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A NEW APPLICATION OF CONVEX SEQUENCES

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Abstract. In this paper, an application of convex sequences dealing with $|C,\alpha|$ summability has been generalized to the $|C,\alpha,\beta;\delta|_k$ summability. Some new results have also been obtained.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by $u_n^{\alpha,\beta}$ and $t_n^{\alpha,\beta}$ the n-th Cesàro means of order (α,β), with $\alpha+\beta>-1$, of the sequence (s_n) and (na_n), respectively, i.e., (see [3])

$$u_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=0}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}s_v \tag{1}$$

$$t_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v, \tag{2}$$

where

$$A_n^{\alpha+\beta}=O(n^{\alpha+\beta}),\quad A_0^{\alpha+\beta}=1\quad and\quad A_{-n}^{\alpha+\beta}=0\quad for\quad n>0. \tag{3}$$

The series $\sum a_n$ is said to be summable $|C,\alpha,\beta|_k$, $k\geqslant 1$, if (see [5])

$$\sum_{n=1}^{\infty}n^{k-1}\mid u_n^{\alpha,\beta}-u_{n-1}^{\alpha,\beta}\mid^k=\sum_{n=1}^{\infty}\frac{1}{n}\mid t_n^{\alpha,\beta}\mid^k<\infty. \tag{4}$$

The series $\sum a_n$ is said to be summable $|C,\alpha,\beta;\delta|_k$, $k\geqslant 1$ and $\delta\geqslant 0$, if (see [2])

$$\sum_{n=1}^{\infty}n^{\delta k+k-1}\mid u_n^{\alpha,\beta}-u_{n-1}^{\alpha,\beta}\mid^k=\sum_{n=1}^{\infty}n^{\delta k-1}\mid t_n^{\alpha,\beta}\mid^k<\infty. \tag{5}$$

If we take $\delta=0$, then $|C,\alpha,\beta;\delta|_k$ summability reduces to $|C,\alpha,\beta|_k$ summability. Also, if we take $\beta=0$, then we get $|C,\alpha;\delta|_k$ summability (see [7]). Furthermore, if we take $\beta=0$ and $\delta=0$, then $|C,\alpha,\beta;\delta|_k$ summability reduces to $|C,\alpha|_k$ summability (see [6]). It should be noted that obviously the $(C,\alpha,0)$ mean is the same as the (C,α) mean. A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n\geqslant 0$, where $\Delta^2\lambda_n=\Delta\lambda_n-\Delta\lambda_{n+1}$.

In [8], Pati has proved the following theorem dealing with $|C,\alpha|$ summability factors.

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THEOREM A. *If* (λ_n) *is a convex sequence such that the series* $\sum n^{-1}\lambda_n$ *is convergent and the sequence* (θ_n^α) *defined by*

$$\theta_n^\alpha = \mid t_n^\alpha \mid, \quad \alpha = 1 \tag{6}$$

$$\theta_n^\alpha = \max_{1 \leqslant v \leqslant n} \mid t_v^\alpha \mid, \quad 0 < \alpha < 1 \tag{7}$$

satisfies the condition

$$\theta_n^\alpha = O(1)(C,1), \tag{8}$$

then the series $\sum a_n \lambda_n$ *is summable* $\mid C, \alpha \mid$ *for* $0 < \alpha \leqslant 1$.

2. The main result

The aim of this paper is to generalize Theorem A to the $\mid C, \alpha, \beta\,; \delta \mid_k$ summability. We shall prove the following theorem.

THEOREM. *If* (λ_n) *is a convex sequence such that the series* $\sum n^{-1}\lambda_n$ *is convergent and the sequence* $(\theta_n^{\alpha,\beta})$ *defined by*

$$\theta_n^{\alpha,\beta} = \mid t_n^{\alpha,\beta} \mid, \quad \alpha = 1, \quad \beta > -1 \tag{9}$$

$$\theta_n^{\alpha,\beta} = \max_{1 \leqslant v \leqslant n} \mid t_v^{\alpha,\beta} \mid, \quad 0 < \alpha < 1, \quad \beta > -1 \tag{10}$$

satisfies the condition

$$(v^\delta \theta_v^{\alpha,\beta})^k = O(1)(C,1), \tag{11}$$

then the series $\sum a_n \lambda_n$ *is summable* $\mid C, \alpha, \beta\,; \delta \mid_k$, $0 < \alpha \leqslant 1$, $\beta > -1$, $k \geqslant 1$, $\delta \geqslant 0$ *and* $\alpha + \beta - \delta > 0$. *It should be noted that if we take* $\delta = 0$, $\beta = 0$ *and k=1, then we get Theorem A.*

We need the following lemmas for the proof of our theorem.

LEMMA 1. ([4]) *If* (λ_n) *is a convex sequence such that the series* $\sum n^{-1}\lambda_n$ *is convergent, then*

$$n\Delta\lambda_n \to 0$$

$$\sum_{n=1}^{\infty} (n+1)\Delta^2\lambda_n$$

is convergent.

LEMMA 2. ([1]) *If* $0 < \alpha \leqslant 1$, $\beta > -1$ *and* $1 \leqslant v \leqslant n$, *then*

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^\beta a_p \mid \leqslant \max_{1 \leqslant m \leqslant v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^\beta a_p \mid . \tag{12}$$

3. Proof of the theorem

Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean of the sequence $(na_n\lambda_n)$. Then, by (2), we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

First applying Abel's transformation and then using Lemma 2, we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v,$$

$$| T_n^{\alpha,\beta} | \leqslant \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} | \Delta\lambda_v || \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p | + \frac{| \lambda_n |}{A_n^{\alpha+\beta}} | \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v |$$

$$\leqslant \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} | \Delta\lambda_v | + | \lambda_n | \theta_n^{\alpha,\beta} = T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}, \quad \text{say.}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\delta k-1} | T_{n,r}^{\alpha,\beta} |^k < \infty, \quad for \quad r=1,2.$$

Whenever $k>1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1} | T_{n,1}^{\alpha,\beta} |^k &\leqslant \sum_{n=2}^{m+1} n^{\delta k-1} | \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \Delta\lambda_v |^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1+(\alpha+\beta-\delta)k}} \{ \sum_{v=1}^{n-1} v^{\alpha k} v^{\beta k} \Delta\lambda_v (\theta_v^{\alpha,\beta})^k \} \times \{ \sum_{v=1}^{n-1} \Delta\lambda_v \}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \Delta\lambda_v (\theta_v^{\alpha,\beta})^k \sum_{n=v+1}^{m+1} \frac{1}{n^{1+(\alpha+\beta-\delta)k}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \Delta\lambda_v (\theta_v^{\alpha,\beta})^k \int_{v}^{\infty} \frac{dx}{x^{1+(\alpha+\beta-\delta)k}} \\
&= O(1) \sum_{v=1}^{m} \Delta\lambda_v (v^{\delta} \theta_v^{\alpha,\beta})^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(\Delta\lambda_v) \sum_{p=1}^{v} (p^{\delta} \theta_p^{\alpha,\beta})^k + O(1) \Delta\lambda_m \sum_{v=1}^{m} (v^{\delta} \theta_v^{\alpha,\beta})^k \\
&= O(1) \sum_{v=1}^{m} v \Delta^2 \lambda_v + O(1) m \Delta\lambda_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

in view of the hypotheses of the theorem and Lemma 1.

Similarly, we have that

$$\begin{aligned}\sum_{n=1}^{m} n^{\delta k-1} \mid \lambda_n \theta_n^{\alpha,\beta} \mid^k &= O(1)\sum_{n=1}^{m} \frac{\lambda_n}{n}(n^\delta \theta_n^{\alpha,\beta})^k \\ &= O(1)\sum_{n=1}^{m-1} \Delta(n^{-1}\lambda_n)\sum_{v=1}^{n}(v^\delta \theta_v^{\alpha,\beta})^k + O(1)\frac{\lambda_m}{m}\sum_{v=1}^{m}(v^\delta \theta_v^{\alpha,\beta})^k \\ &= O(1)\sum_{n=1}^{m-1}\Delta\lambda_n + O(1)\sum_{n=1}^{m-1}\frac{\lambda_{n+1}}{n+1} + O(1)\lambda_m \\ &= O(1)(\lambda_1-\lambda_m) + O(1)\sum_{n=1}^{m-1}\frac{\lambda_{n+1}}{n+1} + O(1)\lambda_m \\ &= O(1) \quad as \quad m\to\infty.\end{aligned}$$

This completes the proof of the theorem. If we take $\beta = 0$, then we get a new result for $\mid C,\alpha;\delta \mid_k$ summability factors. Also, if we take $\beta = 0$ and $\delta = 0$, then we get a result for $\mid C,\alpha \mid_k$ summability.

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