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INCLUSION PROPERTIES FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS INVOLVING MULTIPLIER TRANSFORMATION OPERATOR

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Abstract. A multiplier transformation is used to define certain new subclasses of analytic functions in the open unit disk $\mathbb{U}$. For each of these new function classes, several inclusion relationships are established. Some interesting corollaries and consequences of the main inclusion relationships are also considered.

1. Introduction

Let $\mathscr{A}$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathrm{U} = \{z \in \mathbb{C};\ |z| < 1\}$. Padamanabhan and Parvatham [8] introduced a class $P_k(\alpha)$ of functions $\tau(z)$ which are analytic in U, satisfying the properties, $\tau(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re(\tau(z)) - \alpha}{1-\alpha} \right| d\theta \leqslant k\pi, \tag{1.2}$$

where $z = re^{i\theta}, 0 \leqslant \alpha < 1$ and $k \geqslant 2$. We note that the class $P_k(0) \equiv P_k$ was studied in [9] and $P_2(\alpha) \equiv P(\alpha)$ is the class of functions with positive real part grater then α. In particular $P(0)$ is the class of functions with positive real part. We can write (1.2) as

$$\tau(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1-2\alpha) z e^{-i\theta}}{1 - z e^{-i\theta}} d\mu(\theta),$$

where $\mu(\theta)$ is a function with bounded variation on $[0, 2\pi]$, such that

$$\int_0^{2\pi} d\mu(\theta) = 2\pi \text{ and } \int_0^{2\pi} |d\mu(\theta)| \leqslant k.$$

Also for $\tau(z) \in P_k(\alpha)$, we can write from (1.2), that

$$\tau(z) = \left(\frac{k}{4} + \frac{1}{2}\right) \tau_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) \tau_2(z), \quad z \in \mathrm{U}. \tag{1.3}$$

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where $\tau_1, \tau_2 \in P(\alpha)$. For $0 \leqslant \alpha < 1$ and $0 \leqslant \beta < 1$ we define following subclasses of the class $\mathscr{A}$

$$R_k(\alpha) = \left\{ f : f \in \mathscr{A} \ \ and \ \ \frac{zf'}{f} \in P_k(\alpha), \ z \in \mathrm{U} \right\}, \tag{1.4}$$

$$V_k(\alpha) = \left\{ f : f \in \mathscr{A} \ \ and \ \ \frac{(zf')'}{f'} \in P_k(\alpha), \ z \in \mathrm{U} \right\}, \tag{1.5}$$

$$T_k(\beta, \alpha) = \left\{ f : f \in \mathscr{A}, \ g \in R_2(\alpha) \ \ and \ \ \frac{zf'}{g} \in P_k(\beta), \ z \in \mathrm{U} \right\}, \tag{1.6}$$

and

$$\overline{T}_k(\beta, \alpha) = \left\{ f : f \in \mathscr{A}, \ \ g \in V_2(\alpha) \ \ and \ \ \frac{(zf')'}{g'} \in P_k(\beta), \ z \in \mathrm{U} \right\}. \tag{1.7}$$

The classes $R_2(\alpha) = \mathscr{S}^*(\alpha)$ and $V_2(\alpha) = \mathscr{K}(\alpha)$, are respectively, the classes of starlike functions and convex functions, each of order α $(0 \leqslant \alpha < 1)$ (see, for more details [12]). Also note that, the class $\overline{T}_2(\beta, \alpha) = C^*(\beta, \alpha)$ was considered by Noor [5] and the class $\overline{T}_2(0,0) = C^*$ is the class of quasi-convex univalent functions which was first studied in [6]. It can be easily seen from the above definitions that

$$f \in V_k(\alpha) \Longleftrightarrow zf' \in R_k(\alpha) \quad \text{and} \quad f \in \overline{T}_k(\beta, \alpha) \Longleftrightarrow zf' \in T_k(\beta, \alpha).$$

Komatu [3] introduced and investigated a family of integral operator $\mathscr{Q}_a^\lambda : \mathscr{A} \to \mathscr{A}$, which is defined as follows.

$$\begin{aligned} \mathscr{Q}_a^\lambda f(z) &= \frac{a^\lambda}{\Gamma(\lambda) z^{a-1}} \int_0^z t^{a-2} \left(\log \frac{z}{t} \right)^{\lambda - 1} f(t) dt \\ &= z + \sum_{n=2}^{\infty} \left(\frac{a}{a+n-1} \right)^\lambda a_n z^n \quad (z \in \mathrm{U}, a > 0, \lambda \geqslant 0). \end{aligned} \tag{1.8}$$

We note that

(i) For $a = 1$ and $\lambda = k$ (k is an integer), the multiplier transformation operator $\mathscr{Q}_1^k f(z) = I^k f(z)$ was studied by Flett [1] and Salagean [11];

(ii) For $a = 2$ and $\lambda = k$ (k is an integer), the operator $\mathscr{Q}_2^k f(z) = L^k f(z)$ was studied by Uralegaddi and Somanatha [13];

(iii) For $a = 2$ the operator $\mathscr{Q}_2^k f(z) = I^\lambda f(z)$ was studied by Jung *et al.* [2].

Following the recent investigation by Noor [7] and Prajapat [10], we define

DEFINITION 1.1. Let $f \in \mathscr{A}$. Then $f \in R_a^\lambda(k, \alpha)$ if and only if $\mathscr{Q}_a^\lambda f \in R_k(\alpha)$, for $z \in \mathrm{U}$.

DEFINITION 1.2. Let $f \in \mathscr{A}$. Then $f \in V_a^\lambda(k, \alpha)$ if and only if $\mathscr{Q}_a^\lambda f \in V_k(\alpha)$, for $z \in \mathrm{U}$.

DEFINITION 1.3. Let $f \in \mathscr{A}$. Then $f \in T_a^\lambda(k,\beta,\alpha)$ if and only if $\mathscr{Q}_a^\lambda f \in T_k(\beta,\alpha)$, for $z \in \mathrm{U}$.

DEFINITION 1.4. Let $f \in \mathscr{A}$. Then $f \in \overline{T}_a^\lambda(k,\beta,\alpha)$ if and only if $\mathscr{Q}_a^\lambda f \in \overline{T}_k(\beta,\alpha)$, for $z \in \mathrm{U}$.

In order to derive our main results, we shall need following lemma.

LEMMA 1.1. [4] *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and suppose that $\phi(u,v)$ be a continuous complex valued function in a domain $\mathbb{D} \subset \mathbb{C}^2$ such that*

(i) $(1,0) \in \mathbb{D}$ *and* $\phi(1,0) > 0$,

(ii) $\Re(\phi(iu_2,v_1)) \leqslant 0,$ *whenever* $(iu_2,v_1) \in \mathbb{D}$ *and* $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$.

If $h(z) = 1 + \sum_{m=2}^{\infty} b_m z^m$, is a function analytic in U such that $(h(z), zh'(z)) \in \mathbb{D}$ and $\Re(\phi(h(z), zh'(z))) > 0$ for $z \in \mathrm{U}$, then $\Re(h(z)) > 0$ for $z \in \mathrm{U}$.

In this paper we shall establish certain inclusion relationships for the above mentioned function classes. Some corollaries and consequences of our main inclusion relationships are also mentioned.

2. Main results

Unless otherwise mentioned, we assumed throughout this section that $\lambda \geqslant 0$ and $a > 0$. Our first main inclusion relationship is given by Theorem 2.1 below.

THEOREM 2.1. $R_a^\lambda(k,0) \subset R_a^{\lambda+1}(k,\alpha)$, *where*

$$\alpha = \frac{2}{2a - 1 + \sqrt{4a^2 - 4a + 9}}. \tag{2.1}$$

Proof. Let $f \in R_a^\lambda(k,0)$. Then, upon setting

$$\frac{z\left(\mathscr{Q}_a^{\lambda+1} f(z)\right)'}{\mathscr{Q}_a^{\lambda+1} f(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad z \in \mathrm{U}, \tag{2.2}$$

we see that the function $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = 1$ in $z \in \mathrm{U}$. It can be seen from (1.8) that operator $\mathscr{Q}_a^\lambda$, satisfies the differential formula

$$z(\mathscr{Q}_a^{\lambda+1} f(z))' = a\mathscr{Q}_a^\lambda f(z) - (a-1)\mathscr{Q}_a^{\lambda+1} f(z). \tag{2.3}$$

Using (2.3) in (2.2), we get

$$\frac{z\left(\mathscr{Q}_a^\lambda f(z)\right)'}{\mathscr{Q}_a^\lambda f(z)} = p(z) + \frac{zp'(z)}{p(z) + a - 1}$$
$$\in P_k, \quad z \in \mathrm{U}.$$

We have

$$p(z)*\frac{\phi(z)}{z}=p(z)+\frac{zp'(z)}{p(z)+a-1}$$
$$=\sum_{i=1}^{2}(-1)^{i-1}\left(\frac{k}{4}+\frac{(-1)^{i-1}}{2}\right)\left(p_i(z)*\frac{\phi(z)}{z}\right),$$
$$\text{where } \phi(z)=\sum_{j=1}^{\infty}\frac{a-1+j}{a}z^j,$$

this implies that

$$p_i(z)+\frac{zp_i'(z)}{p_i(z)+a-1}\in P,\quad z\in \mathrm{U};\ i=1,2. \tag{2.4}$$

Now, we want to show that $p_i(z)\in P(\alpha)$, where α is given by (2.1), this will show that $p(z)\in P_k(\alpha)$ for $z\in \mathrm{U}$. Let

$$p_i(z)=(1-\alpha)h_i(z)+\alpha\quad z\in \mathrm{U};\ i=1,2. \tag{2.5}$$

Then in view of (2.4) and (2.5), we obtain

$$\Re\left((1-\alpha)h_i(z)+\alpha+\frac{(1-\alpha)zh_i'(z)}{(1-\alpha)zh_i(z)+\alpha+a-1}\right)>0\quad z\in \mathrm{U};\ i=1,2. \tag{2.6}$$

We now form a function $\phi(u,v)$ by choosing $u=h_i(z)$ and $v=zh_i'(z)$ in (2.6). Thus

$$\phi(u,v)=(1-\alpha)u+\alpha+\frac{(1-\alpha)v}{(1-\alpha)u+\alpha+a-1}. \tag{2.7}$$

We can easily see that the first two conditions of Lemma 1.1, are easily satisfied as $\phi(u,v)$ is continuous in $\mathbb{D}=\left(\mathbb{C}-\left(-\frac{\alpha+a-1}{1-\alpha}\right)\right)\times\mathbb{C}$, $(1,0)\in\mathbb{D}$ and $\mathrm{R}(\phi(1,0))>0$. Now for $v_1\leqslant-\frac{1}{2}(1+u_2^2)$, we obtain

$$\mathrm{R}(\phi(iu_2,v_1))=\alpha+\mathrm{R}\left(\frac{(1-\alpha)v_1}{(1-\alpha)iu_2+\alpha+a-1}\right)$$
$$=\alpha+\frac{(1-\alpha)(\alpha+a-1)v_1}{(\alpha+a-1)^2+(1-\alpha)^2u_2^2}.$$
$$\leqslant\alpha-\frac{1}{2}\frac{(1-\alpha)(\alpha+a-1)(1+u_2^2)}{(\alpha+a-1)^2+(1-\alpha)^2u_2^2}=\frac{A+Bu_2^2}{2C}$$

where $A=(\alpha+a-1)[2\alpha(\alpha+a-1)-(1-\alpha)]$, $B=(1-\alpha)[2\alpha(1-\alpha)-(\alpha+a-1)]$, $C=(\alpha+a-1)^2+(1-\alpha)^2u_2^2>0$. Note that $\mathrm{R}(\phi(iu_2,v_1))\leqslant 0$ if and only if, $A\leqslant 0$ and $B\leqslant 0$. From $A\leqslant 0$, we obtain α as given by (2.1) and $B\leqslant 0$ gives us $0\leqslant\alpha<1$. This completes the proof of Theorem 2.1. □

On setting $\lambda=1$ and $a=1$, Theorem 2.1 would yield the following result.

COROLLARY 2.1. *Let $f(z) \in \mathscr{A}$. If $f(z)$ satisfies the inequality.*

$$\int_0^{2\pi} \left| \Re\left(\frac{f(z)}{\int_0^z \frac{f(t)}{t} dt} \right) \right| d\theta \leqslant k\pi, \quad z \in \mathrm{U},$$

then

$$\int_0^z \frac{1}{t} \left(\log \frac{z}{t} \right) f(t) dt \in R_k(1/2), \quad z \in \mathrm{U}.$$

In its *further* special case when $k = 2$, Corollary 2.1 reduces to the

COROLLARY 2.2. *Let $f(z) \in \mathscr{A}$. If $f(z)$ satisfies the inequality*

$$\Re\left(\frac{f(z)}{\int_0^z \frac{f(t)}{t} dt} \right) > 0, \; z \in \mathrm{U},$$

then the function $\int_0^z \frac{1}{t} \left(\log \frac{z}{t} \right) f(t) dt$ *is starlike function of order* $1/2$.

Next, if we set $\lambda = 1$ and $a = 2$, in Theorem 2.1, we get

COROLLARY 2.3. *Let $f(z) \in \mathscr{A}$. If $f(z)$ satisfies the inequality*

$$\int_0^{2\pi} \left| \Re\left(\frac{zf(z)}{\int_0^z f(t) dt} - 1 \right) \right| d\theta \leqslant k\pi, \; z \in \mathbb{U},$$

then

$$\frac{4}{z} \int_0^z \left(\log \frac{z}{t} \right) f(t) dt \in R_k \left(\frac{2}{3 + \sqrt{17}} \right), \; z \in \mathbb{U}.$$

THEOREM 2.2. $V_a^\lambda(k, 0) \subset V_a^{\lambda+1}(k, \alpha)$, *where* α *is given by (2.1).*

Proof. To prove the inclusion relationship, we observe (in view of Theorem 2.1) that

$$f(z) \in V_a^\lambda(k, 0) \Longleftrightarrow zf'(z) \in R_a^\lambda(k, 0) \Longrightarrow zf'(z) \in R_a^{\lambda+1}(k, \alpha) \Longleftrightarrow f(z) \in V_a^{\lambda+1}(k, \alpha),$$

which establishes Theorem 2.2. $\square$

By putting $\lambda = 1$ and $a = 2$ in Theorem 2.2, we arrive at

COROLLARY 2.4. *Let $f(z) \in \mathscr{A}$. If $f(z)$ satisfies the inequality*

$$\left| \Re\left(\frac{z^2 f'(z)}{zf(z) - \int_0^z f(t) dt} - 1 \right) \right| \leqslant k\pi, \; z \in \mathrm{U},$$

then

$$\frac{4}{z} \int_0^z \left(\log \frac{z}{t} \right) f(t) dt \in V_k \left(\frac{2}{3 + \sqrt{17}} \right), \; z \in \mathrm{U}.$$

THEOREM 2.3. $T_a^{\lambda}(k,\beta,0) \subset T_a^{\lambda+1}(k,\gamma,\alpha)$, *where* α *is given by (2.1) and* $\gamma \leqslant \beta$ *is defined in the proof.*

Proof. Let $f(z) \in T_a^{\lambda}(k,\beta,0)$. Then there exists $g(z) \in R_a^{\lambda}(2,0)$ such that

$$\frac{z\big(\mathscr{Q}_a^{\lambda} f(z)\big)'}{\mathscr{Q}_a^{\lambda} g(z)} \in P_k(\beta), \quad z \in \mathbb{U};\ 0 \leqslant \beta < 1.$$

Let

$$\begin{aligned}\frac{z\big(\mathscr{Q}_a^{\lambda+1} f(z)\big)'}{\mathscr{Q}_a^{\lambda+1} g(z)} &= (1-\gamma)p(z)+\gamma \\ &= \sum_{i=1}^{2}(-1)^{i-1}\left(\frac{k}{4}+\frac{(-1)^{i-1}}{2}\right)[(1-\gamma)p_i(z)+\gamma],\ \ 1=1,2, \qquad (2.8)\end{aligned}$$

where $p(z)$ is analytic in $\mathbb{U}$, with $p(0)=1$. Using the identity (2.3) in (2.8), and after some computation, we obtain

$$\frac{z\big(\mathscr{Q}_a^{\lambda} f(z)\big)'}{\mathscr{Q}_a^{\lambda} g(z)} = [(1-\gamma)p(z)+\gamma] + \frac{(1-\gamma)zp'(z)}{a}\frac{\mathscr{Q}_a^{\lambda+1} g(z)}{\mathscr{Q}_a^{\lambda} g(z)}. \qquad (2.9)$$

Since $g(z) \in R_a^{\lambda}(2,0)$, then by Theorem 2.1, we know that $g(z) \in R_a^{\lambda+1}(2,\alpha)$, where α is given by (2.1). Hence there exist an analytic function $q(z)$ with $q(0)=1$, such that

$$\frac{z\big(\mathscr{Q}_a^{\lambda+1} g(z)\big)'}{\mathscr{Q}_a^{\lambda+1} g(z)} = (1-\alpha)q(z)+\alpha, \quad z \in \mathbb{U}. \qquad (2.10)$$

Then, by using identity (2.3) once again for the function $g(z)$, we have

$$a\frac{\mathscr{Q}_a^{\lambda} g(z)}{\mathscr{Q}_a^{\lambda+1} g(z)} = (1-\alpha)q(z)+\alpha+a-1. \qquad (2.11)$$

From (2.9) and (2.11), we obtain

$$\begin{aligned}\frac{z\big(\mathscr{Q}_a^{\lambda} f(z)\big)'}{\mathscr{Q}_a^{\lambda} g(z)} - \beta &= (1-\gamma)p(z)+(\gamma-\beta)+\frac{(1-\gamma)zp'(z)}{(1-\alpha)q(z)+\alpha+a-1} \\ &\in P_k. \qquad (2.12)\end{aligned}$$

We now form a function $\phi(u,v)$ by taking $u=p(z)$, $v=zp'(z)$ in (2.12) as

$$\phi(u,v) = (1-\gamma)u+(\gamma-\beta)+\frac{(1-\gamma)v}{(1-\alpha)q(z)+\alpha+a-1}. \qquad (2.13)$$

It can be easily seen that the function $\phi(u,v)$ defined by (2.13) satisfies the conditions of Lemma 2. We verify the condition (ii) as follows:

$$\begin{aligned}\Re\big(\phi(iu_2,v_1)\big) &= \gamma-\beta+\Re\left(\frac{(1-\gamma)v_1}{(1-\alpha)(q_1+iq_2)+\alpha+a-1}\right)\\ &= \gamma-\beta+\frac{(1-\gamma)[(1-\alpha)q_1+\alpha+a-1]v_1}{[(1-\alpha)q_1+\alpha+a-1]^2+(1-\alpha)^2q_2^2}\\ &\leqslant \gamma-\beta-\frac{1}{2}\frac{(1-\gamma)[(1-\alpha)+\alpha+a-1](1+u_2^2)}{[(1-\alpha)q_1+\alpha+a-1]^2+(1-\alpha)^2q_2^2}\leqslant 0,\end{aligned}$$

for $0\leqslant\beta<1$. Therefore applying Lemma 1.1, $p_i\in P$, $i=1,2$, and consequently $p\in P_k$ and thus $f\in T_a^{\lambda+1}(k,\beta,\alpha)$. □

By setting $\lambda=1$ and $a=1$ in Theorem 2.3, we immediately get the following result.

COROLLARY 2.5. *Let $f(z)\in\mathscr{A},\beta<1$ and $g\in\mathscr{S}^*$. If $f(z)$ satisfies the following inequality.*

$$\int_0^{2\pi}\left|\Re\left(\frac{\frac{f(z)}{g(z)}-\beta}{1-\beta}\right)\right|d\theta\leqslant k\pi,\ \ z\in\mathrm{U},$$

then

$$\int_0^z\frac{1}{t}\left(\log\frac{z}{t}\right)f(t)dt\in T_k(\gamma,\alpha),\ \ z\in\mathrm{U},$$

where α is given by (2.1) and $\gamma\leqslant\beta$.

THEOREM 2.4. $\overline{T}_a^{\lambda}(k,\beta,0)\subset\overline{T}_a^{\lambda+1}(k,\gamma,\alpha)$, *where γ and α are as given in Theorem 2.3.*

Proof. Proof is analogous to that of Theorem 2.3. We, therefore, choose to omit it. □

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