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ON p–ADIC INTERPOLATING FUNCTION ASSOCIATED WITH MODIFIED DIRICHLET'S TYPE OF TWISTED q–EULER NUMBERS AND POLYNOMIALS WITH WEIGHT α

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Abstract. In the present paper, we introduce modified Dirichlet's type of twisted q-Euler polynomials with weight α. We apply the method of generating function and p-adic q-integral representation on $\mathbb{Z}_p$, which are exploited to derive further classes of q-Euler numbers and polynomials. Our new generating function possess a number of interesting properties which we state in this paper.

1. Introduction, Definitions and Notations

p-adic numbers and L-functions theory plays a vital and important role in mathematics. p-adic numbers were invented by the German mathematician Kurt Hensel, around the end of the nineteenth century. In spite of their being already one hundred years old, these numbers are still today enveloped in an aura of mystery within the scientific community. The p-adic integral was used in mathematical physics, for instance, the functional equation of the q-Zeta function, q-Stirling numbers and q-Mahler theory of integration with respect to the ring $\mathbb{Z}_p$ together with Iwasawa's p-adic q-L functions. A p-adic zeta function, or more generally a p-adic L-function, is a function analogous to the Riemann zeta function, or more general L-functions, but whose domain and target are p-adic (where p is a prime number). For example, the domain could be the p-adic integers $\mathbb{Z}_p$, a profinite p-group, or a p-adic family of Galois representations, and the image could be the p-adic numbers $\mathbb{Q}_p$ or its algebraic closure. For a prime number p and for a Dirichlet character defined modulo some integer, the p-adic L-function was constructed by interpolating the values of complex analytic L-function at non-positive integers. For powers of the Teichmüller character, many mathematicians used the integral representation to extend the L-function to the large domain, in which it is a meromorphic function in the first variable and an analytic element in the second. These integral representations imply systems of congruences for the generalized Euler polynomials (see [9], [16], [17], [18], [19], [20]). In this paper our main focus will be on p-adic interpolation of modified Dirichlet's type of twisted q-Euler polynomials with weight α. Actually interpolation is the process of defining a continuous function that takes on specified values at specified points. During

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the development of p-adic analysis, researchers were made to derive a meromorphic function, defined over the p-adic number field, which would interpolate the same or at least similar values as the Dirichlet L-function at non-positive integers. Finding the interpolation functions of special orthogonal numbers and polynomials started by Tsumura [25] for the Bernoulli and Euler polynomials. After Kim [4]–[11] and Simsek [17]–[20], studied on p-adic interpolation functions of these numbers and polynomials. L. C. Washington [24], constructed one-variable p-adic-L-function which interpolates generalized classical Bernoulli numbers at negative integers. Kim [13], constructed the p-adic q-L-function, which is interpolation function of the generalized q-Bernoulli polynomials. Kim and Rim [9], introduced twisted q-Euler numbers and polynomials associated with basic twisted q-l-functions by using p-adic q-invariant integral on $\mathbb{Z}_p$ in the fermionic sense. Also, Srivastava developed many different techniques to evaluate various families of series involving Zeta and related functions (see [21]–[23]). In this paper, we will construct a p-adic interpolation function of modified Dirichlet's type of twisted q-Euler polynomials with weight α.

Imagine that p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will be denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* := \mathbb{N} \cup \{0\}$. Let q be variously considered as an indeterminate a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, we assume that $|q-1|_p < 1$ for $|x|_p \leqslant 1$ in the p-adic case. If $q \in \mathbb{C}$, we assume that $|q| < 1$.

The p-adic absolute value $|.|_p$, is normally defined by

$$|x|_p = \frac{1}{p^r},$$

where $x = p^r \frac{s}{t}$ with $(p,s) = (p,t) = (s,t) = 1$ and $r \in \mathbb{Q}$.

As well-known definition, Euler polynomials are defined by

$$\sum_{n=0}^{\infty} \boldsymbol{E}_n(x) \frac{t^n}{n!} = \frac{2}{e^t+1} e^{xt}, \tag{1.1}$$

with the usual convention about replacing $\boldsymbol{E}^n(x)$ by $\boldsymbol{E}_n(x)$ (for more information, see [1], [3], [7]–[9], [12]–[14], [21]–[23]).

Let f be uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$. For $f \in UD(\mathbb{Z}_p)$, we can begin with the following expression:

$$\frac{1}{[p^n]_q} \sum_{j=0}^{p^n-1} f(j) q^j = \sum_{j=0}^{p^n-1} f(j) \mu_q(j + p^n \mathbb{Z}_p)$$

which stands for p-adic q-analogue of Riemann sums for f. The integral of f on $\mathbb{Z}_p$ will be defined as the limit $(n \to \infty)$ of these sums, when it exists. The following p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim [5], [6]

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{n\to\infty} \frac{1}{[p^n]_q} \sum_{j=0}^{p^n-1} f(j) q^j.$$

Note that if $f_n \to f$ in $UD(\mathbb{Z}_p)$, then

$$\int_{\mathbb{Z}_p} f_n(x)\, d\mu_q(x) \to \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x).$$

In [6], [7], [8], Kim showed that p-adic q-integral on $\mathbb{Z}_p$ in the fermionic sense is as follows:

$$\lim_{t \to -q} I_t(f) = I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = \lim_{n\to\infty} \frac{1}{[p^n]_{-q}} \sum_{j=0}^{p^n-1} (-1)^j f(j)\, q^j \qquad (1.2)$$

in which $[x]_q$ is a q-extension of x defined by

$$[x]_q = \frac{1-q^x}{1-q},$$

note that $\lim_{q\to 1} [x]_q = x$ cf. [1-23].

If we take $f_1(x) = f(x+1)$ in (1.2), then we easily see that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \text{ (see [6]).} \qquad (1.3)$$

By expression (1.3), we readily see that,

$$(-1)^{n-1} I_{-q}(f) + q^n I_{-q}(f_n) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \qquad (1.4)$$

where $f_n(x) = f(x+n)$.

Recently, Rim and Jeong [14] defined the modified weighted q-Euler numbers $\boldsymbol{E}_{n,q}^{(\alpha)}$ and the modified weighted q-Euler polynomials $\boldsymbol{E}_{n,q}^{(\alpha)}(x)$ by using p-adic q-integral on $\mathbb{Z}_p$ in the form

$$\boldsymbol{E}_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} q^{-\xi} [\xi]_{q^\alpha}^n\, d\mu_{-q}(\xi), \text{ for } n \in \mathbb{N}^* \text{ and } \alpha \in \mathbb{Z}.$$

Let $C_{p^n} = \left\{ w \mid w^{p^n} = 1 \right\}$ be the Cylic group of order p^n, and let

$$\mathbf{T_p} = \lim_{n\to\infty} C_{p^n} = C_{p^\infty} = \bigcup_{n \geqslant 0} C_{p^n},$$

note that $\mathbf{T_p}$ is locally constant space (for details, see [3], [9], [12], [16], [17], [18], [19], [20]).

In [3], let χ be a Dirichlet's character with conductor $d\,(= odd) \in \mathbb{N}$ and $w \in \mathbf{T_p}$. If we take $f(x) = \chi(x)\, w^x e^{tx}$, then we have $f(x+d) = \chi(x)\, w^x w^d e^{tx} e^{td}$. From (1.4), we see that

$$\int_X \chi(x)\, w^x e^{tx} d\mu_{-q}(x) = \frac{[2]_q \sum_{i=0}^{d-1} (-1)^{d-1-i} q^i \chi(i)\, w^i e^{ti}}{q^d w^d e^{td} + 1}. \qquad (1.5)$$

In view of (1.5), it is considered by

$$F_{w,\chi}^{q}(t) = \frac{[2]_q \sum_{i=0}^{d-1} (-1)^{d-1-i} q^i \chi(i) w^i e^{ti}}{q^d w^d e^{td} + 1} = \sum_{n=0}^{\infty} E_{n,\chi,w}^{q} \frac{t^n}{n!},\ |t + \log q| < \frac{\pi}{d}, \text{ (see [3]).} \tag{1.6}$$

Let us consider the modified twisted q-Euler polynomials with weight α as follows:

$$\boldsymbol{E}_{n,q}^{(\alpha,w)}(x) = \int_{\mathbb{Z}_p} q^{-\xi} w^{\xi} [x+\xi]_{q^\alpha}^{n} \, d\mu_{-q}(\xi), \text{ for } n \in \mathbb{N}^*. \tag{1.7}$$

By (1.7), and applying combinatorial techniques we have,

$$\begin{aligned} \boldsymbol{E}_{n,q}^{(\alpha,w)}(x) &= \sum_{k=0}^{n} \binom{n}{k} q^{\alpha(n-k)x} \boldsymbol{E}_{n-k,q}^{(\alpha,w)} [x]_{q^\alpha}^{k} \\ &= \sum_{k=0}^{n} \binom{n}{k} q^{\alpha k x} \boldsymbol{E}_{k,q}^{(\alpha,w)} [x]_{q^\alpha}^{n-k}, \end{aligned} \tag{1.8}$$

where $\boldsymbol{E}_{n,q}^{(\alpha,w)}(0) := \boldsymbol{E}_{n,q}^{(\alpha,w)}$ are called modified twisted q-Euler numbers with weight α.

By (1.7), we get generating function of modified twisted q-Euler polynomials as follows:

$$\begin{aligned} F^{(\alpha)}(t,x,q,w) &= \sum_{n=0}^{\infty} \boldsymbol{E}_{n,q}^{(\alpha,w)}(x) \frac{t^n}{n!} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m w^m e^{t[x+m]_{q^\alpha}}. \end{aligned} \tag{1.9}$$

By using a complex contour integral representation and (1.9), we get modified twisted Hurwitz-zeta function as follows:

For $q, s \in \mathbb{C}$ $(|q^\alpha| < 1)$,

$$\begin{aligned} \widetilde{\zeta}_q^{(\alpha,w)}(s,x) &= \frac{1}{\Gamma(s)} \int_0^{\infty} F^{(\alpha)}(-t,x,q,w) t^{s-1} dt \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m w^m \left(\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-t[x+m]_{q^\alpha}} \right) \\ &= [2]_q \sum_{m=0}^{\infty} \frac{(-1)^m w^m}{[m+x]_{q^\alpha}^{s}} \qquad (\Re e(s) > 1 : x \neq 0, -1, -2, \cdots), \end{aligned} \tag{1.10}$$

in which $\widetilde{\zeta}_q^{(\alpha,w)}(s,x)$ can be continued meromorphically everywhere in the complex s-plane expect for a simple pole at $s = 1$.

By using same method as in [23], we arrive at equation (1.11):

$$\widetilde{\zeta}_q^{(\alpha,w)}(-n,x) = \boldsymbol{E}_{n,q}^{(\alpha,w)}(x) \tag{1.11}$$

which can be computed in order to evaluate $\widetilde{\zeta}_q^{(\alpha,w)}(-n,x)$ whereby the special values of $n \in \mathbb{N}^*$.

In this paper, we construct the generating function of modified Dirichlet's type twisted q-Euler polynomials with weight α in the p-adic case. Also, we give Witt's formula for this type polynomials. Moreover, we obtain a new p-adic q-Euler L-function with weight α associated with Dirichlet's character χ, as follows:

$$l_{p,q}^{(\alpha,w)}(-n \mid \chi) = \widetilde{E}_{n,\chi_n}^{(\alpha,w)} - \frac{1}{[p^{-1}]_{q^{\alpha F}}^n} \chi_n(p) \widetilde{E}_{n,\chi_n}^{*(\alpha,w)}$$

where $n \in \mathbb{N}^*$.

2. Properties of modified Dirichlet's type of twisted q-Euler numbers and polynomials

In this section, by using fermionic p-adic q-integral equations on $\mathbb{Z}_p$, some interesting identities and relations of the modified Dirichlet's type of twisted q-Euler numbers and polynomials with weight α, are given. Throughout this part, we assume that $q \in \mathbb{C}$ with $|q^\alpha| < 1$.

DEFINITION 1. Let χ be a Dirichlet's character with conductor $d\,(=odd) \in \mathbb{N}$. For each $n \in \mathbb{N}^*$ and $w \in T_p$. Then modified Dirichlet's type of twisted q-Euler polynomials with weight α are defined by means of the following generating function:

$$F^{(\alpha)}(t,x,q,w \mid \chi) = \sum_{n=0}^{\infty} \widetilde{E}_{n,q}^{(\alpha,w)}(x \mid \chi) \frac{t^n}{n!} \tag{2.1}$$

also

$$F^{(\alpha)}(t,x,q,w \mid \chi) = [2]_q \sum_{m=0}^{\infty} (-1)^m w^m \chi(m) e^{t[x+m]_{q^\alpha}}. \tag{2.2}$$

where $q \in \mathbb{C}$ $(|q^\alpha| < 1)$.

REMARK 1. Setting as $\chi \equiv 1$ into Definition 1, leads to (1.9).

REMARK 2. Setting as $\chi \equiv 1, q \to 1$ and $w \to 1$ into Definition 1, reduces to (1.1).

From (2.1) and (2.2) we obtain,

$$\sum_{n=0}^{\infty} \widetilde{E}_{n,q}^{(\alpha,w)}(x \mid \chi) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left([2]_q \sum_{m=0}^{\infty} (-1)^m w^m \chi(m) [x+m]_{q^\alpha}^n \right) \frac{t^n}{n!}.$$

Therefore, we state the following theorem:

THEOREM 1. *Let χ be a Dirichlet's character with conductor $d\,(=odd) \in \mathbb{N}$. For each $n \in \mathbb{N}^*$ and $w \in T_p$ we have*

$$\widetilde{E}_{n,q}^{(\alpha,w)}(x \mid \chi) = [2]_q \sum_{m=0}^{\infty} (-1)^m w^m \chi(m) [x+m]_{q^\alpha}^n . \tag{2.3}$$

By using (2.3), we can write

$$\begin{aligned}
&\widetilde{E}_{n,q}^{(\alpha,w)}(x \mid \chi) \\
&= [2]_q \sum_{m=0}^{\infty} \sum_{l=0}^{d-1} (-1)^{l+md} w^{l+md} \chi(l+md) [x+l+md]_{q^\alpha}^n \\
&= \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^{d-1} (-1)^l w^l \chi(l) \sum_{m=0}^{\infty} (-1)^m \left(w^d\right)^m \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{\alpha k(x+l+md)} \\
&= \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^{d-1} (-1)^l w^l \chi(l) \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{\alpha k(x+l)} \sum_{m=0}^{\infty} (-1)^m \left(w^d\right)^m \left(q^{\alpha k d}\right)^m \\
&= \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^{d-1} (-1)^l w^l \chi(l) \sum_{k=0}^{n} \frac{\binom{n}{k} (-1)^k q^{\alpha k(x+l)}}{q^{\alpha k d} w^d + 1}.
\end{aligned}$$

So, we obtain the following corollary:

COROLLARY 1. *Let χ be a Dirichlet's character with conductor $d\,(=odd) \in \mathbb{N}$. For each $n \in \mathbb{N}^*$ and $w \in T_p$ we have*

$$\begin{aligned}
\widetilde{E}_{n,q}^{(\alpha,w)}(x \mid \chi) &= [2]_q \sum_{m=0}^{\infty} (-1)^m w^m \chi(m) [x+m]_{q^\alpha}^n \\
&= \frac{[2]_q}{[\alpha]_q^n (1-q)^n} \sum_{l=0}^{d-1} (-1)^l w^l \chi(l) \sum_{k=0}^{n} \frac{\binom{n}{k} (-1)^k q^{\alpha k(x+l)}}{q^{\alpha k d} w^d + 1}.
\end{aligned}$$

By applying $f(\xi) = q^{-\xi} \chi(\xi) w^{\xi} [x+\xi]_{q^\alpha}^n$ into (1.2),

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-\xi} \chi(\xi) w^{\xi} [x+\xi]_{q^\alpha}^n \, d\mu_{-q}(\xi) \\
&= \frac{1}{(1-q^\alpha)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{\alpha k x} \int_{\mathbb{Z}_p} \chi(\xi) w^{\xi} q^{a \xi k - \xi} \, d\mu_{-q}(\xi),
\end{aligned} \tag{2.4}$$

where from (1.4), we easily see that

$$\int_{\mathbb{Z}_p} \chi(\xi) w^{\xi} q^{\xi \alpha k - \xi} \, d\mu_{-q}(\xi) = \frac{[2]_q \sum_{l=0}^{d-1} (-1)^l q^{\alpha k l} w^l \chi(l)}{q^{\alpha k d} w^d + 1}. \tag{2.5}$$

By using (2.4) and (2.5) we obtain

$$\begin{aligned}&\int_{\mathbb{Z}_p} q^{-\xi}\chi(\xi)\, w^{\xi}\,[x+\xi]_{q^\alpha}^n\, d\mu_{-q}(\xi)\\ &= \frac{1}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{k}(-1)^k q^{\alpha kx}\frac{[2]_q\sum_{l=0}^{d-1}(-1)^l q^{\alpha kl}w^l\chi(l)}{q^{\alpha kd}w^d+1}\\ &= \widetilde{E}_{n,q}^{(\alpha,w)}(x\mid\chi). \end{aligned} \tag{2.6}$$

Last from equivalent, we obtain Witt's type formula of modified Dirichlet's type of twisted q-Euler polynomials with weight α as follows:

THEOREM 2. *Let χ be a Dirichlet's character with conductor $d\,(=odd)\in\mathbb{N}$. For each $n\in\mathbb{N}^*$ and $w\in T_p$ we obtain*

$$\widetilde{E}_{n,q}^{(\alpha,w)}(x\mid\chi) = \int_{\mathbb{Z}_p} q^{-\xi}\chi(\xi)\, w^{\xi}\,[x+\xi]_{q^\alpha}^n\, d\mu_{-q}(\xi). \tag{2.7}$$

By (2.2), we obtain *functional equation* as follows:

$$F^{(\alpha)}(t,x,q,w\mid\chi) = e^{t[x]_{q^\alpha}}F^{(\alpha)}(q^x t,q,w\mid\chi).$$

By using the definition of the generating function $F^{(\alpha)}(t,x,q,w\mid\chi)$ as follows:

$$\sum_{n=0}^{\infty}\widetilde{E}_{n,q}^{(\alpha,w)}(x\mid\chi)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty}[x]_{q^\alpha}^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}q^{n\alpha x}\widetilde{E}_{n,q}^{(\alpha,w)}(\chi)\frac{t^n}{n!}\right),$$

by the Cauchy product in the above equation, we have

$$\sum_{n=0}^{\infty}\widetilde{E}_{n,q}^{(\alpha,w)}(x\mid\chi)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}q^{\alpha lx}\widetilde{E}_{l,q}^{(\alpha,w)}(\chi)\,[x]_{q^\alpha}^{n-l}\right)\frac{t^n}{n!}.$$

Therefore, by comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we can state following theorem:

THEOREM 3. *Let χ be a Dirichlet's character with conductor $d\,(=odd)\in\mathbb{N}$. For each $n\in\mathbb{N}^*$ and $w\in T_p$ we have*

$$\widetilde{E}_{n,q}^{(\alpha,w)}(x\mid\chi) = \sum_{l=0}^{n}\binom{n}{l}q^{\alpha xl}\widetilde{E}_{l,q}^{(\alpha,w)}(\chi)\,[x]_{q^\alpha}^{n-l}. \tag{2.8}$$

So, by using *umbral calculus* convention in equality (2.8), we get

$$\widetilde{E}_{n,q}^{(\alpha,w)}(x\mid\chi) = \left(q^{\alpha x}\widetilde{E}_{q}^{(\alpha,w)}(\chi)+[x]_{q^\alpha}\right)^n, \tag{2.9}$$

where $\left(\widetilde{E}_{q}^{(\alpha,w)}(\chi)\right)^n$ is replaced by $\widetilde{E}_{n,q}^{(\alpha,w)}(\chi)$.

From (1.4) we arrive at the following theorem:

THEOREM 4. *Let* χ *be a Dirichlet's character with conductor* $d\,(=odd)\in\mathbb{N}$, $w\in T_p$ *and* $m\in\mathbb{N}^*$ *we get*

$$w^n\widetilde{E}_{m,q}^{(\alpha,w)}(n\mid\chi)+(-1)^{n-1}\widetilde{E}_{m,q}^{(\alpha,w)}(\chi)=[2]_q\sum_{l=0}^{n-1}(-1)^{n-1-l}\chi(l)\,w^l\,[l]_{q^\alpha}^m .$$

So, from (1.4), and some combinatorial techniques we can write

$$\begin{aligned}&\int_{\mathbb{Z}_p}q^{-\xi}\chi(\xi)\,w^{\xi}\,[x+\xi]_{q^\alpha}^n\,d\mu_{-q}(\xi)\\&=\frac{[d]_{q^\alpha}^n}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^a\chi(a)\,w^a\int_{\mathbb{Z}_p}q^{-d\xi}w^{d\xi}\left[\frac{x+a}{d}+\xi\right]_{q^{d\alpha}}^n d\mu_{(-q)^d}(\xi)\\&=\frac{[d]_{q^\alpha}^n}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^a w^a\chi(a)\,\boldsymbol{E}_{n,q^d}^{(\alpha,w^d)}\left(\frac{x+a}{d}\right).\end{aligned}\tag{2.10}$$

Therefore, by (2.10), we obtain the following theorem:

THEOREM 5. *Let* χ *be a Dirichlet's character with conductor* $d\,(=odd)\in\mathbb{N}$, $w\in T_p$ *and* $n\in\mathbb{N}^*$ *we have*

$$\widetilde{E}_{n,q}^{(\alpha,w)}(x\mid\chi)=\frac{[d]_{q^\alpha}^n}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^a w^a\chi(a)\,\boldsymbol{E}_{n,q^d}^{(\alpha,w^d)}\left(\frac{x+a}{d}\right).$$

3. Modified Dirichlet's type of twisted q-Euler L-function with weight α

In this section, our goal is to consider interpolation function of the generating functions of modified Dirichlet's type of twisted q-Euler polynomials with weight α. For $q,s\in\mathbb{C}$ $(|q^\alpha|<\mathbf{1})$, $w\in T_p$ and χ be a Dirichlet's character with conductor $d(=odd)\in\mathbb{N}$, by applying the Mellin transformation in equation (2.2), we obtain

$$\begin{aligned}\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(x,s\mid\chi)&=\frac{1}{\Gamma(s)}\oint t^{s-1}F^{(\alpha)}(-t,x,q,w\mid\chi)\,dt\\&=[2]_q\sum_{m=0}^{\infty}(-1)^m w^m\chi(m)\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t[m+x]_{q^\alpha}}dt\right),\end{aligned}$$

so, from the above equality, we have

$$\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(x,s\mid\chi)=[2]_q\sum_{m=0}^{\infty}\frac{(-1)^m\chi(m)\,w^m}{[m+x]_{q^\alpha}^s}.$$

Consequently, we are now in a position to define modified Dirichlet's type of twisted q-Euler L-function as follows:

DEFINITION 2. Let χ be a Dirichlet's character with conductor $d\,(=odd) \in \mathbb{N}$ and $w \in T_p$ we have

$$\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(x,s \mid \chi) = [2]_q \sum_{m=0}^{\infty} \frac{(-1)^m \chi(m) w^m}{[m+x]_{q^\alpha}^s}, \tag{3.1}$$

for all $s \in \mathbb{C}$. We note that $\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(x,s \mid \chi)$ is analytic function in the whole complex s-plane.

By substituting $s = -n$ into (3.1) we easily get

$$\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(x,-n \mid \chi) = \widetilde{E}_{n,q}^{(\alpha,w)}(x \mid \chi),$$

which led to stating following theorem:

THEOREM 6. *Let χ be a Dirichlet's character with conductor $d\,(=odd) \in \mathbb{N}$, $w \in T_p$ and $n \in \mathbb{N}^*$, we define*

$$\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(x,-n \mid \chi) = \widetilde{E}_{n,q}^{(\alpha,w)}(x \mid \chi). \tag{3.2}$$

$\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(1,s \mid \chi) = \widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(s \mid \chi)$ which is the modified Dirichlet's type of twisted q-Euler L-function with weight α. Now, by applying combinatorial techniques we can write,

$$\begin{aligned}
\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(s \mid \chi) &= [2]_q \sum_{m=1}^{\infty} \frac{(-1)^m \chi(m) w^m}{[m]_{q^\alpha}^s} \\
&= [2]_q \sum_{m=1}^{\infty} \sum_{a=0}^{d-1} \frac{(-1)^{a+dm} \chi(a+dm) w^{a+dm}}{[a+dm]_{q^\alpha}^s} \\
&= \frac{[2]_q}{[2]_{q^d}} [d]_{q^\alpha}^{-s} \sum_{a=0}^{d-1} (-1)^a \chi(a) w^a \left[[2]_{q^d} \sum_{m=1}^{\infty} \frac{(-1)^m \left(w^d\right)^m}{\left[\left(\frac{a}{d}+m\right)\right]_{q^{d\alpha}}^s} \right] \\
&= \frac{[2]_q}{[2]_{q^d}} [d]_{q^\alpha}^{-s} \sum_{a=0}^{d-1} (-1)^a \chi(a) w^a \widetilde{\zeta}_{q^d}^{\left(\alpha,w^d\right)}\left(s, \frac{a}{d}\right).
\end{aligned} \tag{3.3}$$

So, by previous calculation we can state following theorem:

THEOREM 7. *Let χ be a Dirichlet's character with conductor $d\,(=odd) \in \mathbb{N}$ and $w \in T_p$ we have*

$$\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(s \mid \chi) = \frac{[2]_q}{[2]_{q^d}} [d]_{q^\alpha}^{-s} \sum_{a=0}^{d-1} (-1)^a \chi(a) w^a \widetilde{\boldsymbol{\zeta}}_{q^d}^{\left(\alpha,w^d\right)}\left(s, \frac{a}{d}\right). \tag{3.4}$$

We now consider the partial-zeta function $\widetilde{\boldsymbol{H}}_q^{(\alpha)}(s,a,w \mid F)$ as follows:

$$\widetilde{\boldsymbol{H}}_q^{(\alpha)}(s,a,w \mid F) = [2]_q \sum_{\substack{m \equiv a (\mathrm{mod}\ F) \\ m>0}} \frac{(-1)^m w^m}{[m]_{q^\alpha}^s}. \tag{3.5}$$

If $F \equiv 1 \,(\mathrm{mod}\ 2)$, then we have

$$\begin{aligned}
\widetilde{\boldsymbol{H}}_q^{(\alpha)}(s,a,w \mid F) &= [2]_q \sum_{\substack{m \equiv a (\mathrm{mod}\ F) \\ m>0}} \frac{(-1)^m w^m}{[m]_{q^\alpha}^s} \\
&= [2]_q \sum_{m>0} \frac{(-1)^{mF+a} w^{mF+a}}{[mF+a]_{q^\alpha}^s} \\
&= \frac{[2]_q}{[2]_{q^F}} \frac{(-1)^a w^a}{[F]_{q^\alpha}^s} \left[[2]_{q^F} \sum_{m>0} \frac{(-1)^m \left(w^F\right)^m}{\left[m+\frac{a}{F}\right]_{q^{\alpha F}}^s} \right] \\
&= \frac{[2]_q}{[2]_{q^F}} \frac{(-1)^a w^a}{[F]_{q^\alpha}^s} \widetilde{\zeta}_{q^F}^{(\alpha, w^F)}\left(s, \frac{a}{F}\right)
\end{aligned} \tag{3.6}$$

By expressions (3.2) and (3.6) we get the following theorem:

THEOREM 8. *Let* $F \equiv 1 \,(\mathrm{mod}\ 2)$, $w \in T_p$, q, $s \in \mathbb{C}$, $|q| < 1$ *and* $n \in \mathbb{N}^*$ *we have*

$$\widetilde{\boldsymbol{H}}_q^{(\alpha)}(-n,a,w \mid F) = \frac{[2]_q}{[2]_{q^F}} (-1)^a w^a [F]_{q^\alpha}^n \boldsymbol{E}_{n,q^F}^{(\alpha, w^F)}\left(\frac{a}{F}\right). \tag{3.7}$$

By expressions (3.4) and (3.7), we obtain the following corollary:

COROLLARY 2. *Let* χ *be a Dirichlet's character with conductor* $d\,(=odd) \in \mathbb{N}$, $w \in T_p$ *and* $F \equiv 1 \,(\mathrm{mod}\ 2)$ *we have*

$$\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(s \mid \chi) = \sum_{a=0}^{F-1} \chi(a) \widetilde{\boldsymbol{H}}_q^{(\alpha)}(s,a,w \mid F). \tag{3.8}$$

By (1.8) and (3.7), we modify the q-analogue of the partial zeta function with weight α as follows:

$$\widetilde{\boldsymbol{H}}_q^{(\alpha)}(s,a,w \mid F) = \frac{[2]_q}{[2]_{q^F}} (-1)^a w^a [a]_{q^\alpha}^{-s} \sum_{l=0}^{\infty} \binom{-s}{l} q^{\alpha a l} \left(\frac{[F]_{q^\alpha}}{[a]_{q^\alpha}}\right)^l \boldsymbol{E}_{l,q^F}^{(\alpha, w^F)}. \tag{3.9}$$

Let f (=odd) and a be the positive integer with $0 \leqslant a < F$. Then, (3.8) reduces to

$$\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(s \mid \chi) = \frac{[2]_q}{[2]_{q^F}} \sum_{a=0}^{F-1} \chi(a) (-1)^a w^a [a]_{q^\alpha}^{-s} \sum_{l=0}^{\infty} \binom{-s}{l} q^{\alpha a l} \left(\frac{[F]_{q^\alpha}}{[a]_{q^\alpha}}\right)^l \boldsymbol{E}_{l,q^F}^{(\alpha, w^F)}. \tag{3.10}$$

By expression (3.10), we see that $\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(s \mid \chi)$ is an analytic function $s \in \mathbb{C}$, with except $s = 0$. Furthermore, for each $n \in \mathbb{Z}$, with $n \geqslant 0$, we get

$$\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(-n \mid \chi) = \widetilde{E}_{n,q}^{(\alpha,w)}(\chi). \tag{3.11}$$

By using (3.9), (3.10) and (3.11) we derive behavior of the modified Dirichlet's type of twisted q-Euler L-function with weight α at $s = 0$ as follows:

THEOREM 9. *The following likeable identity*

$$\widetilde{\boldsymbol{L}}_q^{(\alpha,w)}(0 \mid \chi) = \frac{1+q}{1+w^F} \sum_{a=0}^{F-1} (-1)^a \chi(a) w^a,$$

is true.

4. Modified p-adic twisted interpolation q-l-function with weight α

In this section, we construct modified p-adic twisted q-Euler l-function, which interpolate modified Dirichlet's type of twisted q-Euler polynomials at negative integers. Firstly, Washington constructed p-adic l-function which interpolates generalized classical Bernoulli numbers. Throughout this final section, we assume that $q \in \mathbb{C}_p$ with $|q-1|_p < 1$.

Here, we use some the following notations, which will be useful in reminder of paper.

Let ω denote the *Kummer* character by the conductor $f_\omega = p$. For an arbitrary character χ, we set $\chi_n = \chi\omega^{-n}$, $n \in \mathbb{Z}$, in the sense of product of characters.

Let

$$\langle a \rangle = \omega^{-1}(a)a = \frac{a}{\omega(a)},$$
$$\langle a \rangle_q = \frac{[a]_q}{\omega(a)}.$$

Thus, we note that $\langle a \rangle \equiv 1 \pmod{p\mathbb{Z}_p}$. Let

$$A_j(x) = \sum_{n=0}^{\infty} a_{n,j} x^n,\ a_{n,j} \in \mathbb{C}_p,\ j = 0, 1, 2, \ldots$$

be a sequence of power series, each convergent on a fixed subset

$$T = \left\{ s \in \mathbb{C}_p \mid |s|_p < p^{-\frac{2-p}{p-1}} \right\},$$

of $\mathbb{C}_p$ such that

(1) $a_{n,j} \to a_{n,0}$ as $j \to \infty$ for any n;

(2) for each $s \in T$ and $\varepsilon > 0$, there exists an $n_0 = n_0(s, \varepsilon)$ such that

$$\left| \sum_{n \geqslant n_0} a_{n,j} s^n \right|_p < \varepsilon \text{ for } \forall j.$$

So,

$$\lim_{j \to \infty} A_j(s) = A_0(s), \text{ for all } s \in T.$$

This was constructed by Washington [24] to indicate that each functions $\omega^{-s}(a) a^s$ and

$$\sum_{l=0}^{\infty} \binom{s}{l} \left(\frac{F}{a}\right)^l B_l,$$

where F is multiple of p and f and B_l is the l-th Bernoulli numbers, is analytic on T (for more information, see [24]).

Assume that χ be a Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1 (\text{mod}\, 2)$. Thus, we consider the modified Dirichlet's type of twisted p-adic q-Euler l-function with weight α, $l_{p,q}^{(\alpha,w)}(s \mid \chi)$, which interpolate the modified Dirichlet's type of twisted q-Euler polynomials with weight α at negative integers.

For $f \in \mathbb{N}$ with $f \equiv 1 (\text{mod}\, 2)$, let us assume that F is positive integral multiple of p and $f = f_\chi$. We are now ready to give definition of $l_{p,q}^{(\alpha,w)}(s \mid \chi)$ as follows:

$$l_{p,q}^{(\alpha,w)}(s \mid \chi) = \sum_{a=0}^{F-1} \chi(a) (-1)^a w^a \langle a \rangle_{q^\alpha}^{-s} \sum_{l=0}^{\infty} \binom{-s}{l} q^{\alpha a l} \left(\frac{[F]_{q^\alpha}}{[a]_{q^\alpha}} \right)^l \boldsymbol{E}_{l,q^F}^{(\alpha, w^F)}. \tag{4.1}$$

By (4.1), we note that $l_{p,q}^{(\alpha,w)}(s \mid \chi)$ is analytic for $s \in T$.
For $n \in \mathbb{N}$, we have

$$\widetilde{E}_{n,\chi_n}^{(\alpha,w)} = [F]_{q^\alpha}^n \sum_{a=0}^{F-1} (-1)^a \chi_n(a) \boldsymbol{E}_{n,q}^{(\alpha,w)}\left(\frac{a}{F}\right). \tag{4.2}$$

If $\chi_n(p) \neq 0$, then $(p, f_{\chi_n}) = 1$, and thus the ratio $\frac{F}{p}$ is a multiple of f_{χ_n}.
Let

$$\lambda = \left\{ \frac{a}{p} \mid a \equiv 0 (\text{mod}\, p) \text{ for } a_i \in \mathbb{Z} \text{ with } 0 \leqslant a_i < F \right\}.$$

Thus, we have

$$[F]_{q^\alpha}^n \sum_{\substack{a=0 \\ p \mid a}}^{F-1} (-1)^a \chi_n(a) \boldsymbol{E}_{n,q}^{(\alpha,w)}\left(\frac{a}{F}\right) \tag{4.3}$$

$$= \frac{1}{[p^{-1}]_{q^{\alpha F}}^n} \left[\frac{F}{p}\right]_{q^\alpha}^n \chi_n(p) \sum_{\substack{a=0 \\ \eta \in \lambda}}^{\frac{F}{p}} (-1)^\eta \chi_n(\eta) \boldsymbol{E}_{n,q}^{(\alpha,w)}\left(\frac{\eta}{F/p}\right).$$

By (4.3), we can define the second modified twisted generalized Euler numbers attached to χ as follows:

$$\widetilde{E}^{*(\alpha,w)}_{n,\chi_n} = \left[\frac{F}{p}\right]^n_{q^\alpha} \sum_{\substack{a=0 \\ \eta\in\lambda}}^{\frac{F}{p}} (-1)^\eta \chi_n(\eta) \boldsymbol{E}^{(\alpha,w)}_{n,q}\left(\frac{\eta}{F/p}\right). \tag{4.4}$$

By (4.2), (4.3) and (4.4), we readily get that

$$\widetilde{E}^{(\alpha,w)}_{n,\chi_n} - \frac{1}{[p^{-1}]^n_{q^{\alpha F}}}\chi_n(p)\widetilde{E}^{*(\alpha,w)}_{n,\chi_n} = [F]^n_{q^\alpha} \sum_{\substack{a=0 \\ p\nmid a}}^{F-1} (-1)^a \chi_n(a) \boldsymbol{E}^{(\alpha,w)}_{n,q}\left(\frac{a}{F}\right) \tag{4.5}$$

By (4.1) and (1.9), we readily see that

$$\begin{aligned} l^{(\alpha,w)}_{p,q}(-n \mid \chi) &= [F]^n_{q^\alpha} \sum_{\substack{a=0 \\ p\nmid a}}^{F-1} (-1)^a \chi_n(a) \boldsymbol{E}^{(\alpha,w)}_{n,q}\left(\frac{a}{F}\right) \\ &= \widetilde{E}^{(\alpha,w)}_{n,\chi_n} - \frac{1}{[p^{-1}]^n_{q^{\alpha F}}}\chi_n(p)\widetilde{E}^{*(\alpha,w)}_{n,\chi_n}. \end{aligned}$$

Consequently, we state the following Theorem:

THEOREM 10. *Let $n \in \mathbb{N}$, the following equalities*

$$l^{(\alpha,w)}_{p,q}(s \mid \chi) = \sum_{a=1}^{F} \chi(a)(-1)^a w^a \langle a\rangle^{-s}_{q^\alpha} \sum_{l=0}^{\infty} \binom{-s}{l} q^{\alpha a l} \left(\frac{[F]_{q^\alpha}}{[a]_{q^\alpha}}\right)^l \boldsymbol{E}^{\left(\alpha,w^F\right)}_{l,q^F},$$

and

$$l^{(\alpha,w)}_{p,q}(-n \mid \chi) = \widetilde{E}^{(\alpha,w)}_{n,\chi_n} - \frac{1}{[p^{-1}]^n_{q^{\alpha F}}}\chi_n(p)\widetilde{E}^{*(\alpha,w)}_{n,\chi_n},$$

are true.

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