\alpha - 1}$.

(H3) The functionals $g_i : C([0,1];\mathbb{R}) \to \mathbb{R}$ are continuous and non-decreasing for $i = 1,2$.

The first main result of this work is

THEOREM 7. *Let $\underline{u}$ and $\overline{u}$ be lower and upper solutions respectively for problem (1) such that $\underline{u} \leqslant \overline{u}$ in $[0,1]$. Assume that the conditions (Hi) for $i = 1,2,3$ are satisfied. Then the problem (1) has a maximal solution u^* and a minimal solution u_* such that for every solution u of (1) with $\underline{u} \leqslant u \leqslant \overline{u}$ in $[0,1]$, we have*

$$\underline{u} \leqslant u_* \leqslant u \leqslant u^* \leqslant \overline{u} \text{ in } [0,1].$$

Proof. The proof will be given in several steps.
We take $\underline{u}_0 = \underline{u}$, and we define the sequence of functions $(\underline{u}_n)_{n \in \mathbb{N}}$ by

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0+}^{\alpha}\underline{u}_{n+1}\right)(x) + M\underline{u}_{n+1}(x) = f\left(x,\underline{u}_n\right) + M\underline{u}_n(x), \ x \in [0,1], \\ \underline{u}_{n+1}(0) - a\underline{u}'_{n+1}(0) = g_1(\underline{u}_n), \\ \underline{u}_{n+1}(1) + b\underline{u}_{n+1}(1) = g_2(\underline{u}_n). \end{cases} \tag{6}$$

Analogously, we take $\overline{u}_0 = \overline{u}$ and we define the sequence of functions $(\overline{u}_n)_{n \in \mathbb{N}}$ by

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0+}^{\alpha}\overline{u}_{n+1}\right)(x) + M\overline{u}_{n+1}(x) = f\left(x,\overline{u}_n\right) + M\overline{u}_n(x), \ x \in [0,1], \\ \overline{u}_{n+1}(0) - a\overline{u}'_{n+1}((0) = g_1(\overline{u}_n), \\ \overline{u}_{n+1}(1) + b\overline{u}'_{n+1}(1) = g_2(\overline{u}_n). \end{cases} \tag{7}$$

Step 1: For all $n \in \mathbb{N}^*$, we have

$$\underline{u} \leqslant \underline{u}_1 \leqslant \ldots \leqslant \underline{u}_n \leqslant \underline{u}_{n+1} \leqslant \overline{u}_{n+1} \leqslant \overline{u}_n \leqslant \ldots \leqslant \overline{u}_1 \leqslant \overline{u} \text{ in } [0,1].$$

Let

$$w_0(x) := \underline{u}_0(x) - \underline{u}_1(x), \quad x \in [0,1].$$

By (6) and using the definition of lower solution, we have

$$\begin{cases} -\left({}^C\mathfrak{D}^{\alpha}_{0+} w_0\right)(x) + M w_0(x) \leqslant 0, \quad x \in (0,1), \\ w_0(0) - a w_0'(0) \leqslant 0, \\ w_0(1) + b w_0(1) \leqslant 0. \end{cases}$$

Then from Proposition 3, we obtain

$$w_0(x) \leqslant 0, \quad \text{for all } \; x \in [0,1].$$

That is

$$\underline{u}_0 \leqslant \underline{u}_1 \;\text{ in }\; [0,1]. \tag{8}$$

Similarly, we can prove that

$$\overline{u}_1 \leqslant \overline{u}_0 \;\text{ in }\; [0,1]. \tag{9}$$

Now, we put by definition

$$p_1(x) = \underline{u}_1(x) - \overline{u}_1(x), \quad x \in [0,1].$$

Combining (6) and (7), we obtain

$$-\left({}^C\mathfrak{D}^{\alpha}_{0+} p_1\right)(x) + M p_1(x) = f_0(x), \quad \text{for all } \; x \in [0,1],$$

where

$$f_0(x) = f(x, \underline{u}) - f(x, \overline{u}) + M\left(\underline{u}(x) - \overline{u}(x)\right).$$

Since $\underline{u}_0 = \underline{u} \leqslant \overline{u} = \overline{u}_0$ in $[0,1]$ and using the hypothesis (H1), we deduce that

$$-\left({}^C\mathfrak{D}^{\alpha}_{0+} p_1\right)(x) + M p_1(x) \leqslant 0, \quad x \in [0,1]. \tag{10}$$

On the other hand, we have

$$p_1(0) - a p_1'((0) \leqslant g_1(\underline{u}) - g_1(\overline{u}).$$

Since $\underline{u} \leqslant \overline{u}$ and g_1 is increasing, we obtain

$$p_1(0) - a p_1'(0) \leqslant 0. \tag{11}$$

Similarly, we have

$$p_1(1) + b p_1'(1) \leqslant 0. \tag{12}$$

Combining (10), (11) and (12), we obtain

$$\begin{cases} -\left({}^C\mathfrak{D}^{\alpha}_{0+} p_1\right)(x) + M p_1(x) \leqslant 0, \quad x \in [0,1], \\ p_1(0) - a p_1'(0) \leqslant 0, \\ p_1(1) + b p_1'(1) \leqslant 0. \end{cases}$$

Then from Proposition 3, we obtain

$$p_1(x) \leqslant 0, \text{ for all } x \in [0,1],$$

That is

$$\underline{u}_1 \leqslant \overline{u}_1 \text{ in } [0,1], \tag{13}$$

and then by (8), (9) and (13), we obtain

$$\underline{u}_0 \leqslant \underline{u}_1 \leqslant \overline{u}_1 \leqslant \overline{u}_0 \text{ in } [0,1].$$

Assume for fixed $n \geqslant 1$, we have

$$\underline{u}_n \leqslant \underline{u}_{n+1} \leqslant \overline{u}_{n+1} \leqslant \overline{u}_n \text{ in } [0,1],$$

and we show that

$$\underline{u}_{n+1} \leqslant \underline{u}_{n+2} \leqslant \overline{u}_{n+2} \leqslant \overline{u}_{n+1} \text{ in } [0,1].$$

We put by definition

$$w_{n+1}(x) := \underline{u}_{n+1}(x) - \underline{u}_{n+2}(x), \; x \in [0,1].$$

By (6), we have

$$\begin{cases} -\left({}^C\mathfrak{D}_{0^+}^{\alpha} w_{n+1}\right)(x) + M w_{n+1}(x) = \widehat{f}_n(x), \; x \in [0,1], \\ w_{n+1}(0) - a w_{n+1}'(0) = g_1(\underline{u}_n) - g_1(\underline{u}_{n+1}), \\ w_{n+1}(1) + b w_{n+1}'(1) = g_2(\underline{u}_n) - g_2(\underline{u}_{n+1}), \end{cases}$$

where

$$\widehat{f}_n(x) = \left(f(x,\underline{u}_n) - f\left(x,\underline{u}_{n+1}\right)\right) + M\left(\underline{u}_n(x) - \underline{u}_{n+1}(x)\right), \text{ for all } x \in [0,1].$$

Since by the hypothesis of recurrence, we have $\underline{u}_n \leqslant \underline{u}_{n+1}$ in $[0,1]$ and using the hypotheses (H1) and (H3) we obtain

$$\begin{cases} -\left({}^C\mathfrak{D}_{0^+}^{\alpha} w_{n+1}\right)(x) + M w_{n+1}(x) \leqslant 0, \; x \in [0,1], \\ w_{n+1}(0) - a w_{n+1}'(0) \leqslant 0, \\ w_{n+1}(1) + b w_{n+1}'(1) \leqslant 0. \end{cases}$$

Then from Proposition 3, it follows that

$$w_{n+1}(x) \leqslant 0, \text{ for all } x \in [0,1].$$

That is

$$\underline{u}_{n+1}(x) \leqslant \underline{u}_{n+2}(x), \text{ for all } x \in [0,1]. \tag{14}$$

Similarly, we can prove that

$$\overline{u}_{n+2} \leqslant \overline{u}_{n+1} \text{ in } [0,1], \tag{15}$$

and

$$\underline{u}_{n+2} \leqslant \overline{u}_{n+2} \text{ in } [0,1]. \tag{16}$$

Combining (14), (15) and (16), we obtain

$$\underline{u}_{n+1} \leqslant \underline{u}_{n+2} \leqslant \overline{u}_{n+2} \leqslant \overline{u}_{n+1} \text{ in } [0,1],$$

and consequently for all $n \in \mathbb{N}$, we have

$$\underline{u}_{n} \leqslant \underline{u}_{n+1} \leqslant \overline{u}_{n+1} \leqslant \overline{u}_{n} \text{ in } [0,1].$$

The proof of Step 1 is complete.

Step 2: The sequence of functions $(\underline{u}_n)_{n\in\mathbb{N}}$ and $(\overline{u}_n)_{n\in\mathbb{N}}$ are uniformly bounded in $C^1([0,1];\mathbb{R})$.

Let's $n \in \mathbb{N}$ and $x \in [0,1]$, we have

$$-\left({}^{C}\mathfrak{D}^{\alpha}_{0^+}\underline{u}_{n+1}\right)(x) = f(x,\underline{u}_n) + M\left(\underline{u}_n(x) - \underline{u}_{n+1}(x)\right).$$

Then from Lemma 2, one has

$$\underline{u}_{n+1}(x) = \underline{u}_{n+1}(0) + x\underline{u}'_{n+1}(0) - \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} F_n(t)\,dt.$$

where

$$F_n(x) = f(x,\underline{u}_n) + M\left(\underline{u}_n(x) - \underline{u}_{n+1}(x)\right).$$

Which implies that

$$\underline{u}'_{n+1}(x) = \underline{u}'_{n+1}(0) - \frac{1}{\Gamma(\alpha-1)}\int_0^x (x-t)^{\alpha-2} F_n(t)\,dt.$$

That is

$$\underline{u}'_{n+1}(x) = \frac{\underline{u}_{n+1}(0) - g_1(\underline{u}_n)}{a} - \frac{1}{\Gamma(\alpha-1)}\int_0^x (x-t)^{\alpha-2} F_n(t)\,dt.$$

Since $\underline{u} \leqslant \underline{u}_n \leqslant \overline{u}$, for all $n \in \mathbb{N}$, f and g_1 are continuous, we obtain

$$\left|\underline{u}'_n(x)\right| \leqslant \frac{C_1 + C_2}{a} + \frac{C_3 + 2MC_1}{\Gamma(\alpha)},$$

where

$$C_1 = \max\left(\max_{x\in[0,1]}|\underline{u}(x)|, \max_{x\in[0,1]}|\overline{u}(x)|\right),$$

$$C_2 = \max\{|g_1(u)|,\ \underline{u} \leqslant u \leqslant \overline{u}\},$$

and

$$C_3 = \max\{|f(x,u)|,\ \ x \in [0,1] \text{ and } \underline{u} \leqslant u \leqslant \overline{u}\}.$$

If we put by definition $C = \dfrac{C_1 + C_2}{a} + \dfrac{C_3 + 2MC_1}{\Gamma(\alpha)}$, we get

$$\forall n \in \mathbb{N},\ \ \forall x \in [0,1],\ \ \left|\underline{u}_n'(x)\right| \leqslant C.$$

That is

$$\forall n \in \mathbb{N},\ \ \left\|\underline{u}_n'\right\|_0 = \max_{x\in[0,1]} \left|\underline{u}_n'(x)\right| \leqslant C.$$

Similarly, we can prove that the sequence of functions $(\overline{u}_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C^1([0,1];\mathbb{R})$.

The proof of Step 2 is complete.

Step 3: The sequence of functions $(\underline{u}_n)_{n\in\mathbb{N}}$ converges to a minimal solution of (1).
By Step 2 the sequence $(\underline{u}_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C^1([0,1];\mathbb{R})$.
Now let $\varepsilon > 0$ and $t, s \in [0,1]$ such that $t < s$, then for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\left|\underline{u}_n'(s) - \underline{u}_n'(t))\right| \\
&= \frac{1}{\Gamma(\alpha-1)} \left| \int_0^s (s-\tau)^{\alpha-2} F_n(\tau)\, d\tau - \int_0^t (t-\tau)^{\alpha-2} F_n(\tau)\, d\tau \right| \\
&= \frac{1}{\Gamma(\alpha-1)} \left| \int_0^t ((s-\tau)^{\alpha-2} - (t-\tau)^{\alpha-2}) F_n(\tau)\, d\tau + \int_t^s (s-\tau)^{\alpha-2} F_n(\tau)\, d\tau \right| \\
&\leqslant \frac{C_3 + 2MC_1}{\Gamma(\alpha-1)} \Big(\int_0^t ((t-\tau)^{\alpha-2} - (s-\tau)^{\alpha-2}) d\tau + \int_t^s (s-\tau)^{\alpha-2} d\tau \Big) \\
&= \frac{C_3 + 2MC_1}{\Gamma(\alpha-1)} \left(\frac{t^{\alpha-1}}{\alpha-1} + \frac{(s-t)^{\alpha-1}}{\alpha-1} - \frac{s^{\alpha-1}}{\alpha-1} + \frac{(s-t)^{\alpha-1}}{\alpha-1} \right) \\
&= \frac{C_3 + 2MC_1}{\Gamma(\alpha)} \left(t^{\alpha-1} - s^{\alpha-1} + 2(s-t)^{\alpha-1} \right) \\
&< \frac{2(C_3 + 2MC_1)}{\Gamma(\alpha)} (s-t)^{\alpha-1}.
\end{aligned}$$

That is

$$\left|\underline{u}_n'(s) - \underline{u}_n'(t))\right| < \frac{2(C_3 + 2MC_1)}{\Gamma(\alpha)} (s-t)^{\alpha-1}. \tag{17}$$

Now since the functions $t \mapsto t^{\alpha-1}$ is continuous, then there exists $\delta(\varepsilon) > 0$ such that if $s - t < \delta(\varepsilon)$, we have

$$(s-t)^{\alpha-1} < \frac{\Gamma(\alpha)\,\varepsilon}{2(C_3 + 2MC_1 + 1)}. \tag{18}$$

Then by (17), (18) and if we choose $s-t<\delta(\varepsilon)$, we obtain

$$\left|\underline{u}_n'(s)-\underline{u}_n'(t))\right|<\varepsilon,$$

and therefore the sequence of functions $(\underline{u}_n')_{n\in\mathbb{N}}$ is equicontinuous on $[0,1]$.

Hence by Arzelá–Ascoli theorem, there exists a subsequence $\left(\underline{u}_{n_j}\right)_{n_j\in\mathbb{N}}$ of $(\underline{u}_n)_{n\in\mathbb{N}}$ which converges in $C^1([0,1];\mathbb{R})$.

Let

$$u=\lim_{n_j\to+\infty}\underline{u}_{n_j},$$

then

$$u'=\lim_{n_j\to+\infty}\underline{u}_{n_j}'.$$

But by Step 1, the sequence $(\underline{u})_{n\in\mathbb{N}}$ is decreasing and bounded from below, then the pointwise limit of this sequence exists and it is denoted by u_*.

Hence we have $u=u_*$ and moreover, the whole sequence converges in $C^1([0,1];\mathbb{R})$ to u_*.

Let $x\in[0,1]$, then from Lemma 2 we have

$$\underline{u}_{n+1}(x)=\underline{u}_{n+1}(0)+x\underline{u}_{n+1}'(0)-\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}F_n(t)\,dt.$$

Now as $n\longrightarrow+\infty$, one has

$$\lim_{n\to+\infty}F_n(x)=f(s,u_*(s)).$$

Which implies that

$$u_*(s)=u_*(0)+xu_*'(0)-\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(s,u_*(s))ds,$$

and from Lemma 1, we obtain

$$-\left({}^C\mathfrak{D}_{0^+}^{\alpha}u_*\right)(x)=f(x,u_*),\ \ \forall x\in[0,1].$$

Also, we have

$$u_*(0)-au_*'(0)=g_1(u_*),$$

and

$$u_*(1)+bu_*'(1)=g_2(u_*).$$

Consequently u_* is a solution of (1).

Now, we are going to prove that u_* is a minimal solution; i.e., if u is another solution of (1) such that $\underline{u}\leqslant u\leqslant\overline{u}$ on $[0,1]$, then $u_*\leqslant u$ on $[0,1]$.

Since u is a lower solution of (1), then by Step 1, we have

$$\forall n \in \mathbb{N}, \ \ u \leqslant \underline{u}_n.$$

Letting $n \longrightarrow +\infty$, we obtain

$$u \leqslant \lim_{n \to +\infty} \underline{u}_n = u_*.$$

Which means that u_* is a minimal solution of problem (1).

The proof of Step 3 is complete.

Similarly, we can prove that the sequence $(\overline{u}_n)_{n \in \mathbb{N}}$ converges to a maximal solution of (1).

The proof of our result is complete. □

To prove the uniqueness of solutions for the problem (1), it is necessary to impose additional conditions on f and g_i.

On the nonlinearity f and the functional g_i, we shall impose the following additional conditions.

(H4) The function $y \longmapsto f(x,y)$ is decreasing for all $x \in [0,1]$, $\underline{u} \leqslant y \leqslant \overline{u}$.

(H5) $g_i(u) = \int_0^1 \mathfrak{h}_i(x)\, u(x)\, dx$ with $\mathfrak{h}_i : \ [0,1] \to \mathbb{R}^+$ continuous and $\int_0^1 \mathfrak{h}_i(x)\, dx < 1$ for $i = 1, 2$.

We have the following result.

THEOREM 8. *Assume that hypotheses (Hi) for $i = 1,2,4,5$ are satisfied and $\underline{u}$ and $\overline{u}$ be lower and upper solutions respectively for problem* (1) *such that $\underline{u} \leqslant \overline{u}$ in $[0,1]$. Then the problem* (1) *admits a unique solution u such that $\underline{u} \leqslant u \leqslant \overline{u}$ in $[0,1]$.*

Proof. By Theorem 7, the problem (1) admits a minimal solution u_* and a maximal solution u^* such that

$$\underline{u} \leqslant u_* \leqslant u^* \leqslant \overline{u} \ \text{ in } \ [0,1].$$

We put by definition

$$\widehat{u}(x) = u^*(x) - u_*(x), \ \ x \in [0,1].$$

We have

$$\widehat{u}(x) \geqslant 0 \ \text{ for all } \ x \in [0,1]. \tag{19}$$

Now we are going to prove that

$$\widehat{u}(x) \leqslant 0 \ \text{ for all } \ x \in [0,1].$$

We have

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0^+}^{\alpha}\widehat{u}\right)(x) = f(x,u^*(x)) - f(x,u_*(x)), \quad x\in[0,1], \\ \widehat{u}(0) - a\widehat{u}'(0) = \int\limits_0^1 \mathfrak{h}_1(x)\,\widehat{u}(x)\,dx, \\ \widehat{u}(1) + b\widehat{u}'(1) = \int\limits_0^1 \mathfrak{h}_2(x)\,\widehat{u}(x)\,dx. \end{cases}$$

Using the hypothesis (H4), we get

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0^+}^{\alpha}\widehat{u}\right)(x) \leqslant 0, \quad x\in[0,1], \\ \widehat{u}(0) - a\widehat{u}'(0) = \int\limits_0^1 \mathfrak{h}_1(x)\,\widehat{u}(x)\,dx, \\ \widehat{u}(1) + b\widehat{u}'(1) = \int\limits_0^1 \mathfrak{h}_2(x)\,\widehat{u}(x)\,dx. \end{cases}$$

Now using the hypothesis (H5), then from Proposition 6, we obtain

$$\widehat{u}(x) \leqslant 0, \text{for all } x\in[0,1].$$

and by the inequality (19), it follows that

$$\widehat{u}(x) = 0, \ \text{ for all } \ x\in[0,1].$$

That is

$$u^*(x) = u_*(x) \ \text{ for all } \ x\in[0,1],$$

and consequently it follows that the problem (1) admits a unique solution. □

REMARK 3. Theorem 8 remains valid if we replace the hypothesis (H5) by the following hypothesis

(H6) $g_i(u) = \int\limits_0^1 \mathfrak{h}_i(x)\,u(x)\,dx + \sum\limits_{j=1}^{m_i} \delta_{i,j}u(\sigma_{i,j})$, where $\mathfrak{h}_i : I \to \mathbb{R}^+$ continuous, $\delta_{i,j} \geqslant 0$ for all $1 \leqslant j \leqslant m_i$, $0 < \sigma_{i,1} < \ldots \sigma_{i,m_i} < 1$ and $\int\limits_0^1 \mathfrak{h}_i(x)\,dx + \sum\limits_{j=1}^{m_i} \delta_{i,j} < 1$ for $i = 1,2$.

4. Examples

In this section, we give some examples illustrating the application of our results.

4.1. Example 1

We consider the following problem

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0^+}^{\alpha}u\right)(x) = x - \dfrac{\Gamma(\alpha)}{2}xu(x), \quad x \in [0,1], \\ u(0) - au'(0) = \displaystyle\int_0^1 xu(x)\,dx, \\ u(1) = \displaystyle\int_0^1 xu(x)\,dx, \end{cases} \tag{20}$$

where ${}^{C}\mathfrak{D}_{0}^{\alpha}$ is the Caputo fractional derivative of order α with $1 < \alpha < 2$ and a a positive real number such that $a \geqslant \dfrac{1}{\alpha - 1}$.

We put by definition

$$\underline{u}(x) = 0 \text{ and } \overline{u}(x) = \frac{2}{\Gamma(\alpha)}, \text{ for all } x \in [0,1],$$

First it is easy to check that $\underline{u}$ and $\overline{u}$ are lower and upper solutions for the problem (20).

On the other hand it is not difficult to see that the function $(x,u) \mapsto -\dfrac{\Gamma(\alpha)}{2}xu(x) + x$ satisfies the other assumptions of Theorem 8 and consequently it follows that the problem (20) admits a unique solution u such that

$$0 \leqslant u(x) \leqslant \frac{2}{\Gamma(\alpha)}, \text{ for all } x \in [0,1].$$

4.2. Example 2

We consider the following problem

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0^+}^{\alpha}u\right)(x) = x^2 + x + 1 - e^{u(x)}, \quad x \in [0,1], \\ u(0) - au'(0) = \displaystyle\int_0^1 x^{\beta}u(x)\,dx, \\ u(1) + bu'(1) = \displaystyle\int_0^1 x^{\beta}u(x)\,dx, \end{cases} \tag{21}$$

where ${}^{C}\mathfrak{D}_{0}^{\alpha}$ is the Caputo fractional derivative of order α with $1<\alpha<2$, a a positive real number such that $a\geqslant \dfrac{1}{\alpha-1}$ and $\beta>0$.

We put by definition

$$\underline{u}(x)=0 \ \text{ and } \ \overline{u}(x)=\frac{11}{10}, \ \text{ for all } \ x\in[0,1].$$

First it is not difficult to see that $\underline{u}$ is a lower solution for the problem (21).
Now $\overline{u}$ is an upper solution for the problem (21) if we have

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0^+}^{\alpha}\overline{u}\right)(x)\geqslant x^2+x+1-e^{\overline{u}(x)}, \ x\in[0,1], \\ \overline{u}(0)-a\overline{u}'(0)\geqslant \displaystyle\int_0^1 x^{\beta}\overline{u}(x)\,dx, \\ \overline{u}(1)+b\overline{u}'(1)\geqslant \displaystyle\int_0^1 x^{\beta}\overline{u}(x)\,dx. \end{cases}$$

That is

$$\begin{cases} 0\geqslant x^2+x+1-e^{1.1}, \ x\in[0,1], \\ \dfrac{11}{10}\geqslant \dfrac{11}{10(\beta+1)}, \\ \dfrac{11}{10}\geqslant \dfrac{11}{10(\beta+1)}. \end{cases}$$

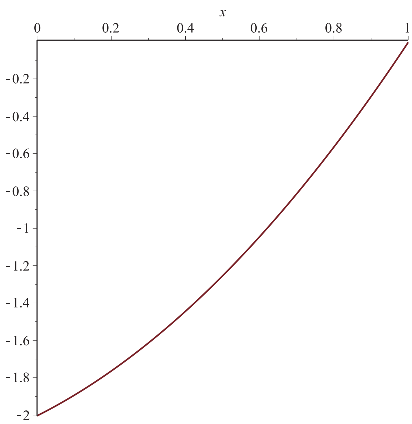

Figure 1: *Graph of the function* ϕ_1.

Since

$$\phi_1(x):=x^2+x+1-e^{1.1}\leqslant 0, \ \text{ for all } \ x\in[0,1],$$

we obtain $\overline{u}$ is an upper solution for the problem (21).

On the other hand it is not difficult to see that the function $(x,u) \mapsto x^2+x+1-e^u$ satisfies the other assumptions of Theorem 8 and consequently it follows that the problem (21) admits a unique solution u such that

$$0 \leqslant u(x) \leqslant \frac{11}{10}, \text{ for all } x \in [0,1].$$

4.3. Example 3

We consider the following problem

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0^+}^{\alpha}u\right)(x) = u(x)\left(u(x)-1\right)+\mathfrak{h}(x), \ x \in [0,1], \\ u(0) - au'(0) = \int_0^1 x^{\beta} u(x)\, dx, \\ u(1) + bu'(1) = \int_0^1 x^{\beta} u(x)\, dx, \end{cases} \tag{22}$$

where ${}^{C}\mathfrak{D}_{0}^{\alpha}$ is the Caputo fractional derivative of order α with $1<\alpha<2$, $\mathfrak{h};[0,1] \to \left[0,\frac{1}{4}\right]$ continuous, a a positive real number such that $a \geqslant \dfrac{1}{\alpha-1}$ and $\beta > 0$.

We put by definition

$$\underline{u}(x) = 0 \text{ and } \overline{u}(x) = \frac{1}{2}, \text{ for all } x \in [0,1],$$

It is easy to check that $\underline{u}$ and $\overline{u}$ are lower and upper solutions for the problem (22).

On the other hand it is not difficult to see that the function $(x,u) \mapsto \mathfrak{h}(x) + u(u-1)$ satisfies the other assumptions of Theorem 8 and consequently it follows that the problem (22) admits a unique solution u such that

$$0 \leqslant u(x) \leqslant \frac{1}{2}, \text{ for all } x \in [0,1].$$

4.4. Example 4

We consider the following problem

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0^+}^{\frac{3}{2}}u\right)(x) = u(x)\left(u(x)-\frac{1}{2}\right)(1-u(x)), \ x \in [0,1], \\ u(0) - au'(0) = 0, \\ u(1) = \frac{1}{2}u\left(\frac{4}{5}\right) + \frac{1}{3}u\left(\frac{9}{10}\right). \end{cases} \tag{23}$$

where ${}^{C}\mathfrak{D}_{0}^{\frac{3}{2}}$ is the Caputo fractional derivative of order $\dfrac{3}{2}$ and a a positive real number such that $a \geqslant 2$.

We put by definition

$$\underline{u}(x) = \frac{x^{\frac{3}{2}}}{2}, \text{ for all } x \in [0,1],$$

and

$$\overline{u}(x) = 2 - x, \text{ for all } x \in [0,1].$$

First since

$$\overline{u}(1) = 1 \geqslant \frac{1}{2}\overline{u}\left(\frac{4}{5}\right) + \frac{1}{3}\overline{u}\left(\frac{9}{10}\right) = \frac{1}{2}\left(2 - \frac{4}{5}\right) + \frac{1}{3}\left(2 - \frac{9}{10}\right) = 0.96667,$$

then it is not difficult to prove that $\overline{u}$ is an upper solution for the problem (23).

Now $\underline{u}$ is a lower solution for the problem (23) if we have

$$\begin{cases} -\left({}^{C}\mathfrak{D}_{0+}^{\frac{3}{2}}\underline{u}\right)(x) \leqslant \underline{u}(x)\left(\underline{u}(x) - \frac{1}{2}\right)(1 - \underline{u}(x)), & x \in [0,1], \\ \underline{u}(0) - a\underline{u}'(0) \leqslant 0, \\ \underline{u}(1) \leqslant \frac{1}{2}\underline{u}\left(\frac{4}{5}\right) + \frac{1}{3}\underline{u}\left(\frac{9}{10}\right) \end{cases}$$

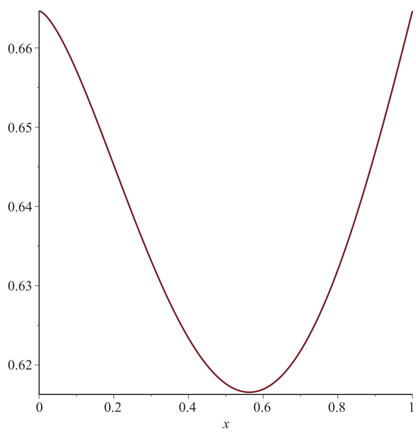

Figure 2: *Graph of the function* ϕ_2.

That is

$$\begin{cases} -\frac{\Gamma\left(\frac{5}{2}\right)}{2} \leqslant \frac{3}{8}x^3 - \frac{1}{4}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{9}{2}}, & x \in [0,1], \\ -\frac{3a}{2} \leqslant 0, \\ \frac{1}{2} \leqslant \frac{1}{2}\left(\frac{4}{5}\right)^{\frac{3}{2}} + \frac{1}{3}\underline{u}\left(\frac{9}{10}\right) = 0.64238. \end{cases}$$

Since

$$\Phi_2(x) := \frac{3}{8}x^3 - \frac{1}{4}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{9}{2}} + \frac{\Gamma\left(\frac{5}{2}\right)}{2} \geqslant 0, \text{ for all } x \in [0,1],$$

it follows that $\underline{u}$ is a lower solution for the problem (23).

On the other hand it is not difficult to see that the function $(x,u) \mapsto u(u-\frac{1}{2})(1-u)$ satisfies the other assumptions of Theorem 8 and consequently it follows that the problem (23) admits a unique solution u such that

$$\frac{x^{\frac{3}{2}}}{2} \leqslant u(x) \leqslant 2-x, \text{ for all } x \in [0,1].$$

Acknowledgements. The authors thanks the anonymous referee for several comments and suggestions which contributed to improve this paper.

REFERENCES

[1] B. AHMAD, A. ALSAEDI AND A. ALSHARIF, *Existence results for fractional-order differential equations with nonlocal multi-point-strip conditions involving Caputo derivative*, Advances in Difference Equations **348** (2015), 11 pages.
[2] B. AHMAD, J. HENDERSON AND R. LUCA, *Boundary Value Problems for Fractional Differential Equations and Systems*, Trends in Abstract and Applied Analysis Volume **9**, World Scientific Publishing Co. Pte. Ltd. 2021.
[3] B. AHMAD, J. J. NIETO AND A. ALSAEDI, *Existence and uniqueness of solutions for nonlinear fractional differential equations with non-separated type integral boundary conditions*, Acta Math. Sci. Ser. B Engl. Ed. **31** (2011), 2122–2130.
[4] A. ALI, M. SARWAR, M. B. ZADA AND K. SHAH, *Existence of solution to fractional differential equation with fractional integral type boundary conditions*, Math. Methods Appl. Sci. **44** (2021), 1615–1627.
[5] M. AL-REFAI, *Basic results on nonlinear eigenvalue problems of fractional order*, Electron. J. Differential Equations **2012** (2012), 1–12.
[6] M. AL-REFAI, *On the fractional derivatives at extreme points*, Electron. J. Qual. Theory Differ. Equ. **2012** (2012), 1–5.
[7] I. AREA AND JUAN J. NIETO, *On the fractional Allee logistic equation in the Caputo sense*, Examples and Counterexamples **4** (2023), 100121.
[8] A. BOUCHERIF AND S. M. BOUGUIMA, *Nonlocal multipoint boundary value problems*, Comm. Appl. Nonlinear Anal. **8** (1998), 73–85.
[9] Z. BAI, S. ZHANG, S. SUN AND C. YIN, *Monotone iterative method for fractional differential equations*, Electron. J. Differential Equations **2016** (2016), 1–8.
[10] A. CABADA AND G. WANG, *Positive solutions of nonlinear fractional differential equations with integral boundary value conditions*, J. Math. Anal. Appl. **389** (2012), 403–411.
[11] M. DERHAB AND M. S. IMAKHLAF, *Existence of minimal and maximal solutions for a fractional differential equation on the half-line with nonlocal initial condition and without condition at infinity*, PanAmer. Math. J. **31** (2021), 79–100.
[12] M. DERHAB AND F. MEZIANE, *Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations with integral boundary conditions*, J. Nonlinear Funct. Anal. **2018** (2018), 1–22.
[13] M. DERHAB AND F. MEZIANE, *Existence results for a class of first order fractional differential equation with a nonlocal initial condition*, PanAmer. Math. J. **32** (2022), 1–16.
[14] K. DIETHELM, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Springer, Berlin, 2010.

[15] Y. JI, W. JIANG AND J. QIU, *Solvability of fractional differential equation with integral boundary conditions at resonance*, Topol. Methods Nonlinear Anal. **42** (2013), 461–479.

[16] R. A. KHAN, M. U REHMAN AND J. HENDERSON, *Existence and uniqueness of solutions for nonlinear fractional differential equations with integral boundary conditions*, Fract. Differ. Calc. **1** (2011), 29–43.

[17] A. A. KILBAS AND S. A. MARZAN, *Cauchy problem for differential equation with Caputo derivative*, Fract. Calc. Appl. Anal. **7** (2004), 297–321.

[18] A. A. KILBAS, M. RIVERO, L. RODRÍGUEZ-GERMÁ AND J. J. TRUJILLO, *Caputo linear fractional differential equations*, Proceedings of the 2nd IFAC Workshop on Fractional Differentiation and its Applications Porto, Portugal, July 19–21, 2006.

[19] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, **204**, Elsevier Science B.V., Amsterdam, 2006.

[20] X. LIU AND Y. LIU, *Fractional differential equations with fractional non-separated integral boundary conditions*, Electron. J. Differential Equations **25** (2013), 1–13.

[21] R. L. MAGIN, *Fractional Calculus in Bioengineering*, Begell House Publishers, Inc. Connectitut, United States of America, 2006.

[22] F. MAINARDI, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, 2010.

[23] C. V. PAO, *Dynamics of reaction–diffusion equations with nonlocal boundary conditions*, Quart. Appl. Math. **53** (1995), 173–186.

[24] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[25] J. D. RAMIREZ AND A. S. VATSALA, *Monotone iterative technique for fractional differential equations with periodic boundary conditions*, Opuscula Math. *29* (2009), 289–304.

[26] S. G. SAMKO, A. A. KILBAS AND O. I. MARICHEV, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[27] Z. SHUQIN, *Monotone iterative method for initial value problem involving Riemann-Liouville fractional derivatives*, Nonlinear Anal. **71** (2009), 2087–2093.

[28] J. A. TENREIRO MACHADO, *Fractional Calculus: Models, Algorithms, Technology*, Discontinuity, Nonlinearity, and Complexity **4**, (2015), 383–389.

[29] O. K. WANASSI, R. BOURGUIBA AND D. F. M. TORRES, *Existence and uniqueness of solution for fractional differential equations boundary conditions and the Adomian decomposition method*, Math. Methods Appl. Sci. **44** (2022), 1–14.

(Received February 28, 2024)

Mohammed Derhab
Dynamic Systems and Applications Laboratory
Department of Mathematics, Faculty of Sciences
University Abou-Bekr Belkaid Tlemcen
B.P. 119, Tlemcen, 13000, Algeria
e-mail: derhab@yahoo.fr

Fatiha Meziane
Mathematics, Statistics and Physics Department
P.O. Box: 2713 Doha – Qatar
e-mail: fmeziane@qu.edu.qa
tihameziane@hotmail.fr