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APPROXIMATION BY MEANS OF HEXAGONAL FOURIER SERIES IN HÖLDER NORMS

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Abstract. In [7], it was proved that the Cesàro $(C,1)$ means and the Abel-Poisson means of Fourier series of an H-periodic continuous function f converge to it uniformly on the closure of the regular hexagon Ω. In [3], the order of convergence of these was estimated in the uniform norm, where the function belongs to the Hölder class $H_\alpha\left(\overline{\Omega}\right)$, $0<\alpha\leqslant 1$. In this work, the order of approximation of $(C,1)$ and Abel-Poisson means of functions in $H_\alpha\left(\overline{\Omega}\right)$, $0<\alpha\leqslant 1$ is investigated in the Hölder norm $\|\cdot\|_\beta$, $0\leqslant\beta<\alpha$.

1. Introduction

The theory of approximation by trigonometric polynomials is a very rich theory. There are several results on approximation of $2\pi-$periodic functions by trigonometric polynomials, in particular, the order of approximation was studied by many authors. These results can be found in the monographs [1] and [8]. The most important trigonometric polynomials used in approximation theory are the partial sums and means of Fourier series of $2\pi-$periodic functions on the real line (Cesàro means, Abel-Poisson means, de la Vallèe-Poussin means, etc.). It is known that much of the advance in the theory of trigonometric approximation is due to the periodicity of the functions.

Approximation of functions of several variables, in the tensor product case, is usually studied by assuming that the functions are $2\pi-$periodic in each of their variables. But in the case of non tensor product domain another definition of periodicity is needed. For such domains there are other definitions of periodicity, and the most notable one is the periodicity defined by the lattices. A lattice is the discrete subgroup $A\mathbb{Z}^d$ of the $d-$dimensional Euclidean space $\mathbb{R}^d$, where A is a nonsingular matrix (the generator matrix of the lattice), and the periodic function satisfies $f(x+Ak)=f(x)$ for all $k\in\mathbb{Z}^d$. With such periodicity, one works with exponentials of the form $e^{2\pi i\langle\alpha,x\rangle}$, where α and x are in proper sets of $\mathbb{R}^d$, not necessarily the usual trigonometric polynomials.

A theorem of Fuglede ([2]) states that a set tiles $\mathbb{R}^d$ by lattice translation if and only if it has an orthonormal basis of exponentials $e^{2\pi i\langle\alpha,x\rangle}$ with α in the dual lattice. Such a set is called a spectral set. This Theorem suggests that one can study Fourier series and approximation problems on a spectral set. For the simplest spectral sets, cubes in $\mathbb{R}^d$, the Fourier series with respect to the lattice coincides with the classical Fourier series of functions of d variables. Besides the usual rectangular domain in $\mathbb{R}^2$, the simplest spectral set is the regular hexagon.

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Discrete Fourier analysis on lattices was developed in [4]. In the paper [4], the case of hexagon lattice was studied in details; in particular, Lagrange interpolation and cubature formulas by trigonometric functions on a regular hexagon and on an equilateral triangle were studied. In [7], the author studied Cesàro and Abel summability of Fourier series over the regular hexagon, and deduced compact formulas for the Fejèr and Poisson kernels of hexagonal Fourier series. Furthermore, in the same paper, the direct and inverse approximation theorems were established in terms of a modulus of smoothness.

In [3], the order of convergence of Cesàro and Abel-Poisson means of functions belong to the Hölder class $H_\alpha\left(\overline{\Omega}\right)$, $0<\alpha\leqslant 1$ was studied in the uniform norm.

The aim of this work is to study the order of convergence of $(C,1)$ and Abel-Poisson means of functions belong to $H_\alpha\left(\overline{\Omega}\right)$, $0<\alpha\leqslant 1$ in the Hölder norm $\|\cdot\|_\beta$, where $0\leqslant\beta<\alpha$.

2. Hexagonal Fourier series

In this section, we shall give the definition and basic properties of hexagonal Fourier series, and functions periodic with respect to the hexagon lattice. The detailed information can be found in [4] and [7].

The generator matrix and the spectral set of the hexagonal lattice $H\mathbb{Z}^2$ are given by

$$H=\begin{bmatrix}\sqrt{3} & 0\\ -1 & 2\end{bmatrix}$$

and

$$\Omega_H=\left\{(x_1,x_2)\in\mathbb{R}^2: -1\leqslant x_2, \frac{\sqrt{3}}{2}x_1\pm\frac{1}{2}x_2<1\right\}.$$

It is more convenient to use the homogeneous coordinates (t_1,t_2,t_3) that satisfies $t_1+t_2+t_3=0$. If we define

$$t_1:=-\frac{x_2}{2}+\frac{\sqrt{3}x_1}{2},\quad t_2:=x_2,\quad t_3:=-\frac{x_2}{2}-\frac{\sqrt{3}x_1}{2}, \tag{1}$$

the hexagon Ω_H becomes

$$\Omega=\left\{(t_1,t_2,t_3)\in\mathbb{R}^3: -1\leqslant t_1,t_2,-t_3<1,\ t_1+t_2+t_3=0\right\},$$

which is the intersection of the plane $t_1+t_2+t_3=0$ with the cube $[-1,1]^3$.

We use bold letters $\mathbf{t}$ for homogeneous coordinates and we denote by $\mathbb{R}^3_H$ the plane $t_1+t_2+t_3=0$, that is

$$\mathbb{R}^3_H=\left\{\mathbf{t}=(t_1,t_2,t_3)\in\mathbb{R}^3: t_1+t_2+t_3=0\right\}.$$

Also we use the notation $\mathbb{Z}^3_H$ for the set of points in $\mathbb{R}^3_H$ with integer components, that is $\mathbb{Z}^3_H=\mathbb{Z}^3\cap\mathbb{R}^3_H$.

It follows from (1) that the Jacobian determinant of the change of variables $x = (x_1, x_2) \to \mathbf{t} = (t_1, t_2, t_3)$ is $dx_1 dx_2 = \frac{2\sqrt{3}}{3} dt_1 dt_2$.

The inner product on the hexagonal domain is defined by

$$\langle f, g \rangle_H = \frac{1}{|\Omega_H|} \int\limits_{\Omega_H} f(x_1, x_2) \overline{g(x_1, x_2)} dx_1 dx_2 = \frac{1}{|\Omega|} \int\limits_{\Omega} f(\mathbf{t}) \overline{g(\mathbf{t})} d\mathbf{t}, \tag{2}$$

where $|\Omega|$ denotes the area of Ω.

If we set

$$\phi_{\mathbf{j}}(\mathbf{t}) := e^{\frac{2\pi i}{3} \langle \mathbf{j}, \mathbf{t} \rangle}, \quad \mathbf{j} \in \mathbb{Z}_H^3, \ \mathbf{t} \in \mathbb{R}_H^3,$$

where $\langle \mathbf{j}, \mathbf{t} \rangle$ is the Euclidean inner product of $\mathbf{j}$ and $\mathbf{t}$, we have the following result.

THEOREM A. ([2]) *The set* $\left\{ \phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}_H^3 \right\}$ *is an orthonormal basis of* $L^2(\Omega)$ *with respect to the inner product (2).*

A function f is called periodic with respect to the hexagonal lattice or $H-$periodic if

$$f(x + Hk) = f(x), \quad k \in \mathbb{Z}^2.$$

If we define $\mathbf{t} \equiv \mathbf{s} \pmod 3$ as

$$t_1 - s_1 \equiv t_2 - s_2 \equiv t_3 - s_3 \pmod 3,$$

it follows that, in homogeneous coordinates, f is H-periodic if and only if $f(\mathbf{t}) = f(\mathbf{t} + \mathbf{s})$ whenever $\mathbf{s} \equiv \mathbf{0} \pmod 3$. If the function f is H-periodic then

$$\int\limits_{\Omega} f(\mathbf{t} + \mathbf{s}) d\mathbf{t} = \int\limits_{\Omega} f(\mathbf{t}) d\mathbf{t}, \quad \mathbf{s} \in \mathbb{R}_H^3.$$

It is clear that the functions $\phi_{\mathbf{j}}(\mathbf{t})$ are H-periodic.

For every natural number n, we define two subsets of $\mathbb{Z}_H^3$ by

$$\mathbb{H}_n := \left\{ \mathbf{j} = (j_1, j_2, j_3) \in \mathbb{Z}_H^3 : -n \leqslant j_1, j_2, j_3 \leqslant n \right\}$$

and

$$\mathbb{J}_n := \mathbb{H}_n \backslash \mathbb{H}_{n-1}.$$

$\mathbb{H}_n$ consists of all integer points inside the hexagon $n\overline{\Omega}$ and $\mathbb{J}_n$ is the intersection of $\mathbb{H}_n$ with the boundary of $n\Omega$. The elements of the set

$$\mathscr{H}_n := \operatorname{span}\left\{ \phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{H}_n \right\}, \quad n \in \mathbb{N}$$

are called the hexagonal trigonometric polynomials. It is clear that the dimension of $\mathscr{H}_n$ is $\#\mathbb{H}_n = 3n^2 + 3n + 1$.

The hexagonal Fourier series of an H-periodic function $f \in L^1(\Omega)$ is

$$f(\mathbf{t}) \sim \sum_{\mathbf{j} \in \mathbb{Z}_H^3} \widehat{f}_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}), \tag{3}$$

where

$$\widehat{f_{\mathbf{j}}}=\frac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t})\, e^{-\frac{2\pi i}{3}\langle \mathbf{j},\mathbf{t}\rangle}\, d\mathbf{t},\quad \mathbf{j}\in\mathbb{Z}_H^3.$$

In the study of the summability of hexagonal Fourier series it is more convenient to write the series (3) as blocks are groupped according to $\mathbb{J}_n$:

$$f(\mathbf{t})\sim\sum_{k=0}^{\infty}\sum_{\mathbf{j}\in\mathbb{J}_k}\widehat{f_{\mathbf{j}}}\phi_{\mathbf{j}}(\mathbf{t}). \tag{4}$$

The nth partial sums of the series (3) are defined by

$$S_n(f)(\mathbf{t}):=\sum_{\mathbf{j}\in\mathbb{H}_n}\widehat{f_{\mathbf{j}}}\phi_{\mathbf{j}}(\mathbf{t})=\sum_{k=0}^{n}\sum_{\mathbf{j}\in\mathbb{J}_k}\widehat{f_{\mathbf{j}}}\phi_{\mathbf{j}}(\mathbf{t}).$$

It is easy to show that

$$S_n(f)(\mathbf{t})=\frac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t}-\mathbf{s})\, D_n(\mathbf{s})\, d\mathbf{s},$$

where D_n is the Dirichlet kernel, defined by

$$D_n(\mathbf{t}):=\sum_{\mathbf{j}\in\mathbb{H}_n}\phi_{\mathbf{j}}(\mathbf{t})=\sum_{k=0}^{n}\sum_{\mathbf{j}\in\mathbb{J}_k}\phi_{\mathbf{j}}(\mathbf{t}).$$

It is known that ([6], [4]) the Dirichlet kernel has the compact formula

$$D_n(\mathbf{t})=\Theta_n(\mathbf{t})-\Theta_{n-1}(\mathbf{t}),$$

where

$$\Theta_n(\mathbf{t})=\frac{\sin\frac{(n+1)(t_1-t_2)\pi}{3}\sin\frac{(n+1)(t_2-t_3)\pi}{3}\sin\frac{(n+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}},\quad \mathbf{t}=(t_1,t_2,t_3)\in\mathbb{R}_H^3.$$

We denote by $C_H\left(\overline{\Omega}\right)$ the Banach space of $H-$periodic complex valued continuous functions, whose norm is the uniform norm:

$$\|f\|_\infty=\sup\left\{|f(\mathbf{t})|:\mathbf{t}\in\overline{\Omega}\right\}.$$

A function $f\in C_H\left(\overline{\Omega}\right)$ is said to belongs to the Hölder space $H_\alpha\left(\overline{\Omega}\right)$, $0<\alpha\leqslant 1$ if

$$\sup_{\mathbf{t}\neq\mathbf{s}}\frac{|f(\mathbf{t})-f(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^\alpha}<\infty,$$

where $\|\mathbf{t}\|=\max\{|t_1|,|t_2|,|t_3|\}$. $H_\alpha\left(\overline{\Omega}\right)$, $0<\alpha\leqslant 1$ is a Banach space with respect to the norm

$$\|f\|_\alpha:=\|f\|_\infty+\sup_{\mathbf{t},\mathbf{s}\in\mathbb{R}_H^3}\frac{|f(\mathbf{t})-f(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^\alpha}.$$

3. Approximation by Cesàro means

The Cesàro $(C,\delta),\delta\geqslant 0$ means of the Fourier series (4) are defined by

$$S_n^{(\delta)}(f)(\mathbf{t}) := \frac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t}-\mathbf{s})\,K_n^{(\delta)}(\mathbf{s})\,d\mathbf{s},$$

where

$$K_n^{(\delta)}(\mathbf{t}) := \frac{1}{A_n^{\delta}}\sum_{k=0}^{n} A_{n-k}^{\delta}\sum_{\mathbf{j}\in\mathbb{J}_k}\phi_{\mathbf{j}}(\mathbf{t}), \quad A_n^{\delta} = \binom{n+\delta}{n}.$$

It is evident that $K_n^{(0)}(\mathbf{t}) = D_n(\mathbf{t})$, hence $S_n^{(0)}(f)(\mathbf{t}) = S_n(f)(\mathbf{t})$, where

$$K_n^{(1)}(\mathbf{t}) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(\mathbf{t}) = \frac{1}{n+1}\Theta_n(\mathbf{t}).$$

By orthogonality of $\phi_{\mathbf{j}}$'s it follows that

$$\frac{1}{|\Omega|}\int_{\Omega} K_n^{(1)}(\mathbf{t})\,d\mathbf{t} = 1.$$

THEOREM B. ([3]) *If* $f\in H_\alpha\left(\overline{\Omega}\right)$ *then*

$$\left\|f-S_n^{(1)}(f)\right\|_{\infty} = \begin{cases} O\left(n^{-\alpha}\right), & 0<\alpha<1 \\ O\left(n^{-1}(\log n)^2\right), & \alpha=1. \end{cases} \tag{5}$$

The main theorem of this section is the following. Note that, in the proof of Theorem 1 and also in the proof of Theorem 2 in the next section, c will denote the positive constants which are not important for the questions involve in the paper, and in general different at each occurrence.

THEOREM 1. *Let* $f\in H_\alpha\left(\overline{\Omega}\right)$ $(0<\alpha\leqslant 1)$ *and* $0\leqslant\beta<\alpha$. *Then*

$$\left\|f-S_n^{(1)}(f)\right\|_{\beta} = \begin{cases} O\left(n^{\beta-\alpha}\right), & 0<\alpha<1 \\ O\left(n^{\beta-1}(\log n)^2\right), & \alpha=1. \end{cases}$$

Proof. $f\in H_\alpha\left(\overline{\Omega}\right)$ implies

$$|f(\mathbf{t})-f(\mathbf{t}-\mathbf{u})-f(\mathbf{s})+f(\mathbf{s}-\mathbf{u})| \leqslant M\|\mathbf{t}-\mathbf{s}\|^{\alpha} \tag{6}$$

and

$$|f(\mathbf{t})-f(\mathbf{t}-\mathbf{u})-f(\mathbf{s})+f(\mathbf{s}-\mathbf{u})| \leqslant M\|\mathbf{u}\|^{\alpha}, \tag{7}$$

where M is a positive constant.

If we set $R_n(\mathbf{t}) := f(\mathbf{t}) - S_n^{(1)}(f)(\mathbf{t})$, then we get

$$R_n(\mathbf{t}) - R_n(\mathbf{s}) = \frac{1}{|\Omega|}\int_\Omega \left(f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})\right) K_n^{(1)}(\mathbf{u})\, d\mathbf{u}.$$

Thus

$$\begin{aligned}|R_n(\mathbf{t}) - R_n(\mathbf{s})| &\leqslant \frac{1}{|\Omega|}\int_\Omega |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \left|K_n^{(1)}(\mathbf{u})\right| d\mathbf{u} \\ &= \frac{1}{(n+1)|\Omega|}\int_\Omega |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \, |\Theta_n(\mathbf{u})|\, d\mathbf{u}.\end{aligned}$$

It is known that ([7], [3])

$$\int_\Omega |\Theta_n(\mathbf{u})|\, d\mathbf{u} \leqslant cn \tag{8}$$

and

$$\int_\Omega \|\mathbf{u}\|^\alpha |\Theta_n(\mathbf{u})|\, d\mathbf{u} \leqslant c \begin{cases} n^{-\alpha+1}, & 0<\alpha<1 \\ (\log n)^2, \alpha = 1. \end{cases} \tag{9}$$

Let

$$I_n := \int_\Omega |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \, |\Theta_n(\mathbf{u})|\, d\mathbf{u}.$$

By (6),

$$\begin{aligned}(I_n)^{\frac{\beta}{\alpha}} &= \left(\int_\Omega |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \, |\Theta_n(\mathbf{u})|\, d\mathbf{u}\right)^{\frac{\beta}{\alpha}} \\ &\leqslant \left(\int_\Omega M\|\mathbf{t}-\mathbf{s}\|^\alpha |\Theta_n(\mathbf{u})|\, d\mathbf{u}\right)^{\frac{\beta}{\alpha}} = M^{\beta/\alpha}\|\mathbf{t}-\mathbf{s}\|^\beta \left(\int_\Omega |\Theta_n(\mathbf{u})|\, d\mathbf{u}\right)^{\frac{\beta}{\alpha}},\end{aligned}$$

and by (7),

$$\begin{aligned}(I_n)^{1-\frac{\beta}{\alpha}} &= \left(\int_\Omega |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \, |\Theta_n(\mathbf{u})|\, d\mathbf{u}\right)^{1-\frac{\beta}{\alpha}} \\ &\leqslant \left(\int_\Omega M\|\mathbf{u}\|^\alpha |\Theta_n(\mathbf{u})|\, d\mathbf{u}\right)^{1-\frac{\beta}{\alpha}} = M^{1-\frac{\beta}{\alpha}} \left(\int_\Omega \|\mathbf{u}\|^\alpha |\Theta_n(\mathbf{u})|\, d\mathbf{u}\right)^{1-\frac{\beta}{\alpha}}.\end{aligned}$$

Let $0<\alpha<1$. By considering (8) we obtain

$$(I_n)^{\frac{\beta}{\alpha}} \leqslant c\|\mathbf{t}-\mathbf{s}\|^\beta n^{\frac{\beta}{\alpha}},$$

and taking into account (9) yields

$$(I_n)^{1-\frac{\beta}{\alpha}} \leqslant c\left(n^{-\alpha+1}\right)^{1-\frac{\beta}{\alpha}} = cn^{1-\frac{\beta}{\alpha}} n^{\beta-\alpha}.$$

Hence we get

$$I_n = (I_n)^{\frac{\beta}{\alpha}} (I_n)^{1-\frac{\beta}{\alpha}} \leqslant c \|\mathbf{t}-\mathbf{s}\|^{\beta} n^{\beta-\alpha+1},$$

which implies

$$|R_n(\mathbf{t}) - R_n(\mathbf{s})| \leqslant c \|\mathbf{t}-\mathbf{s}\|^{\beta} n^{\beta-\alpha}.$$

Taking into account (5) and the last inequality we get

$$\begin{aligned} \left\|f - S_n^{(1)}(f)\right\|_{\beta} &= \left\|f - S_n^{(1)}(f)\right\|_{\infty} + \sup_{\mathbf{t},\mathbf{s}\in\mathbb{R}_H^3} \frac{|R_n(\mathbf{t}) - R_n(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^{\beta}} \\ &\leqslant cn^{-\alpha} + cn^{\beta-\alpha} \leqslant cn^{\beta-\alpha}. \end{aligned}$$

Now let $\alpha = 1$. The inequality (8) yields

$$(I_n)^{\beta} \leqslant c \|\mathbf{t}-\mathbf{s}\|^{\beta} \left(\int_{\Omega} |\Theta_n(\mathbf{u})| \, d\mathbf{u} \right)^{\beta} \leqslant c \|\mathbf{t}-\mathbf{s}\|^{\beta} n^{\beta},$$

where (9) gives

$$(I_n)^{1-\beta} \leqslant c \left(\int_{\Omega} \|\mathbf{u}\| \, |\Theta_n(\mathbf{u})| \, d\mathbf{u} \right)^{1-\beta} \leqslant c(\log n)^{2(1-\beta)}.$$

Thus

$$I_n = (I_n)^{\beta} (I_n)^{1-\beta} \leqslant c \|\mathbf{t}-\mathbf{s}\|^{\beta} n^{\beta} (\log n)^{2(1-\beta)},$$

and hence

$$|R_n(\mathbf{t}) - R_n(\mathbf{s})| \leqslant c \|\mathbf{t}-\mathbf{s}\|^{\beta} n^{\beta-1} (\log n)^{2(1-\beta)}.$$

By combining the last inequality with (5) we obtain

$$\begin{aligned} \left\|f - S_n^{(1)}(f)\right\|_{\beta} &= \left\|f - S_n^{(1)}(f)\right\|_{\infty} + \sup_{\mathbf{t},\mathbf{s}\in\mathbb{R}_H^3} \frac{|R_n(\mathbf{t}) - R_n(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^{\beta}} \\ &\leqslant c\frac{(\log n)^2}{n} + cn^{\beta-1} (\log n)^{2(1-\beta)} \\ &\leqslant cn^{\beta-1} (\log n)^2. \end{aligned}$$

The analogue of Theorem 1 for $(C,1)$ means of Classical Fourier series was obtained in [5].

4. Approximation by Abel-Poisson means

The Abel-Poisson means of an H-periodic function $f \in L^1(\Omega)$ are defined by

$$U_r(f)(\mathbf{t}) := \frac{1}{|\Omega|} \int\limits_{\Omega} f(\mathbf{t}-\mathbf{s}) P_r(\mathbf{s})\, d\mathbf{s},$$

where

$$P_r(\mathbf{t}) := \sum_{k=0}^{\infty} \sum_{\mathbf{j} \in \mathbb{J}_k} r^k \phi_{\mathbf{j}}(\mathbf{t}), \quad 0 \leqslant r < 1$$

is the Poisson kernel. It is clear that if the function f has the Fourier series (4) then

$$U_r(f)(\mathbf{t}) = \sum_{k=0}^{\infty} \sum_{\mathbf{j} \in \mathbb{J}_k} r^k \widehat{f}_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}).$$

The Poisson kernel is nonnegative and satisfies

$$\frac{1}{|\Omega|} \int\limits_{\Omega} P_r(\mathbf{t})\, d\mathbf{t} = 1.$$

Also,

$$\begin{aligned} P_r(\mathbf{t}) \leqslant & \frac{2(1-r)^2}{q_r\left(\frac{2\pi(t_1-t_2)}{3}\right) q_r\left(\frac{2\pi(t_2-t_3)}{3}\right)} + \frac{2(1-r)^2}{q_r\left(\frac{2\pi(t_2-t_3)}{3}\right) q_r\left(\frac{2\pi(t_3-t_1)}{3}\right)} \\ & + \frac{2(1-r)^2}{q_r\left(\frac{2\pi(t_3-t_1)}{3}\right) q_r\left(\frac{2\pi(t_1-t_2)}{3}\right)} =: Q_r(\mathbf{t}), \end{aligned}$$

where $q_r(t) = 1 + r^2 - 2r\cos t$ (see [7]).

THEOREM C. ([3]) *If $f \in H_\alpha(\overline{\Omega})$ then*

$$\|f - U_r(f)\|_\infty = \begin{cases} O\left((1-r)^\alpha\right), & 0 < \alpha < 1 \\ O\left((1-r)(\log(1-r))^2\right), & \alpha = 1 \end{cases} \tag{10}$$

for $r \to 1-$.

Our new result is the following.

THEOREM 2. *Let $f \in H_\alpha(\overline{\Omega})$ $(0 < \alpha \leqslant 1)$ and $0 \leqslant \beta < \alpha$. Then*

$$\|f - U_r(f)\|_\beta = \begin{cases} O\left((1-r)^{\alpha-\beta}\right), & 0 < \alpha < 1 \\ O\left((1-r)^{1-\beta}(\log(1-r))^2\right), & \alpha = 1 \end{cases}$$

for $r \to 1-$.

Proof. $R_r(\mathbf{t}) := f(\mathbf{t}) - U_r(f)(\mathbf{t})$. Since

$$R_r(\mathbf{t}) - R_r(\mathbf{s}) = \frac{1}{|\Omega|}\int_{\Omega} (f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u}))\, P_r(\mathbf{u})\, d\mathbf{u},$$

we have

$$|R_r(\mathbf{t}) - R_r(\mathbf{s})| \leqslant \frac{1}{|\Omega|}\int_{\Omega} |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})|\, P_r(\mathbf{u})\, d\mathbf{u} =: J_r.$$

It is known that ([3])

$$\int_{\Omega} \|\mathbf{u}\|^{\alpha} Q_r(\mathbf{u})\, d\mathbf{u} \leqslant c \begin{cases} (1-r)^{\alpha}, & 0<\alpha<1 \\ (1-r)(\log(1-r))^2, & \alpha = 1. \end{cases} \tag{11}$$

$$\begin{aligned}(J_r)^{\frac{\beta}{\alpha}} &= \left(\frac{1}{|\Omega|}\int_{\Omega} |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})|\, P_r(\mathbf{u})\, d\mathbf{u}\right)^{\frac{\beta}{\alpha}} \\ &\leqslant M^{\frac{\beta}{\alpha}} \|\mathbf{t}-\mathbf{s}\|^{\beta} \left(\frac{1}{|\Omega|}\int_{\Omega} P_r(\mathbf{u})\, d\mathbf{u}\right)^{\frac{\beta}{\alpha}} = M^{\frac{\beta}{\alpha}} \|\mathbf{t}-\mathbf{s}\|^{\beta}.\end{aligned}$$

$$\begin{aligned}(J_r)^{1-\frac{\beta}{\alpha}} &= \left(\frac{1}{|\Omega|}\int_{\Omega} |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})|\, P_r(\mathbf{u})\, d\mathbf{u}\right)^{1-\frac{\beta}{\alpha}} \\ &\leqslant \left(\frac{M}{|\Omega|}\right)^{1-\frac{\beta}{\alpha}} \left(\int_{\Omega} \|\mathbf{u}\|^{\alpha} P_r(\mathbf{u})\, d\mathbf{u}\right)^{1-\frac{\beta}{\alpha}} \\ &\leqslant \left(\frac{M}{|\Omega|}\right)^{1-\frac{\beta}{\alpha}} \left(\int_{\Omega} \|\mathbf{u}\|^{\alpha} Q_r(\mathbf{u})\, d\mathbf{u}\right)^{1-\frac{\beta}{\alpha}}.\end{aligned}$$

Let $0<\alpha<1$. By (11) we get

$$(J_r)^{1-\frac{\beta}{\alpha}} \leqslant c\left(\frac{M}{|\Omega|}\right)^{1-\frac{\beta}{\alpha}} \left((1-r)^{\alpha}\right)^{1-\frac{\beta}{\alpha}} = c(1-r)^{\alpha-\beta}.$$

Hence

$$J_r = (J_r)^{\frac{\beta}{\alpha}} (J_r)^{1-\frac{\beta}{\alpha}} \leqslant c\|\mathbf{t}-\mathbf{s}\|^{\beta} (1-r)^{\alpha-\beta}.$$

Taking into account (10) and this inequality we obtain

$$\|f-U_r(f)\|_\beta = \|f-U_r(f)\|_\infty + \sup_{\mathbf{t},\mathbf{s}\in\mathbb{R}^3_H} \frac{|R_r(\mathbf{t})-R_r(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^\beta}$$
$$\leqslant c(1-r)^\alpha + c(1-r)^{\alpha-\beta} \leqslant c(1-r)^{\alpha-\beta}.$$

In the case $\alpha=1$, (11) yields

$$J_r = (J_r)^\beta (J_r)^{1-\beta} \leqslant c\|\mathbf{t}-\mathbf{s}\|^\beta \left((1-r)(\log(1-r))^2\right)^{1-\beta}.$$

Thus, using (10), we obtain

$$\|f-U_r(f)\|_\beta = \|f-U_r(f)\|_\infty + \sup_{\mathbf{t},\mathbf{s}\in\mathbb{R}^3_H} \frac{|R_r(\mathbf{t})-R_r(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^\beta}$$
$$\leqslant c\left((1-r)(\log(1-r))^2\right) + c\left((1-r)(\log(1-r))^2\right)^{1-\beta}$$
$$\leqslant c(1-r)^{1-\beta}(\log(1-r))^2,$$

which finishes the proof.

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