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MOMENTS OF A q–BASKAKOV–BETA OPERATORS IN CASE $0<q<1$

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Abstract. In this paper we obtain the estimates of the central moments for the recently defined q-analogue of Baskakaov-beta operators. We obtain the evaluation for the rate of convergence in term of the first modulus of smoothness and Voronovskaja-type theorem for these operators.

1. Introduction

Recently, Phillips [19] proposed the following generalization of celebrated Bernstein polynomials operators based on q-integers

$$B_{n,q}(f,x)=\sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) p_{n,k}(q;x),\ f\in C[0,1]$$

where $p_{n,k}(q;x)=\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]x^k\prod_{r=0}^{n-k-1}(1-q^rx)$. These operators have been studied by several authors (cf. [12], [19]–[25]). The Bernstein polynomials were suitably modified by Durrmeyer to approximate Lebesgue integrable functions (cf. [7], [15]). Derriennic [5] introduced a q-analogue of the Durrmeyer operators wherein she established some approximation properties of the q-Durrmeyer operators. Similar to these modification, the q-analogue of some well known positive linear operators e.g. Bernstein, Baskakov and Szász operators were introduced and studied by several authors, many of which have been introduced by Gupta (see [2], [8], [10], [11]). Motivated by these modifications the authors in [3] introduced the q-Baskakov-beta operators $\mathscr{B}_{n,q}(f,x)$ as follows:

Let $\mathbb{N}$ be the set of positive integer and $f\in C_B[0,\infty)$ (the class of the continuous and bounded functions on $[0,\infty)$). For any $n\in\mathbb{N}$, the operator $\mathscr{B}_{n,q}:C_B[0,\infty)\to C^\infty[0,\infty)$ is defined by

$$\mathscr{B}_{n,q}(f,x)=\frac{[n-1]}{[n]}\sum_{k=0}^{\infty} b_{n,k}(q;x)\int_0^{\infty/A} q^k p_{n,k}(q;u)f(u)\,d_qu,$$

where $b_{n,k}(q;x)=\dfrac{q^{k(k-1)/2}x^k}{B_q(k+1,n)(1+x)^{(n+k+1)}}$, $p_{n,k}(q;x)=\begin{bmatrix}n+k-1\\k\end{bmatrix}\dfrac{q^{k(k-1)/2}x^k}{(1+x)^{(n+k)}}$ and $(1+x)^{(n)}=\prod_{j=0}^{n-1}(1+q^jx)$.

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The operators $\mathscr{B}_{n,q}$ are linear positive and reproduce constant functions. Some approximation properties was established by the authors in [3]. In order to make the paper self contained we recall some definitions and properties of q-calculus. (see [14], [20]).

Let q be a real number satisfying $0<q<1$. For $n\in\mathbb{N}$, we define

$$[n]=\begin{cases}\frac{1-q^n}{1-q}, & q\neq 1\\ n, & q=1.\end{cases}$$

$$[n]!=\begin{cases}[n][n-1][n-2].....[1], & n=1,2,......\\ 1, & n=0.\end{cases}$$

The q-binomial coefficients are defined by

$$\begin{bmatrix}n\\k\end{bmatrix}=\frac{[n]!}{[k]![n-k]!},\quad 0\leqslant k\leqslant n.$$

And by

$$(a+b)^{(n)}=\prod_{j=0}^{n-1}(a+q^jb)$$

we denote the q-rising factorial. The q-analogue E_q^x of classical exponential function which we shall use in this paper is given by

$$E_q^x=\sum_{j=0}^{\infty}q^{j(j-1)/2}\frac{x^j}{[j]!}.$$

For further properties see [14]. The q-Jackson integrals and q-improper integrals are given by [13] and [17].

$$\int_0^a f(x)\,d_qx=(1-q)\,a\sum_{n=0}^{\infty}f(aq^n)\,q^n$$

and

$$\int_0^{\infty/A} f(x)\,d_qx=(1-q)\sum_{n=-\infty}^{\infty}f\left(\frac{q^n}{A}\right)\frac{q^n}{A},\quad A>0$$

respectively, whenever the sums converge absolutely.

For $q\in(0,1)$ and any arbitrary real function $f:\mathbb{R}\to\mathbb{R}$, the q-derivative $D_qf(t)$ is defined as

$$D_qf(x)=\begin{cases}\frac{f(x)-f(qx)}{(1-q)x}; & x\neq 0\\ \lim_{x\to 0}D_qf(t); & x=0.\end{cases}$$

The q-derivative of the product is given by the formula

$$D_q(f(x)g(x))=f(qx)D_q(g(x))+g(x)D_q(f(x))$$

Analogous to the well known gamma and beta function the q-gamma and q-beta functions are introduced.

We define q-gamma function

$$\Gamma_q(t) = \int_0^{1/(1-q)} x^{t-1} E_q^{-qx} d_q x.$$

The q-beta function is given by

$$B_q(t,s) = K(A,t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)^{(t+s)}} d_q x,$$

where $K(x,t) = \frac{1}{1+x} x^t \left(1+\frac{1}{x}\right)_q^t (1+x)_q^{1-t}$. In particular, for any positive integer n

$$K(x,n) = q^{\frac{n(n-1)}{2}}, \qquad K(x,0) = 1$$

and

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}.$$

$\Gamma_q(t)$ and $B_q(t,s)$ are the q-analogues of the gamma and beta functions. In the limit $q \to 1$ they reduce to $\Gamma(t)$ and $B(t,s)$ respectively and also satisfy certain well known properties of classical $\Gamma(t)$ and $B(t,s)$ functions. The space $C_B[0,\infty)$ is endowed with the norm $\|f\| = \sup\{|f(x)| : x \in [0,\infty)\}$. The first order modulus of smoothness of $f \in C_B[0,\infty)$ is defined by

$$\omega_\phi(f,\sqrt{\delta}) = \sup_{0<h\leqslant\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+h) - f(x)|.$$

In what follows, we shall denote $\sqrt{x(1+x)}$ by $\phi(x)$ and $C_B^2[0,\infty)$ will be used for the space of all twice continuously differentiable functions for which f'' is bounded. Further, throughout this paper C is a constant different at each occurrence.

2. Moment Estimates

In this section we shall use the identities (see [3])

$$q^k \phi^2(x) D_q[p_{n,k}(q;x)] = \Big([k] - q^k[n]x\Big) p_{n,k}(q;qx)$$

and

$$q^k \phi^2(x) D_q[b_{n,k}(q;x)] = \Big([k] - q^k[n+1]x\Big) b_{n,k}(q;qx),$$

frequently, where D_q denotes the q-derivative operator.

LEMMA 1. [3] *Let us define* $A_{n,m}(x) = \mathscr{B}_{n,q}(t^m, x)$. *Then, we have*

$$A_{n,0}(x) = 1,$$

$$A_{n,1}(x) = \frac{1}{q[n-2]}\left(1 + \frac{[n+1]x}{q}\right)$$

and

$$A_{n,2}(x) = \frac{\left(q^3[2] + q[2]^2[n+1]x + [n+1][n+2]x^2\right)}{q^6[n-2][n-3]}.$$

Further, there holds the recurrence relation:

$$q^{m+1}[n-m-2]A_{n,m+1}(qx) = \Big([n+1]x + [m+1]\Big)A_{n,m}(qx) + \phi^2(x)D_qA_{n,m}(x) \quad (1)$$

COROLLARY 1. [3] *Since,* $\mathscr{B}_{n,q}(f,x)$ *are linear and preserve constants, it follows that*

$$\mathscr{B}_{n,q}((t-x),x) = \left(\frac{[n+1]}{q^2[n-2]} - 1\right)x + \frac{1}{q[n-2]}.$$

And

$$\mathscr{B}_{n,q}((t-x)^2,x) = \frac{1}{q^6[n-2][n-3]}\Big[q^3(1+q) + \Big(q(q+1)^2[n+1] - 2q^5[n-3]\Big)x + \Big([n+1][n+2] - 2q^4[n+1][n-3] + q^6[n-2][n-3]\Big)x^2\Big].$$

Moreover, there holds the inequality

$$\mathscr{B}_{n,q}((t-x)^2,x) \leqslant \tfrac{8}{q^6[n-2]}\left(\phi^2(x) + \tfrac{1}{[n-3]}\right) = \tfrac{16}{q^6[n-2]}\delta_n^2(x),$$

where $\delta_n^2(x) = \max\{\phi^2(x), \frac{1}{[n-3]}\}$.

LEMMA 2. *Let* $T^q_{n,m}(x) = \mathscr{B}_{n,q}((t-x)^{(m)},x)$ *be the* m*th* q*-central moments of the operators* $\mathscr{B}_{n,q}$, *then there holds the recurrence relation*

$$\begin{aligned}
&\phi^2(x)D_q\left(T^q_{n,m}(x)\right)\\
&= \left(q^{m+1}[n-m-2]\right)T^q_{n,m+1}(qx) + \Bigg[\left([n]xq^{m-1} - [n+1]x + [n]xq^m - [n]xq^{-1}\right)\\
&\quad -[m+1](1+2q^mx + q^{m-1}x + q^{2m-1}x^2) - [m+2]x\left(q^m - \frac{1}{q}\right)\Bigg]T^q_{n,m}(qx)\\
&\quad -\Bigg[\left([m]\phi^2(x) + [n+1](q^m-1)x^2 - q^{m-1}[n]x^2(q^m-1)\right)\\
&\quad +[m+1]x(q^m-1)(1+q^mx+q^{m-1}x)\\
&\quad +[m+1]q^mx^2(q^m-1)(1+q^{m-1}x)\Bigg]T^q_{n,m-1}(qx).
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\phi^2(x)D_q\left(T^q_{n,m}(x)\right) &= \frac{[n-1]}{[n]}\sum_{k=0}^{\infty}\phi^2(x)\int_0^{\infty/A} q^k p_{n,k}(q;t)D_q\left(b_{n,k}(q;x)(t-x)^{(m)}\right)d_qt\\
&= \frac{[n-1]}{[n]}\sum_{k=0}^{\infty}\phi^2(x)\int_0^{\infty/A} q^k p_{n,k}(q;t)b_{n,k}(q;qx)\left(D_q(t-x)^{(m)}d_qt\right)\\
&\quad+\frac{[n-1]}{[n]}\sum_{k=0}^{\infty}\phi^2(x)\int_0^{\infty/A} q^k p_{n,k}(q;t)(t-x)^{(m)}\left(D_q b_{n,k}(q;x)\right)d_qt\\
&= E_1+E_2 \text{ say.}
\end{aligned}$$

$$\begin{aligned}
E_1 &= -[m]\frac{[n-1]}{[n]}\sum_{k=0}^{\infty}\phi^2(x)b_{n,k}(q;qx)\int_0^{\infty/A} q^k p_{n,k}(q;t)(t-qx)^{(m-1)}d_qt\\
&= -[m]\phi^2(x)T_{n,m-1}(qx).
\end{aligned}$$

And in view of $q^k\phi^2(x)D_q[b_{n,k}(q;x)] = \Big([k]-q^k[n+1]x\Big)b_{n,k}(q;qx)$ we get

$$\begin{aligned}
E_2 &= \frac{[n-1]}{[n]}\sum_{k=0}^{\infty}q^{-k}b_{n,k}(q;qx)\int_0^{\infty/A}[k]q^k p_{n,k}(q;t)(t-x)^{(m)}d_qt\\
&\quad-[n+1]x\frac{[n-1]}{[n]}\sum_{k=0}^{\infty}b_{n,k}(q;qx)\int_0^{\infty/A}q^k p_{n,k}(q;t)(t-x)^{(m)}d_qt\\
&= E_3+E_4 \text{ say.}
\end{aligned}$$

It is obtained easily that $(t-x)^{(m)} = x(q^m-1)(t-qx)^{(m-1)}+(t-qx)^{(m)}$. Therefore, $E_4 = -[n+1]x\left[x(q^m-1)T^q_{n,m-1}(qx)+T^q_{n,m}(qx)\right]$. Next,

$$\begin{aligned}
E_3 &= \frac{[n-1]}{[n]}\sum_{k=0}^{\infty}q^{-k}b_{n,k}(q;qx)\int_0^{\infty/A}\left([k]-q^{k-1}[n]t\right)q^k p_{n,k}(q;t)(t-x)^{(m)}d_qt\\
&\quad+\frac{[n-1]}{[n]}\sum_{k=0}^{\infty}q^{-k}b_{n,k}(q;qx)\int_0^{\infty/A}\left(q^{k-1}[n]t\right)q^k p_{n,k}(q;t)(t-x)^{(m)}d_qt\\
&= E_5+E_6 \text{ say.}
\end{aligned}$$

Again,

$$E_6 = q^{-1}[n]\frac{[n-1]}{[n]}\sum_{k=0}^{\infty}b_{n,k}(q;qx)\int_0^{\infty/A}q^k p_{n,k}(q;t)(t-q^mx)(t-x)^{(m)}d_qt$$

$$+q^{-1}[n]\frac{[n-1]}{[n]}\sum_{k=0}^{\infty}b_{n,k}(q;qx)\int_0^{\infty/A}q^m x q^k p_{n,k}(q;t)(t-x)^{(m)}\,d_qt$$
$$=E_7+E_8 \text{ say.}$$

Clearly, $E_8=q^{m-1}[n]x\left[x(q^m-1)T^q_{n,m-1}(qx)+T^q_{n,m}(qx)\right]$. Next,

$$E_7=q^{-1}[n]\frac{[n-1]}{[n]}\sum_{k=0}^{\infty}b_{n,k}(q;qx)\int_0^{\infty/A}q^k p_{n,k}(q;t)(t-x)^{(m+1)}\,d_qt$$
$$=q^{-1}[n]\left[x(q^{m+1}-1)T^q_{n,m}(qx)+T^q_{n,m+1}(qx)\right].$$

In order to simplify E_5 we take the transformation $t=qz$ which is valid in the case of q-integration.

$$E_5=\frac{[n-1]}{[n]}\sum_{k=0}^{\infty}b_{n,k}(q;qx)\int_0^{\infty/A}q\left([k]-q^k[n]z\right)(qz-x)^{(m)}p_{n,k}(q;qz)\,d_qz$$
$$=\frac{[n-1]}{[n]}\sum_{k=0}^{\infty}b_{n,k}(q;qx)\int_0^{\infty/A}q\phi^2(z)D_q(p_{n,k}(q;z))(qz-x)^{(m)}\,d_qz. \quad (2)$$

Next, we write the function $\phi^2(z)=z+z^2$ as $\phi^2(z)=\frac{(q^mx+q^{m-1}x+1)(qz-q^mx)}{q}+q^{m-1}x(1+q^{m-1}x)+\frac{(qz-q^mx)(qz-q^{m+1}x)}{q^2}$ and substitute in (2). We obtain three terms corresponding to the three terms in (2), namely E_9, E_{10} and E_{11}.

$$E_9=\left(\frac{q^mx+q^{m-1}x+1}{q}\right)\frac{[n-1]}{[n]}\sum_{k=0}^{\infty}q^{-k}b_{n,k}(q;qx)$$
$$\times\int_0^{\infty/A}q^k.qD_q(p_{n,k}(q;z))(qz-x)^{(m+1)}\,d_qz.$$

Now, we integrate by parts and then make the inverse transformation $z=q^{-1}t$ which gives

$$E_9=\left(\frac{q^mx+q^{m-1}x+1}{q}\right)\frac{[n-1]}{[n]}\sum_{k=0}^{\infty}q^{-k}b_{n,k}(q;qx)$$
$$\times\left\{\left[p_{n,k}(q;z)(qz-x)^{(m+1)}\right]_0^{\infty/A}-\int_0^{\infty/A}p_{n,k}(q;qz)D_q\left((qz-x)^{(m+1)}\right)d_qz\right\}$$
$$=-[m+1]q^2\left(\frac{q^mx+q^{m-1}x+1}{q}\right)\frac{[n-1]}{[n]}\sum_{k=0}^{\infty}b_{n,k}(q;qx)q^k$$

$$\times \int_0^{\infty/A} p_{n,k}(q;qz)(qz-x)^{(m)}\,d_qz$$

$$= -[m+1]q\left(\frac{q^m x+q^{m-1}x+1}{q}\right)\frac{[n-1]}{[n]}\sum_{k=0}^{\infty} b_{n,k}(q;qx)q^k$$

$$\times \int_0^{\infty/A} p_{n,k}(q;t)(t-x)^{(m)}\,d_qz$$

$$= -[m+1](q^m x+q^{m-1}x+1)\Big[x(q^m-1)T_{n,m-1}(qx)+T_{n,m}(qx)\Big].$$

And

$$E_{10} = q^{m-1}x(1+q^{m-1}x)\frac{[n-1]}{[n]}\sum_{k=0}^{\infty} b_{n,k}(q;qx)\int_0^{\infty/A} q^{k+1}D_q\left(p_{n,k}(q;z)\right)(qz-x)^{(m+1)}\,d_qz$$

$$= -[m+1]q^m x(1+q^{m-1}x)\frac{[n-1]}{[n]}\sum_{k=0}^{\infty} b_{n,k}(q;qx)\int_0^{\infty/A} q^k p_{n,k}(q;t)(t-x)^{(m)}\,d_qt$$

$$= -[m+1]q^m x(1+q^{m-1}x)\Big[x(q^m-1)T_{n,m-1}(qx)+T_{n,m}(qx)\Big].$$

Similarly,

$$E_{11} = -\frac{[m+2]}{q}\Big[x(q^{m+1}-1)T_{n,m}(qx)+T_{n,m+1}(qx)\Big].$$

Combining the estimates E_1-E_{11} the lemma is established. $\square$

Now, we are in a position to state the following lemma

LEMMA 3. *Let $m\in N$, $0<q<1$. There exists a constant $C=C(m)>0$ independent of x and n such that for any $x\in(0,\infty)$ we have*

$$\mathscr{B}_{n,q}((t-x)^{(m)},x)\leqslant C\left(\frac{1}{[n]^{\lfloor (m+1)/2\rfloor}}\right).$$

Proof. The proof follows by induction on m and the Lemma 2. $\square$

LEMMA 4. *Suppose the functions $A_{n,m}(x)$ are expressed as*

$$A_{n,m}(x)=a_0^m+a_1^m x+a_2^m x^2+\ldots a_m^m x^m,$$

where a_i^m are corresponding coefficients. Then the recurrence relations hold

$$q^{2m+2}[n-m-2]a_{m+1}^{m+1}=[n+m+1]a_m^m; \tag{3}$$

$$q^{2m+1}[n-m-2]a_m^{m+1} = [n+m]a_{m-1}^m + [2m+1]a_m^m; \tag{4}$$

$$q^{2m-r+1}[n-m-2]a_{m-r}^{m+1} = [n+m-r]a_{m-r-1}^m + [2m-r]a_{m-r}^m, \quad 1 \leqslant r < m \tag{5}$$

$$q^{m+1}[n-m-2]a_0^{m+1} = [m+1]a_0^m. \tag{6}$$

Proof. Using $A_{n,m}(x) = \sum_{i=0}^m a_i^m x^i$ in (1) and equating the coefficients of various powers of x the we get the recurrence relations

$$\begin{aligned}
q^{2m+2}[n-m-2]a_{m+1}^{m+1} &= \Big([m]+q^m[n+1]\Big)a_m^m;\\
q^{2m+1}[n-m-2]a_m^{m+1} &= \Big([m-1]+q^{m-1}[n+1]\Big)a_{m-1}^m + \Big([m]+q^m[m+1]\Big)a_m^m;\\
q^{2m-r+1}[n-m-2]a_{m-r}^{m+1} &= \Big([m-r-1]+q^{m-r-1}[n+1]\Big)a_{m-r-1}^m\\
&\quad + \Big([m-r]+q^{m-r}[m+1]\Big)a_{m-r}^m, \quad 1 \leqslant r < m\\
q^{m+1}[n-m-2]a_0^{m+1} &= [m+1]a_0^m.
\end{aligned}$$

Now, using $[n] = q^k[n-k]+[k]$ in these relations (3)–(6) are established. □

LEMMA 5. *Let $m \in N$, $0 < q < 1$, $n > 4$. There exists a constant C independent of x and n and $\hat{q} \in (0,1)$ such that*

$$\mathscr{B}_{n,q}((t-x)^4, x) \leqslant C\left(\frac{16}{q^6[n-2]}\delta_n^2(x)\right)^2 \ \forall q \in (0,\hat{q}].$$

Proof. The quantities $A_{n,k}(x)$, $k = 0,1,2$ are obtained in Lemma 1. Now, using (1) we obtain

$$\begin{aligned}
A_{n,3}(x) &= \big[(([n+1]xq^{-1}+[3])(q^3[2]+q[2]^2[n+1]x+[n+1][n+2]x^2)\\
&\quad +(\phi(x/q))(q[2]^2[n+1]+[2][n+1][n+2]xq^{-1})\big]\frac{1}{q^9[n-2][n-3][n-4]}
\end{aligned}$$

and

$$\begin{aligned}
A_{n,4}(x) = \Bigg[&\Big[(([n+1]xq^{-1}+[3])([n+1]xq^{-1}+[4])(q^3[2]+q[2]^2[n+1]\\
&+[n+1][n+2])\Big] + \phi(x/q)\Big\{q[2]^2[n+1]+[2][n+1][n+2]xq^{-1}\\
&+[2]q^2[n+1]+[2]^2[3]q[n+1]+[2]xq^{-1}\Big([2][n+1][n+2]q^{-2}\\
&+[2]^2q^{-1}[n+1]+[2]^2[n+1]^2+[3][n+1][n+2]\Big)\\
&+[3]x^2q^{-2}\Big([2][n+1][n+2]q^{-3}+[n+1]^2[n+2]q^{-1}\Big)+[2]^2[n+1]\Big\}\Bigg]
\end{aligned}$$

$$\times \frac{1}{q^{13}[n-2][n-3][n-4][n-5]}$$

We write $\mathscr{B}_{n,q}((t-x)^4,x)=\sum_{k=0}^{4}\binom{4}{k}(-1)^k x^{4-k}A_{n,k}(x)$ and substitute $A_{n,k}$, $k=0,1,\dots 4$. The coefficient of x^4 in the quantity $C\left(\frac{16}{q^6[n-2]}\delta_n^2(x)\right)^2-\mathscr{B}_{n,q}((t-x)^4,x)$ is obtained as

$$\left(256Cq^8\prod_{k=2}^{5}[n-k]+4q^{18}(q^2[n+1]+1)[n+1][n+2][n-2][n-5]\right)$$
$$-\left(q^{20}[n-2]\prod_{k=2}^{5}[n-k]+6q^{14}\prod_{k=1}^{2}[n+k][n-2][n-4][n-5]+[3]\prod_{k=1}^{3}[n+k]\right)$$
$$\times\frac{1}{q^{20}[n-2]\prod_{k=2}^{5}[n-k]}$$

Now the coefficient of the terms of $O([n]^5)$ in numerator is $\frac{4-q^{10}+6q^{14}}{q^{10}}$ which is positive iff $q\leqslant\left(\frac{2}{3+q^6}\right)^{1/4}$. Further for sufficiently large n, remaining terms in the numerator are positive. Therefore the coefficient of x^4 becomes positive for $q\in(0,\hat{q})$, where $\hat{q}=\left(\frac{2}{3+q^6}\right)^{1/4}$. Similarly the coefficients of x^i, $i=1,2,3$ are positive for large n. In case $q\to 1$, we have $[n]\to n$ as $[n]$ is continuous function of q, the lemma is established easily in this case again. This completes the proof. □

LEMMA 6. *For the coefficients a_i^m defined in lemma 4 there holds*

$$a_{m-r}^m=O\left(\left(\frac{1}{[n]}\right)^r\right).$$

Proof. For $r=1$, we need to prove $a_{m-1}^m=O\left(\frac{1}{[n]}\right)$. In case $m=1$, we have $a_0^1=\frac{1}{q[n-2]}$ and using recurrence relation $q^{2m+1}[n-m-2]=[n+m]a_{m-1}^m+[2m+1]a_m^m$ with $a_m^m=\frac{[n+1]....[n+m]}{q^{m(m+1)}[n-2]....[n-m-1]}$, the lemma is proved for $r=1$. Suppose the lemma is true for a certain r, then from the recurrence relation (5), we get

$$[n+m-r]a_{m-(r+1)}^m=q^{2m-r+1}[n-m-2]a_{m-r}^{m+1}-[2m-r+1]a_{m-r}^m$$
$$=q^{2m-r+1}[n-m-2]O\left(\frac{1}{[n]}\right)^{r+1}-[2m-r+1]O\left(\frac{1}{[n]}\right)^r$$
$$\Rightarrow a_{m-(r+1)}^m=O\left(\frac{1}{[n]}\right)^{r+1}.$$

This proves the lemma. □

3. Applications of the moments

The following theorem due to Pop [21] will be used in our asymptotic results.

THEOREM 1. *Let $I \subset R$ be an interval, $a \in I$, $n \in N$ and the function $f : I \Rightarrow R$, f is n times derivable in a. According to Taylor's expansion theorem for the function f around a, we have*

$$f(x) = \sum_{k=0}^{r} \frac{(x-a)^k}{k!} f^{(k)}(a) + (x-a)^r \mu(x-a),$$

where μ is a bounded function and $lim_{t\to x}\mu(x-a) = 0$. If $f^{(r)}$ is a continuous function on I, then for any $\delta > 0$ $|\mu(x-a)| \leqslant \frac{1}{r!}[1+\delta^{-2}(x-a)^2]\omega_1\left(f^{(r)},\delta\right)$.

THEOREM 2. *Let $r \geqslant 0$ and $s \geqslant 1$. Then, for $0 < q < 1$,*

$$\begin{aligned} &D_q^r\left(\mathscr{B}_{n,q}(t^{r+s},x)\right) \\ &= \frac{[r+s]!}{[s]!} \frac{[n+1]....[n+r+s]}{q^{(r+s)(r+s+1)}[n-2]....[n-r-s-1]} x^s \\ &\quad + \frac{[r+s-1]!}{[s-1]!} \frac{[n+1]....[n+r+s]\left(\sum_{j=0}^{r+s-1} q^{r+s-j-1}[2j+1]\right)}{q^{(r+s)^2+3(r+s)+1}[n-2]...[n-r-s-2]} x^{s-1} \\ &\quad + \sum_{j=2}^{r+s} M_j x^{r+s-j} O\left(\frac{1}{[n]^j}\right), \end{aligned}$$

where M_j's are independent of n.

Proof. Using Lemma 1 we get following coefficients

$$a_0^0 = 1, \quad a_0^1 = \frac{1}{q[n-2]},$$

$$a_0^2 = \frac{[2]}{q^6[n-2][n-3]}, \quad a_1^2 = \frac{q[2]^2[n+1]}{q^6[n-2][n-3]}, \quad a_2^2 = \frac{[n+1][n+2]}{q^6[n-2][n-3]}$$

and in general

$$a_0^m = \frac{1}{q^{m(m+1)/2}} \prod_{j=0}^{m-2} \left(\frac{[1+j]}{[n-2-j]}\right) \quad m \geqslant 3.$$

From previous Lemma 6, it is known that

$$a_{r+s}^{r+s} = \frac{[n+1]....[n+r+s]}{q^{(r+s)(r+s+1)}[n-2]....[n-r-s-1]}$$

Simplifying recurrence relation (4), we get

$$q^{2m+1}[n-m-2]a_m^{m+1} = [n+m]a_{m-1}^m + [2m+1]a_m^m.$$

Now, putting $m = 1,2,...$ we obtain from above recurrence relation

$$\begin{aligned} a_m^{m+1} &= \frac{[n+1][n+2]...[n+m]\left(q^m + [3]q^{m-1} + ... + [2m+1]\right)}{q^{m^2+3m+1}[n-2]...[n-m-1]} \\ &= \frac{\prod_{i=1}^m [n+i] \sum_{j=0}^m q^{m-j}[2j+1]}{q^{m^2+3m+1}\prod_{k=2}^{m+2}[n-k]}. \end{aligned}$$

Similarly simplification of (5) gives

$$q^{2m}[n-m-2]a_{m-1}^{m+1} = [n+m-1]a_{m-2}^m + [2m]a_{m-1}^m.$$

Now, in view of Lemma 6, it follows that

$a_{m+r-j}^{m+r} = O\left(\frac{1}{[n]}\right)^j, j = 2,3,....$ This completes the Theorem. □

Following is a Voronovskaya-type theorem for monomials.

THEOREM 3. *Let $r \geqslant 0$ and $s \geqslant 1$ and (q_n) be a sequence in $(0,1)$ such that $q_n \to 1$ as $n \to \infty$, then*

$$\lim_{n\to\infty}[n]\left[D_q^r\left(\mathscr{B}_{n,q}(t^{r+s},x)\right) - \frac{(r+s)!}{s!}x^s\right] = \frac{(r+s)(r+s)!}{(s-1)!}x^{s-1}$$

Proof. It results from Theorem 2 and the limit

$$\begin{aligned} & n\to\infty[n]\frac{[r+s-1]!}{[s-1]!}\frac{[n+1]....[n+r+s]\left([2]^2q^{r+s} + [5]q^{r+s-1} + ... + [2r+2s+1]\right)}{q^{(r+s)^2+3(r+s)+1}[n-2]...[n-r-s-2]} \\ &= \frac{(r+s-1)![(r+s)(r+s+2)-3]}{(s-1)!}. \quad \square \end{aligned}$$

THEOREM 4. *For $f \in C_B^2[0,\infty)$, $q \in (0,\hat{q}]$ we have*

$$\left|\mathscr{B}_{n,q}(f,x) - \sum_{k=0}^2 \frac{\mathscr{B}_{n,q}((t-x)^k,x)}{k!}f^{(k)}(x)\right| \leqslant \omega\left(f'', \frac{4}{q^3\sqrt{[n-2]}}\delta_n(x)\right)\left[\frac{16}{q^6[n-2]}\delta_n^2(x)\right].$$

Proof. From finite Taylor's expansion for the function f around x, given in Theorem 1, we have

$$\begin{aligned} & \left|\mathscr{B}_{n,q}(f,x) - \sum_{k=0}^2 \frac{\mathscr{B}_{n,q}((t-x)^k,x)}{k!}f^{(k)}(x)\right| \\ &= \left|\mathscr{B}_{n,q}((t-x)^2\mu(t-x),x)\right| \end{aligned}$$

$$\leqslant \frac{1}{2!}\omega(f'',\delta)\left[\left|\mathscr{B}_{n,q}((t-x)^2,x)\right|+\delta^{-2}\left|\mathscr{B}_{n,q}((t-x)^4,x)\right|\right]$$
$$\leqslant \frac{1}{2!}\omega(f'',\delta)\left[\frac{16}{q^6[n-2]}\delta_n^2(x)+\delta^{-2}\left(\frac{16}{q^6[n-2]}\delta_n^2(x)\right)^2\right].$$

Choosing $\delta^2=\frac{16}{q^6[n-2]}\delta_n^2(x)$ the theorem follows. □

THEOREM 5. *If $f\in C_B^2[0,\infty)$, and q_n be a sequence in $(0,1)$ such that $\lim_{n\to\infty}q_n=1$, then we have*

$$\lim_{n\to\infty}[n]\left[\mathscr{B}_{n,q}(f,x)-f(x)\right]=\frac{d}{dx}\left(x(1+x)f'(x)\right).$$

Proof. Throughout the proof it will be assumed that $[n]=[n]_{q_n}$. The proof results from Theorem 4 and the limits $\lim_{n\to\infty}[n]\mathscr{B}_{n,q_n}((t-x)^j,x)$, $j=0,1,2$. We have $\lim_{n\to\infty}[n]\mathscr{B}_{n,q_n}(1,x)=0$ and

$$\begin{aligned}\lim_{n\to\infty}[n]\mathscr{B}_{n,q_n}((t-x),x) &= \lim_{n\to\infty}[n]\left(\frac{[n+1]}{q^2[n-2]}-1\right)x+\frac{1}{q[n-2]}\\ &= \lim_{n\to\infty}\frac{[2](1+q^n)x}{q^2[n-2]}+(1+q^n)x+\frac{[2]}{q[n-2]}+q\\ &= 1+2x,\end{aligned}$$

where we have used the relation $[n]=[2]+q^2[n-2]$.

Writing $\mathscr{B}_{n,q_n}((t-x)^2,x)=c_0+c_1x+c_2x^2$, where $c_i's$ are the coefficients given in Cor. 1. Then,

$$\lim_{n\to\infty}[n]\mathscr{B}_{n,q}((t-x)^2,x)=\lim_{n\to\infty}[n](c_0+c_1x+c_2x^2)$$

Clearly,

$$\lim_{n\to\infty}[n]c_0=0,$$

$$\begin{aligned}\lim_{n\to\infty}[n]c_1x &= \lim_{n\to\infty}[n]\frac{\left(q^3(1+q)+\left(q(q+1)^2[n+1]-2q^5[n-3]\right)\right.}{q^6[n-2][n-3]}x\\ &= \lim_{n\to\infty}\left([2]+q^2[n-2]\right)\left(\frac{[2]^2[4]q}{q^6[n-2][n-3]}+\frac{([2]^2q^5-2q^5)}{q^6[n-2]}\right)x\\ &= 2x.\end{aligned}$$

And simplification yields

$$\begin{aligned}\lim_{n\to\infty}[n]c_2x^2 &= \lim_{n\to\infty}[n]\frac{\Big([n+1][n+2]-2q^4[n+1][n-3]+q^6[n-2][n-3]\Big)x^2}{q^6[n-2][n-3]}\\ &= \lim_{n\to\infty}[n]\left(\frac{([4]-2[3])}{q^2[n-2]}+\frac{[4]}{q^2[n-3]}+\frac{[4]^2}{q^6[n-2][n-3]}\right)x^2\\ &= 2x^2. \quad \square\end{aligned}$$

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