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NEW CRITERIA FOR MULTIVALENTLY MEROMORPHIC CONVEX FUNCTIONS OF ORDER α

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Abstract. Let $\mathscr{J}_{n+p-1}(\alpha)$ $(p \in \mathbb{N},\ n > -p,\ 0 \leqslant \alpha < p)$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \frac{a_1}{z^{p-2}} + \cdots$$

which are regular and p-valent in the punctured unit disc

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$$

and satisfy the condition

$$\operatorname{Re}\left\{ \frac{(D^{n+p} f(z))'}{(D^{n+p-1} f(z))'} - (p+1) \right\} < -\frac{p(n+p-1)+\alpha}{n+p},$$

where

$$D^{n+p-1} f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z).$$

It is proved that $\mathscr{J}_{n+p}(\alpha) \subset \mathscr{J}_{n+p-1}(\alpha)$. Since $\mathscr{J}_0(\alpha)$ is the class of p-valently meromorphic convex functions of order α $(0 \leqslant \alpha < p)$, all functions in $\mathscr{J}_{n+p-1}(\alpha)$ are p-valently meromorphic convex of order α. Further, we consider the integral operators of functions in $\mathscr{J}_{n+p-1}(\alpha)$.

1. Introduction and definitions

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\}$$

be the set of positive integers.

We let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \frac{a_1}{z^{p-2}} + \cdots \quad (p \in \mathbb{N})$$

which are regular and p-valent in the punctured unit disc

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \backslash \{0\}.$$

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In particular, we set

$$\Sigma_1 = \Sigma.$$

A function $f \in \Sigma_p$ is said to be p-valently meromorphic convex of order α $(0 \leqslant \alpha < p)$, that is, $f \in \mathcal{MK}_p(\alpha)$ if it satisfies the following inequality:

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < -\alpha \quad (z \in \mathbb{U}^*).$$

The Hadamard product (or convolution) of two functions $f, g \in \Sigma_p$ will be denoted by $f * g$. Let

$$\begin{aligned} D^{n+p-1} f(z) &= \frac{1}{z^p (1-z)^{n+p}} * f(z) \\ &= \frac{1}{z^p} \left[\frac{z^{n+2p-1} f(z)}{(n+p-1)!} \right]^{(n+p-1)} \\ &= \frac{1}{z^p} + \frac{n+p}{z^{p-1}} a_0 + \frac{(n+p)(n+p+1)}{2! z^{p-2}} a_1 + \cdots, \end{aligned}$$

where n is an integer and $n > -p$.

In [1], Aouf and Srivastava defined the subclass $\mathscr{J}_{n+p-1}(\alpha)$ of Σ_p consisting of functions f which satisfy the following inequality:

$$\operatorname{Re}\left\{ \frac{(D^{n+p} f(z))'}{(D^{n+p-1} f(z))'} - (p+1) \right\} < -\frac{p(n+p-1)+\alpha}{n+p} \tag{1.1}$$

$$(p \in \mathbb{N};\, n > -p;\, 0 \leqslant \alpha < p;\, z \in \mathbb{U}^*).$$

Various special cases of the class $\mathscr{J}_{n+p-1}(\alpha)$ were considered by many earlier researchers on this topic of Geometric Function Theory. For example, we have the following relationships with the classes which were studied in some of these earlier works:

$$\mathscr{J}_{n+p-1}(0) \equiv \mathscr{J}_{n+p-1} \qquad (\text{see } [8]),$$

$$\mathscr{J}_{n+1-1}(\alpha) \equiv \mathscr{J}_{n}(\alpha) \qquad (\text{see } [3]),$$

$$\mathscr{J}_{n+1-1}(0) \equiv \mathscr{J}_{n} \qquad (\text{see } [7]).$$

In this paper, it is proved that the class $\mathscr{J}_{n+p-1}(\alpha)$ consisting of functions in Σ_p satisfying (1.1) holds the relationship

$$\mathscr{J}_{n+p}(\alpha) \subset \mathscr{J}_{n+p-1}(\alpha) \qquad (n > -p;\, 0 \leqslant \alpha < p). \tag{1.2}$$

Since

$$\mathscr{J}_0(\alpha) = \mathcal{MK}_p(\alpha),$$

it follows from (1.2) that all functions in $\mathscr{J}_{n+p-1}(\alpha)$ are p-valently meromorphic convex of order α.

In particular, we set

$$\mathscr{J}_0(0) = \mathscr{M}\mathscr{K}_p(0) = \mathscr{M}\mathscr{K}_p$$

for the p-valently meromorphic convex function class, and

$$\mathscr{M}\mathscr{K}_1 = \mathscr{M}\mathscr{K}$$

for the meromorphic convex function class.

Further, for $c > p-1$, let

$$F(z) = \frac{c-p+1}{z^{c+1}} \int_0^z t^c f(t)\, dt.$$

It is shown that $F \in \mathscr{J}_{n+p-1}(\alpha)$ whenever $f \in \mathscr{J}_{n+p-1}(\alpha)$. Also it is shown that if $f \in \mathscr{J}_{n+p-1}(\alpha)$, then

$$F(z) = \frac{n+p}{z^{n+2p}} \int_0^z t^{n+2p-1} f(t)\, dt$$

belongs to the class $\mathscr{J}_{n+p}(\alpha)$.

2. The class $\mathscr{J}_{n+p-1}(\alpha)$

In order to prove our main results, we shall need the following lemma due to Miller and Mocanu [6]. Aouf and Srivastava [1] have obtained similar results by using lemma due to Jack [5].

LEMMA 2.1. *Let $\Psi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be satisfy the following condition*

$$\operatorname{Re}\{\Psi(is,t)\} \leqslant 0, \qquad \left(s,t \in \mathbb{R};\, t \leqslant -\frac{1+s^2}{2}\right).$$

If the function $h(z) = 1 + h_1 z + h_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ and

$$\operatorname{Re}\left\{\Psi\left(h(z), zh'(z)\right)\right\} > 0 \quad (z \in \mathbb{U}),$$

then

$$\operatorname{Re}\{h(z)\} > 0 \quad (z \in \mathbb{U}).$$

THEOREM 2.2. *$\mathscr{J}_{n+p}(\alpha) \subset \mathscr{J}_{n+p-1}(\alpha)$ where $0 \leqslant \alpha < p$, $n > -p$.*

Proof. Let $f \in \mathscr{J}_{n+p}(\alpha)$. Then

$$\operatorname{Re}\left\{\frac{\left(D^{n+p+1} f(z)\right)'}{\left(D^{n+p} f(z)\right)'} - (p+1)\right\} < -\frac{p(n+p)+\alpha}{n+p+1}. \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$\operatorname{Re}\left\{\frac{\left(D^{n+p}f(z)\right)'}{\left(D^{n+p-1}f(z)\right)'}-(p+1)\right\}<-\frac{p(n+p-1)+\alpha}{n+p}.$$

Let us define the function h in $\mathbb{U}$ by

$$h(z)=\frac{n+2p-\alpha}{p-\alpha}-\frac{n+p}{p-\alpha}\frac{\left(D^{n+p}f(z)\right)'}{\left(D^{n+p-1}f(z)\right)'}. \tag{2.2}$$

Clearly h is regular and $h(0)=1$. The above equation can be written as

$$\frac{\left(D^{n+p}f(z)\right)'}{\left(D^{n+p-1}f(z)\right)'}=\frac{(n+2p-\alpha)-(p-\alpha)h(z)}{n+p}. \tag{2.3}$$

Differentiating (2.3) logarithmically, we obtain

$$\frac{\left(D^{n+p}f(z)\right)''}{\left(D^{n+p}f(z)\right)'}-\frac{\left(D^{n+p-1}f(z)\right)''}{\left(D^{n+p-1}f(z)\right)'}=\frac{-(p-\alpha)h'(z)}{(n+2p-\alpha)-(p-\alpha)h(z)}. \tag{2.4}$$

From the identity

$$z\left(D^{n+p-1}f(z)\right)'=(n+p)D^{n+p}f(z)-(n+2p)D^{n+p-1}f(z), \tag{2.5}$$

we get

$$\begin{aligned} z\left(D^{n+p-1}f(z)\right)''&=(n+p)\left(D^{n+p}f(z)\right)' \\ &\quad-(n+2p+1)\left(D^{n+p-1}f(z)\right)'. \end{aligned} \tag{2.6}$$

Using (2.3), (2.5) and (2.6), the equation (2.4) can be written as

$$\begin{aligned} &(n+p+1)\left\{\frac{\left(D^{n+p+1}f(z)\right)'}{\left(D^{n+p}f(z)\right)'}-(p+1)\right\}+(p(n+p)+\alpha)+(p-\alpha)h(z) \\ &=\frac{-(p-\alpha)zh'(z)}{(n+2p-\alpha)-(p-\alpha)h(z)} \end{aligned}$$

or equivalently

$$\begin{aligned} &h(z)+\frac{zh'(z)}{(n+2p-\alpha)-(p-\alpha)h(z)} \\ &=\frac{n+p+1}{p-\alpha}\left(-\frac{\left(D^{n+p+1}f(z)\right)'}{\left(D^{n+p}f(z)\right)'}+(p+1)-\frac{p(n+p)+\alpha}{n+p+1}\right). \end{aligned} \tag{2.7}$$

Since $f\in\mathscr{I}_{n+p}(\alpha)$, from (2.1) and (2.7) we obtain

$$\operatorname{Re}\left\{h(z)+\frac{zh'(z)}{(n+2p-\alpha)-(p-\alpha)h(z)}\right\}>0.$$

We define the function Ψ by

$$\Psi(u,v) = u + \frac{v}{(n+2p-\alpha)-(p-\alpha)u}.$$

Then we have

$$\mathrm{Re}\left\{\Psi\left(h(z), zh'(z)\right)\right\} > 0$$

and

$$\begin{aligned}\mathrm{Re}\{\Psi(is,t)\} &= \frac{t(n+2p-\alpha)}{(n+2p-\alpha)^2+(p-\alpha)^2 s^2} \\ &\leqslant -\frac{(n+2p-\alpha)}{(n+2p-\alpha)^2+(p-\alpha)^2 s^2}\frac{1+s^2}{2} < 0.\end{aligned}$$

So, by Lemma 2.1, we conclude that $\mathrm{Re}\{h(z)\} > 0$. It follows from (2.2) and (1.1) that

$$f \in \mathscr{J}_{n+p-1}(\alpha),$$

which evidently completes the proof of Theorem 2.2. □

3. Integral operators

THEOREM 3.1. *Let $f \in \Sigma_p$ satisfy the condition*

$$\mathrm{Re}\left\{\frac{(D^{n+p}f(z))'}{(D^{n+p-1}f(z))'} - (p+1)\right\} < \frac{(p-\alpha)-2(p(n+p-1)+\alpha)(c+1-\alpha)}{2(n+p)(c+1-\alpha)} \tag{3.1}$$

for $0 \leqslant \alpha < p$, $n > -p$ and $c > p-1$. Then the integral operator

$$F(z) = \frac{c-p+1}{z^{c+1}} \int_0^z t^c f(t)\,dt \tag{3.2}$$

belongs to the class $\mathscr{J}_{n+p-1}(\alpha)$.

Proof. From the definition of F, we have

$$z\left(D^{n+p-1}F(z)\right)' = (c-p+1)D^{n+p-1}f(z) - (c+1)D^{n+p-1}F(z). \tag{3.3}$$

Using (3.3) and the identity (2.5) for the function F, the condition (3.1) can be written as

$$\begin{aligned}\mathrm{Re}&\left\{\frac{(n+p+1)\frac{(D^{n+p+1}F(z))'}{(D^{n+p}F(z))'} - (n+2p-c)}{(n+p)-(n+2p-c-1)\frac{(D^{n+p-1}F(z))'}{(D^{n+p}F(z))'}} - (p+1)\right\} \\ &< -\frac{(p-\alpha)-2(p(n+p-1)+\alpha)(c+1-\alpha)}{2(n+p)(c+1-\alpha)}.\end{aligned} \tag{3.4}$$

We have to prove that (3.4) implies the inequality

$$\operatorname{Re}\left\{\frac{(D^{n+p}F(z))'}{(D^{n+p-1}F(z))'}-(p+1)\right\}<-\frac{p(n+p-1)+\alpha}{n+p}.$$

Let us define the function h in $\mathbb{U}$ by

$$h(z)=\frac{n+2p-\alpha}{p-\alpha}-\frac{n+p}{p-\alpha}\frac{(D^{n+p}F(z))'}{(D^{n+p-1}F(z))'}. \tag{3.5}$$

Clearly h is regular and $h(0)=1$. The above equation can be written as

$$\frac{(D^{n+p}F(z))'}{(D^{n+p-1}F(z))'}=\frac{(n+2p-\alpha)-(p-\alpha)h(z)}{n+p}. \tag{3.6}$$

Differentiating (3.6) logarithmically, we obtain

$$\frac{(D^{n+p}F(z))''}{(D^{n+p}F(z))'}-\frac{\left(D^{n+p-1}F(z)\right)''}{(D^{n+p-1}F(z))'}=\frac{-(p-\alpha)h'(z)}{(n+2p-\alpha)-(p-\alpha)h(z)}.$$

For the function F, the identity (2.6) becomes

$$z\left(D^{n+p-1}F(z)\right)''=(n+p)\left(D^{n+p}F(z)\right)'-(n+2p+1)\left(D^{n+p-1}F(z)\right)'. \tag{3.7}$$

Using (3.6) and (3.7), we get

$$\begin{aligned}&\frac{(n+p+1)\frac{\left(D^{n+p+1}F(z)\right)'}{(D^{n+p}F(z))'}-(n+2p-c)}{(n+p)-(n+2p-c-1)\frac{\left(D^{n+p-1}F(z)\right)'}{(D^{n+p}F(z))'}}-(p+1)\\ &=\frac{1}{n+p}\Big[-(p(n+p-1)+\alpha)-(p-\alpha)h(z)\\ &\qquad-\frac{(p-\alpha)zh'(z)}{(c+1-\alpha)-(p-\alpha)h(z)}\Big].\end{aligned}$$

So (3.4) is equivalent to

$$\operatorname{Re}\left\{h(z)+\frac{zh'(z)}{(c+1-\alpha)-(p-\alpha)h(z)}+\frac{1}{2(c+1-\alpha)}\right\}>0.$$

We define the function Ψ by

$$\Psi(u,v)=u+\frac{v}{(c+1-\alpha)-(p-\alpha)u}+\frac{1}{2(c+1-\alpha)}.$$

Then we have

$$\operatorname{Re}\left\{\Psi\left(h(z), zh'(z)\right)\right\} > 0$$

and

$$\begin{aligned}\operatorname{Re}\{\Psi(is,t)\} &= \frac{t(c+1-\alpha)}{(c+1-\alpha)^2+(p-\alpha)^2 s^2} + \frac{1}{2(c+1-\alpha)} \\ &\leqslant -\frac{1+s^2}{2}\frac{(c+1-\alpha)}{(c+1-\alpha)^2+(p-\alpha)^2 s^2} + \frac{1}{2(c+1-\alpha)} \\ &= \frac{s^2\left[(p-\alpha)^2-(c+1-\alpha)^2\right]}{2(c+1-\alpha)\left[(c+1-\alpha)^2+(p-\alpha)^2 s^2\right]} < 0.\end{aligned}$$

So, by Lemma 2.1, we conclude that $\operatorname{Re}\{h(z)\} > 0$. It follows from (3.5) and (1.1) that

$$F \in \mathscr{J}_{n+p-1}(\alpha),$$

which evidently completes the proof of Theorem 3.1. □

Putting $p=1$ in Theorem 3.1, we obtain

COROLLARY 3.2. [3] *Let $f \in \Sigma$ satisfy the condition*

$$\operatorname{Re}\left\{\frac{\left(D^{n+1}f(z)\right)'}{\left(D^{n}f(z)\right)'} - 2\right\} < \frac{(1-\alpha)-2(n+\alpha)(c+1-\alpha)}{2(n+1)(c+1-\alpha)}$$

for $0 \leqslant \alpha < 1$, $n > -1$ and $c > 0$. Then the integral operator

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\,dt$$

belongs to the class $\mathscr{J}_n(\alpha)$.

Putting $n=0$ in Corollary 3.2, we obtain

COROLLARY 3.3. *Let $f \in \Sigma$ satisfy the condition*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{(1-\alpha)-2\alpha(c+1-\alpha)}{2(c+1-\alpha)}$$

for $0 \leqslant \alpha < 1$ and $c > 0$. Then the integral operator

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\,dt$$

belongs to the class $\mathscr{MK}_p(\alpha)$.

Putting $\alpha = 0$ in Corollary 3.3, we obtain the following result extends a result of Goel and Sohi [4].

COROLLARY 3.4. *Let $f \in \Sigma$ satisfy the condition*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} < \frac{1}{2(c+1)}$$

for $c > 0$. Then the integral operator

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\,dt$$

belongs to the class $\mathscr{MK}$.

Putting $c = 1$ in Corollary 3.4, we obtain the following result extends a result of Bajpai [2].

COROLLARY 3.5. *Let $f \in \Sigma$ satisfy the condition*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} < \frac{1}{4}.$$

Then the integral operator

$$F(z) = \frac{1}{z^2} \int_0^z t f(t)\,dt$$

belongs to the class $\mathscr{MK}$.

THEOREM 3.6. *If $f \in \mathscr{I}_{n+p-1}(\alpha)$, then the integral operator F defined by (3.2) belongs to the class $\mathscr{I}_{n+p-1}(\alpha)$.*

Proof. Let $f \in \mathscr{I}_{n+p-1}(\alpha)$. Then we have

$$\operatorname{Re}\left\{\frac{\left(D^{n+p}f(z)\right)'}{\left(D^{n+p-1}f(z)\right)'} - (p+1)\right\} < -\frac{p(n+p-1)+\alpha}{n+p}.$$

On the other hand, since

$$-\frac{p(n+p-1)+\alpha}{n+p} < \frac{(p-\alpha)-2(p(n+p-1)+\alpha)(c+1-\alpha)}{2(n+p)(c+1-\alpha)},$$

by Theorem 3.1, we get desired result. □

THEOREM 3.7. *If* $f \in \mathscr{J}_{n+p-1}(\alpha)$, *then*

$$F(z) = \frac{n+p}{z^{n+2p}} \int_0^z t^{n+2p-1} f(t)\, dt$$

belongs to the class $\mathscr{J}_{n+p}(\alpha)$.

Proof. Using (3.3) and the identity (2.5) for the function F, we have

$$(c-p+1) D^{n+p-1} f(z) = (n+p) D^{n+p} F(z) - (n+2p-c-1) D^{n+p-1} F(z)$$

and

$$(c-p+1) D^{n+p} f(z) = (n+p+1) D^{n+p+1} F(z) - (n+2p-c) D^{n+p} F(z).$$

Taking $c = n+2p-1$ in the above relations, we obtain

$$\frac{(n+p+1)\left(D^{n+p+1}F(z)\right)' - \left(D^{n+p}F(z)\right)'}{(n+p)\left(D^{n+p}F(z)\right)'} = \frac{\left(D^{n+p}f(z)\right)'}{\left(D^{n+p-1}f(z)\right)'}$$

which reduces to

$$\frac{(n+p+1)\left(D^{n+p+1}F(z)\right)'}{(n+p)\left(D^{n+p}F(z)\right)'} - \frac{1}{n+p} = \frac{\left(D^{n+p}f(z)\right)'}{\left(D^{n+p-1}f(z)\right)'}.$$

Thus

$$\operatorname{Re}\left\{ \frac{(n+p+1)\left(D^{n+p+1}F(z)\right)'}{(n+p)\left(D^{n+p}F(z)\right)'} - \frac{1}{n+p} - (p+1) \right\}$$
$$= \operatorname{Re}\left\{ \frac{\left(D^{n+p}f(z)\right)'}{\left(D^{n+p-1}f(z)\right)'} - (p+1) \right\} < -\frac{p(n+p-1)+\alpha}{n+p},$$

from which it follows that

$$\operatorname{Re}\left\{ \frac{\left(D^{n+p+1}F(z)\right)'}{\left(D^{n+p}F(z)\right)'} - (p+1) \right\} < -\frac{p(n+p)+\alpha}{n+p+1}.$$

This completes the proof of Theorem 3.7. □

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