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CERTAIN SUBCLASSES OF BI–UNIVALENT FUNCTIONS SATISFYING SUBORDINATE CONDITIONS

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Abstract. In this paper, we introduce and investigate each of the following subclasses:

$$\mathscr{S}_{\Sigma}(\lambda,\gamma;\varphi),\quad \mathscr{H}\mathscr{S}_{\Sigma}(\alpha),\quad \mathscr{R}_{\Sigma}(\eta,\gamma;\varphi)\quad \text{and}\quad \mathscr{B}_{\Sigma}(\mu;\varphi)$$

$$(0\leqslant\lambda\leqslant 1;\ \gamma\in\mathbb{C}\diagdown\{0\};\ \alpha\in\mathbb{C};\ 0\leqslant\eta<1;\ \mu\geqslant 0)$$

of bi-univalent functions, φ is an analytic function with positive real part in the unit disk $\mathbb{D}$, satisfying $\varphi(0)=1$, $\varphi'(0)>0$, and $\varphi(\mathbb{D})$ is symmetric with respect to the real axis. We obtain coefficient bounds involving the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of the function f when f is in these classes. The various results, which are presented in this paper, would generalize and improve those in related works of several earlier authors.

1. Introduction

Let $\mathscr{A}$ be the class of all analytic functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1.1}$$

in the open unit disk $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$ normalized by the conditions $f(0)=0$ and $f'(0)=1$. The well-known Koebe one-quarter theorem [5] ensures that the image of $\mathbb{D}$ under every univalent function $f\in\mathscr{A}$ contains a disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z))=z,\ (z\in\mathbb{D})$ and

$$f^{-1}(f(w))=w,\ \ (|w|<r_0(f),\ r_0(f)\geqslant 1/4)$$

where

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-\ldots.$$

A function $f\in\mathscr{A}$ is said to be *bi-univalent* in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{D}$. A function $f\in\mathscr{A}$ is called starlike, denoted by $f\in\mathscr{S}^*$, if $tw\in f(\mathbb{D})$ whenever $w\in f(\mathbb{D})$ and $t\in[0,1]$. A function $f\in\mathscr{A}$ that maps the unit disk onto a convex domain is called a convex function. Let $\mathscr{C}$ denote the class of all functions $f\in\mathscr{A}$ that are convex. Analytically, these geometric properties are, respectively, equivalent

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to the conditions $\Re zf'(z)/f(z) > 0$ and $1+\Re zf''(z)/f'(z) > 0$. An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. In terms of subordination, these conditions are, respectively, equivalent to $zf'(z)/f(z) \prec (1+z)/(1-z)$ and $1+zf''(z)/f'(z) \prec (1+z)/(1-z)$. Ma and Minda [8] gave a unified presentation of various subclasses of starlike and convex functions by replacing the subordinate function $(1+z)/(1-z)$ by a more general analytic function φ with positive real part and normalized by the conditions $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. They introduced the following general classes that envelopes several well-known classes as special cases:

$$\mathscr{S}^*(\varphi) = \left\{ f \in \mathscr{A} :\ \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \text{ and } \mathscr{C}(\varphi) = \left\{ f \in \mathscr{A} :\ 1+\frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

In literature, the functions belonging to these classes are called Ma-Minda starlike and Ma-Minda convex, respectively. A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are, respectively, Ma-Minda starlike or convex. These classes are denoted, respectively, by $\mathscr{S}^*_{\Sigma}(\varphi)$ and $\mathscr{C}_{\Sigma}(\varphi)$. For β $(0 \leqslant \beta < 1)$, let $\varphi_\beta : \mathbb{D} \to \mathbb{C}$ be defined by $\varphi_\beta(z) = (1+(1-2\beta)z)/(1-z)$. Then the classes $\mathscr{S}^*(\varphi_\beta)$ and $\mathscr{C}(\varphi_\beta)$ reduce to, respectively, the familiar classes $\mathscr{S}^*(\beta)$ and $\mathscr{C}(\beta)$ of univalent starlike of and convex functions of order β which are, respectively, characterized by $\Re zf'(z)/f(z) > \beta$ and $1+\Re zf''(z)/f'(z) > \beta$. Also we say that a function $f \in \mathscr{A}$ is Ma-Minda s*tarlike and convex of complex order* γ $(\gamma \in \mathbb{C}\setminus\{0\})$:

$$\mathscr{S}^*(\gamma;\varphi) = \left\{ f \in \mathscr{A} :\ 1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right) \prec \varphi(z) \right\}$$

and

$$\mathscr{C}(\gamma;\varphi) = \left\{ f \in \mathscr{A} :\ 1+\frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\},$$

respectively. A function f is *bi-starlike* and *bi-convex of complex order* γ $(\gamma \in \mathbb{C}\setminus\{0\})$ of Ma-Minda type if both f and f^{-1} are, respectively, Ma-Minda starlike and Ma-Minda convex of complex order γ. These classes are denoted respectively by $\mathscr{S}^*_{\Sigma}(\gamma;\varphi)$ and $\mathscr{C}_{\Sigma}(\gamma;\varphi)$. Especially the classes $\mathscr{S}^*_{\Sigma}(\gamma;\frac{1+z}{1-z}) = \mathscr{S}^*_{\Sigma}(\gamma)$ and $\mathscr{C}_{\Sigma}(\gamma;\frac{1+z}{1-z}) = \mathscr{C}_{\Sigma}(\gamma)$ are *bi-starlike* and *bi-convex functions of complex order* γ $(\gamma \in \mathbb{C}\setminus\{0\})$, respectively.

We note that:

(1) $\mathscr{S}^*_{\Sigma}((1-\beta)e^{-i\lambda}\cos\lambda) = \mathscr{S}^*_{\Sigma}[\lambda,\beta]$ $(|\lambda| < \pi/2,\ 0 \leqslant \beta < 1)$ is class of bi-λ-spirallike univalent functions of order β

(2) $\mathscr{C}_{\Sigma}((1-\beta)e^{-i\lambda}\cos\lambda) = \mathscr{C}_{\Sigma}[\lambda,\beta]$ $(|\lambda| < \pi/2,\ 0 \leqslant \beta < 1)$ is class of bi-λ-Robertson univalent functions of order β.

The class Σ of bi-univalent functions was introduced in 1967 by Lewin [7] and he showed that, for every functions $f \in \Sigma$ of the form (1.1), the second coefficient of f satisfy the inequality $|a_2| < 1.51$. Subsequently, Brannan and Clunie [4] improved Lewin's result by showing $|a_2| \leqslant \sqrt{2}$. Later, Netanyahu [9] proved that $\max_{f\in\Sigma}|a_2| =$

$4/3$. Since then, various subclasses of the bi-univalent function class Σ were introduced and *non-sharp* estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor–Maclaurin series expansion (1.1) were found in several recent investigations (see [3], [16]; see also [1], [6], [13] and [18])

The object of the present paper is to introduce four new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ. As special cases, estimates on the initial coefficients for bi-starlike and bi-convex of complex order γ $(\gamma \in \mathbb{C} \diagdown \{0\})$ of Ma-Minda type functions are obtained. Several related classes are also considered, and connection to earlier known results are made.

In order to derive our main results, we have to recall here the following lemma [5].

LEMMA 1.1. *If $p \in \mathscr{P}$ then $|b_k| \leqslant 2$ for each k, where $\mathscr{P}$ is the family of all functions p analytic in $\mathbb{D}$ for which $\Re p(z) > 0$, $p(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \ldots$ for $z \in \mathbb{D}$.*

2. Coefficient estimates

In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disk $\mathbb{D}$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, and $\varphi(\mathbb{D})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, \quad B_1 > 0. \tag{2.1}$$

Define the functions p_1 and p_2 in $\mathscr{P}$ given by

$$p_1(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$$

and

$$p_2(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \ldots.$$

It follows

$$u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{c_1}{2} z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \ldots \tag{2.2}$$

and

$$v(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{d_1}{2} z + \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right) z^2 + \ldots. \tag{2.3}$$

Using (2.2) and (2.3) together with (2.1), it is evident that

$$\varphi(u(z)) = 1 + \frac{B_1 c_1}{2} z + \left\{\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) B_1 + \frac{1}{4} c_1^2 B_2\right\} z^2 + \ldots \tag{2.4}$$

and

$$\varphi(v(z)) = 1 + \frac{B_1 d_1}{2} z + \left\{\frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right) B_1 + \frac{1}{4} d_1^2 B_2\right\} z^2 + \ldots \tag{2.5}$$

A function $f \in \Sigma$ is said to be in the class $\mathscr{S}_\Sigma(\lambda,\gamma;\varphi),\ 0 \leqslant \lambda \leqslant 1,\ \gamma \in \mathbb{C}\diagdown\{0\}$ if the following subordinations hold:

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}-1\right) \prec \varphi(z)$$
$$\text{and } 1+\frac{1}{\gamma}\left(\frac{wg'(w)+\lambda w^2 g''(w)}{\lambda w g'(w)+(1-\lambda)g(w)}-1\right) \prec \varphi(w), \quad \left(g(w) := f^{-1}(w)\right).$$

A function in the class $\mathscr{S}_\Sigma(\lambda,\gamma;\varphi)$ is called both bi-$\lambda-$convex function and bi-$\lambda-$starlike function of complex order γ of Ma-Minda type. This class introduced in this paper is motivated by the corresponding class investigated in [17] and [14].

This class unifies the classes $\mathscr{S}^*_\Sigma(\gamma;\varphi)$ and $\mathscr{C}_\Sigma(\gamma;\varphi)$. Note that $\mathscr{S}_\Sigma(0,\gamma;\varphi) \equiv \mathscr{S}^*_\Sigma(\gamma;\varphi)$ and $\mathscr{S}_\Sigma(1,\gamma;\varphi) \equiv \mathscr{C}_\Sigma(\gamma;\varphi)$. Also $\mathscr{S}_\Sigma(0,(1-\alpha)e^{-i\lambda}\cos\lambda;\frac{1+z}{1-z}) \equiv \mathscr{S}^*_\Sigma[\lambda,\alpha]$ and $\mathscr{S}_\Sigma(1,(1-\alpha)e^{-i\lambda}\cos\lambda;\frac{1+z}{1-z}) \equiv \mathscr{C}_\Sigma[\lambda,\alpha]$ $\ (|\lambda| < \pi/2,\ 0 \leqslant \alpha < 1)$.

For functions in the class $\mathscr{S}_\Sigma(\lambda,\gamma;\varphi)$, the following coefficient estimation holds.

THEOREM 2.1. *Let f given by (1.1) be in the class $\mathscr{S}_\Sigma(\lambda,\gamma;\varphi)$. Then*

$$|a_2| \leqslant \frac{|\gamma| B_1\sqrt{B_1}}{\sqrt{\left|\gamma(1+2\lambda-\lambda^2)B_1^2+(1+\lambda)^2(B_1-B_2)\right|}} \tag{2.6}$$

and

$$|a_3| \leqslant \frac{|\gamma|\left(B_1+|B_1-B_2|\right)}{1+2\lambda-\lambda^2}. \tag{2.7}$$

Proof. Let $f \in \mathscr{S}_\Sigma(\lambda,\gamma;\varphi)$. Then there are analytic functions $u,v:\mathbb{D}\to\mathbb{D}$, with $u(0)=v(0)=0$, satisfying

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}-1\right) = \varphi(u(z)) \tag{2.8}$$
$$\text{and } 1+\frac{1}{\gamma}\left(\frac{wg'(w)+\lambda w^2 g''(w)}{\lambda w g'(w)+(1-\lambda)g(w)}-1\right) = \varphi(v(w)), \ \left(\ g := f^{-1}\right).$$

Since

$$\begin{aligned}&1+\frac{1}{\gamma}\left(\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}-1\right)\\ &= 1+\frac{(1+\lambda)a_2}{\gamma}\,z+\frac{2(1+2\lambda)a_3-(1+\lambda)^2a_2^2}{\gamma}z^2+\ldots\end{aligned}$$

and

$$\begin{aligned}&1+\frac{1}{\gamma}\left(\frac{wg'(w)+\lambda w^2 g''(w)}{\lambda w g'(w)+(1-\lambda)g(w)}-1\right)\\ &= 1-\frac{(1+\lambda)a_2}{\gamma}\,w+\frac{-2(1+2\lambda)a_3+(3+6\lambda-\lambda^2)a_2^2}{\gamma}w^2+\ldots\end{aligned}$$

then (2.4), (2.5) and (2.8) yield

$$\frac{(1+\lambda)a_2}{\gamma} = \frac{B_1 c_1}{2}, \tag{2.9}$$

$$\frac{2(1+2\lambda)a_3 - (1+\lambda)^2 a_2^2}{\gamma} = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) B_1 + \frac{1}{4} c_1^2 B_2, \tag{2.10}$$

$$-\frac{(1+\lambda)a_2}{\gamma} = \frac{B_1 d_1}{2} \tag{2.11}$$

and

$$\frac{-2(1+2\lambda)a_3 + (3+6\lambda-\lambda^2)a_2^2}{\gamma} = \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right) B_1 + \frac{1}{4} d_1^2 B_2. \tag{2.12}$$

Now, considering (2.9) and (2.11), we get

$$c_1 = -d_1. \tag{2.13}$$

Also, from (2.10), (2.11), (2.12) and (2.13), we find that

$$a_2^2 = \frac{\gamma^2 B_1^3 (c_2 + d_2)}{4\left[\gamma(1+2\lambda-\lambda^2)B_1^2 + (1+\lambda)^2(B_1 - B_2)\right]}$$

which, in view of the inequalities $|c_2| \leqslant 2$ and $|d_2| \leqslant 2$ from Lemma 1.1, yield

$$|a_2|^2 \leqslant \frac{|\gamma|^2 B_1^3}{\left|\gamma(1+2\lambda-\lambda^2)B_1^2 + (1+\lambda)^2(B_1 - B_2)\right|}.$$

Since $B_1 > 0$, the last inequality gives the desired estimate on $|a_2|$ given in (2.6).

Next, in order to find the bound on $|a_3|$, by further computations from (2.10), (2.11), (2.12) and (2.13) lead to

$$a_3 = \frac{(\gamma B_1/2)\left((3+6\lambda-\lambda^2)c_2 + (1+\lambda)^2 d_2\right) + \gamma(1+2\lambda)d_1^2(B_2 - B_1)}{4(1+2\lambda)(1+2\lambda-\lambda^2)}.$$

Applying Lemma 1.1 for the coefficients d_1, c_2 and d_2, we readily get

$$\begin{aligned} |a_3| &\leqslant \frac{(|\gamma| B_1/2)\left(2(3+6\lambda-\lambda^2) + 2(1+\lambda)^2\right) + 4|\gamma|(1+2\lambda)|B_2 - B_1|}{4(1+2\lambda)(1+2\lambda-\lambda^2)} \\ &= \frac{|\gamma|(B_1 + |B_1 - B_2|)}{1+2\lambda-\lambda^2} \end{aligned}$$

which is the bound on $|a_3|$ as asserted in *(2.7)*. □

COROLLARY 2.2. *Let f given by (1.1) be in the class $\mathscr{C}_\Sigma(\gamma;\varphi)$. Then*

$$|a_2| \leqslant \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{2\left|\gamma B_1^2 + 2(B_1 - B_2)\right|}} \text{ and } |a_3| \leqslant \frac{|\gamma|(B_1 + |B_1 - B_2|)}{2}.$$

COROLLARY 2.3. *Let f given by (1.1) be in the class $\mathcal{S}^*_\Sigma(\gamma;\varphi)$. Then*

$$|a_2| \leqslant \frac{|\gamma| B_1\sqrt{B_1}}{\sqrt{\left|\gamma B_1^2 + B_1 - B_2\right|}} \text{ and } |a_3| \leqslant |\gamma|\left(B_1 + |B_1 - B_2|\right).$$

REMARK 2.4. If the function is bi-λ-spirallike univalent of order β, that is $f \in \mathcal{S}^*_\Sigma[\lambda,\beta]$, then

$$|a_2| \leqslant \sqrt{2(1-\beta)\cos\lambda} \text{ and } |a_3| \leqslant 2(1-\beta)\cos\lambda.$$

If the function is bi-λ-Robertson of order β, that is $f \in \mathcal{C}_\Sigma[\lambda,\beta]$, then

$$|a_2| \leqslant \sqrt{(1-\beta)\cos\lambda} \text{ and } |a_3| \leqslant (1-\beta)\cos\lambda.$$

Let α be a complex number. We say that a function $f \in \Sigma$ belongs to the class $\mathcal{HS}_\Sigma(\alpha)$, $\alpha \in \mathbb{C}$ if the functions F and G defined by

$$\frac{1}{F(z)} = \frac{1-\alpha}{f(z)} + \frac{\alpha}{zf'(z)}$$

and

$$\frac{1}{G(w)} = \frac{1-\alpha}{g(w)} + \frac{\alpha}{wg'(w)},$$

are in the class $\mathcal{S}^*_\Sigma(\varphi)$, $g(w) := f^{-1}(w)$. This class introduced in this paper is motivated by the corresponding class investigated in [10].

Note that $\mathcal{HS}_\Sigma(0) \equiv \mathcal{S}^*_\Sigma(\varphi)$ and $\mathcal{HS}_\Sigma(1) \equiv \mathcal{C}_\Sigma(\varphi)$. For functions in the class $\mathcal{HS}_\Sigma(\alpha)$, the following coefficient estimation holds.

THEOREM 2.5. *Let f given by (1.1) be in the class $\mathcal{HS}_\Sigma(\alpha)$. Then*

$$|a_2| \leqslant \frac{B_1\sqrt{B_1}}{\sqrt{\left|(1+\alpha^2)B_1^2 + (1+\alpha)^2(B_1 - B_2)\right|}} \tag{2.14}$$

and

$$|a_3| \leqslant \frac{(B_1/4)\left(|(\alpha+3)(\alpha+1)| + \left|\alpha^2 - 4\alpha - 1\right|\right) + (1+2\alpha)\,|B_2 - B_1|}{|1+2\alpha|\,|\alpha^2+1|}, \tag{2.15}$$

Proof. Let $f \in \mathcal{HS}_\Sigma(\alpha)$. Simple calculations show that if f is in $\mathcal{HS}_\Sigma(\alpha)$, then

$$\frac{zf'(z)}{f(z)} + \left(1 + \frac{zf''(z)}{f'(z)}\right) - \frac{zf'(z) + (1-\alpha)z^2 f''(z)}{(1-\alpha)zf'(z) + \alpha f(z)} \prec \varphi(z)$$

and

$$\frac{wg'(w)}{g(w)} + \left(1 + \frac{wg''(w)}{g'(w)}\right) - \frac{wg'(w) + (1-\alpha)w^2 g''(w)}{(1-\alpha)wg'(w) + \alpha g(w)} \prec \varphi(w).$$

Consider the analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\frac{zf'(z)}{f(z)} + \left(1 + \frac{zf''(z)}{f'(z)}\right) - \frac{zf'(z) + (1-\alpha)z^2 f''(z)}{(1-\alpha)zf'(z) + \alpha f(z)} = \varphi(u(z)) \quad (2.16)$$

and $$\frac{wg'(w)}{g(w)} + \left(1 + \frac{wg''(w)}{g'(w)}\right) - \frac{wg'(w) + (1-\alpha)w^2 g''(w)}{(1-\alpha)wg'(w) + \alpha g(w)} = \varphi(v(w))$$

where $g := f^{-1}$. Since

$$\begin{aligned}&\frac{zf'(z)}{f(z)} + \left(1 + \frac{zf''(z)}{f'(z)}\right) - \frac{zf'(z) + (1-\alpha)z^2 f''(z)}{(1-\alpha)zf'(z) + \alpha f(z)} \\ &= 1 + (1+\alpha)a_2\, z + \left(2(1+2\alpha)a_3 + (\alpha^2 - 4\alpha - 1)a_2^2\right) z^2 + \ldots\end{aligned}$$

and

$$\begin{aligned}&\frac{wg'(w)}{g(w)} + \left(1 + \frac{wg''(w)}{g'(w)}\right) - \frac{wg'(w) + (1-\alpha)w^2 g''(w)}{(1-\alpha)wg'(w) + \alpha g(w)} \\ &= 1 - (1+\alpha)a_2\, w + \left(-2(1+2\alpha)a_3 + (\alpha+3)(\alpha+1)a_2^2\right) w^2 + ..\end{aligned}$$

then (2.4), (2.5) and (2.16) yield

$$(1+\alpha)a_2 = \frac{B_1 c_1}{2}, \quad (2.17)$$

$$2(1+2\alpha)a_3 + (\alpha^2 - 4\alpha - 1)a_2^2 = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)B_1 + \frac{1}{4}c_1^2 B_2, \quad (2.18)$$

$$-(1+\alpha)a_2 = \frac{B_1 d_1}{2} \quad (2.19)$$

and

$$-2(1+2\alpha)a_3 + (\alpha+3)(\alpha+1)a_2^2 = \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)B_1 + \frac{1}{4}d_1^2 B_2. \quad (2.20)$$

Now, considering (2.17) and (2.19), we get

$$c_1 = -d_1. \quad (2.21)$$

Also, from (2.18), (2.19), (2.20) and (2.21), we find that

$$a_2^2 = \frac{B_1^3(c_2 + d_2)}{4\left[(1+\alpha^2)B_1^2 + (1+\alpha)^2(B_1 - B_2)\right]}$$

which, in view of the inequalities $|c_2| \leqslant 2$ and $|d_2| \leqslant 2$ from Lemma 1.1, yield

$$|a_2|^2 \leqslant \frac{B_1^3}{\left|(1+\alpha^2)B_1^2 + (1+\alpha)^2(B_1 - B_2)\right|}.$$

Since $B_1 > 0$, the last inequality gives the desired estimate on $|a_2|$ given in (2.14).

Next, in order to find the bound on $|a_3|$, by further computations from (2.18), (2.19), (2.20) and (2.21) lead to

$$a_3 = \frac{(B_1/2)\left((\alpha+3)(\alpha+1)c_2 - \left(\alpha^2-4\alpha-1\right)d_2\right) + (1+2\alpha)d_1^2\,(B_2-B_1)}{4\,(1+2\alpha)\,(\alpha^2+1)}.$$

Applying Lemma 1.1 for the coefficients d_1, c_2 and d_2, we readily get

$$\begin{aligned}|a_3| &\leqslant \frac{(B_1/2)\left(2\,|(\alpha+3)(\alpha+1)| + 2\left|\alpha^2-4\alpha-1\right|\right) + 4\,|1+2\alpha|\,|B_2-B_1|}{4\,|1+2\alpha|\,|\alpha^2+1|}\\ &= \frac{(B_1/4)\left(|(\alpha+3)(\alpha+1)| + \left|\alpha^2-4\alpha-1\right|\right) + |1+2\alpha|\,|B_2-B_1|}{|1+2\alpha|\,|\alpha^2+1|}\end{aligned}$$

which is the bound on $|a_3|$ as asserted in *(2.15)*. □

Let $0 \leqslant \eta < 1$ and $\gamma \in \mathbb{C}\setminus\{0\}$. A function $f \in \Sigma$ is in the class $\mathscr{R}_\Sigma(\eta,\gamma;\varphi)$ if

$$1 + \frac{1}{\gamma}\left(f'(z) + \eta z f''(z) - 1\right) \prec \varphi(z)$$

and

$$1 + \frac{1}{\gamma}\left(g'(w) + \eta w g''(w) - 1\right) \prec \varphi(w),$$

$g(w) := f^{-1}(w)$. The class $\mathscr{R}_\Sigma(0,1;\varphi) \equiv H_\Sigma(\varphi)$ was studied by Ali et al [1]. This class introduced in this paper is motivated by the corresponding class investigated in [15].

For functions in the class $\mathscr{R}_\Sigma(\eta,\gamma;\varphi)$, the following coefficient estimation holds.

THEOREM 2.6. *Let f given by (1.1) be in the class $\mathscr{R}_\Sigma(\eta,\gamma;\varphi)$. Then*

$$|a_2| \leqslant \frac{|\gamma|\,B_1\sqrt{B_1}}{\sqrt{\left|3\gamma(1+2\eta)B_1^2 + 4(1+\eta)^2(B_1-B_2)\right|}} \tag{2.22}$$

and

$$|a_3| \leqslant |\gamma|\,B_1\left[\frac{|\gamma|\,B_1}{4(1+\eta)^2} + \frac{1}{3(1+2\eta)}\right] \tag{2.23}$$

Proof. Let $f \in \mathscr{R}_\Sigma(\eta,\gamma;\varphi)$. Consider the analytic functions $u,v:\mathbb{D}\to\mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$1 + \frac{1}{\gamma}\left(f'(z) + \eta z f''(z) - 1\right) = \varphi(u(z)) \tag{2.24}$$

$$\text{and } 1 + \frac{1}{\gamma}\left(g'(w) + \eta w g''(w) - 1\right) = \varphi(v(w)),\ \left(g := f^{-1}\right).$$

Since

$$1+\frac{1}{\gamma}\left(f'(z)+\eta z f''(z)-1\right)=1+\frac{2(1+\eta)}{\gamma}a_2\,z+\frac{3(1+2\eta)}{\gamma}a_3z^2+\ldots$$

and

$$1+\frac{1}{\gamma}\left(g'(w)+\eta w g''(w)-1\right)=1-\frac{2(1+\eta)}{\gamma}a_2\,w+\frac{6(1+2\eta)a_2^2-3(1+2\eta)a_3}{\gamma}w^2+\ldots$$

then (2.4), (2.5) and (2.24) yield

$$\frac{2(1+\eta)}{\gamma}a_2=\frac{B_1c_1}{2}, \tag{2.25}$$

$$\frac{3(1+2\eta)}{\gamma}a_3=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)B_1+\frac{1}{4}c_1^2B_2, \tag{2.26}$$

$$-\frac{2(1+\eta)}{\gamma}a_2=\frac{B_1d_1}{2} \tag{2.27}$$

and

$$\frac{6(1+2\eta)a_2^2-3(1+2\eta)a_3}{\gamma}=\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)B_1+\frac{1}{4}d_1^2B_2. \tag{2.28}$$

Now, considering (2.25) and (2.27), we get

$$c_1=-d_1. \tag{2.29}$$

Also, from (2.26), (2.27), (2.28) and (2.29), we find that

$$a_2^2=\frac{\gamma^2B_1^3(c_2+d_2)}{4\left[3\gamma(1+2\eta)B_1^2+4(1+\eta)^2(B_1-B_2)\right]}$$

which, in view of the inequalities $|c_2|\leqslant 2$ and $|d_2|\leqslant 2$ from Lemma 1.1, yield

$$|a_2|^2\leqslant\frac{|\gamma|^2B_1^3}{\left|3\gamma(1+2\eta)B_1^2+4(1+\eta)^2(B_1-B_2)\right|}.$$

Since $B_1>0$, the last inequality gives the desired estimate on $|a_2|$ given in (2.22).

Next, in order to find the bound on $|a_3|$, by further computations from (2.26), (2.27), (2.28) and (2.29) lead to

$$a_3=\gamma B_1\left[\frac{\gamma B_1d_1^2}{16(1+\eta)^2}+\frac{(c_2-d_2)}{12(1+2\eta)}\right].$$

Applying Lemma 1.1 for the coefficients d_1, c_2 and d_2, we readily get

$$|a_3|\leqslant|\gamma|B_1\left[\frac{|\gamma|B_1}{4(1+\eta)^2}+\frac{1}{3(1+2\eta)}\right]$$

which is the bound on $|a_3|$ as asserted in *(2.15)*. $\square$

For the class of strongly starlike functions and starlike functions of order κ, respectively, the function φ is given by

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\beta} = 1+2\beta z+2\beta^2 z^2+\dots \quad (0<\beta\leqslant 1)$$

which gives $B_1 = 2\beta,\ B_2 = 2\beta^2$ and

$$\varphi(z) = \frac{1+(1-2\kappa)z}{1-z} = 1+2(1-\kappa)z+2(1-\kappa)z^2+\dots \quad (0\leqslant\kappa<1)$$

which gives $B_1 = B_2 = 2(1-\kappa)$.

REMARK 2.7. *i.* Putting $\gamma=1$ and $\eta=0$ in Theorem 2.6, we obtain the corresponding result given earlier by Ali et al [1].

ii. Putting $\gamma=1,\ \eta=0$ and $\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\beta}$ $\left(\text{also } \varphi(z)=\frac{1+(1-2\kappa)z}{1-z}\right)$ in Theorem 2.6, we obtain the corresponding results given earlier by Srivastava et al [13].

iii. Putting $\gamma=1,\ \lambda=0,\ \varphi(z)=\left(\frac{1+z}{1-z}\right)^{\beta}$ $\left(\text{also } \varphi(z)=\frac{1+(1-2\kappa)z}{1-z}\right)$ and $\gamma=1$, $\lambda=1,\ \varphi(z)=\frac{1+(1-2\kappa)z}{1-z}$ in Theorem 2.1, we obtain the corresponding results given earlier by Brannan and Taha [3].

Let $0\leqslant\mu$. A function $f\in\Sigma$ is in the class $\mathscr{B}_{\Sigma}(\mu;\varphi)$ if

$$\frac{z^{1-\mu}f'(z)}{[f(z)]^{1-\mu}} \prec \varphi(z)$$

and

$$\frac{w^{1-\mu}f'(w)}{[f(w)]^{1-\mu}} \prec \varphi(w),$$

$g(w):=f^{-1}(w)$. A function in the class $\mathscr{B}_{\Sigma}(\mu;\varphi)$ is called bi-Bazilevič function of Ma-Minda type. This class introduced in this paper is motivated by the corresponding class investigated in [11] (also see [2], [12]). This class unifies the classes $\mathscr{S}^*_{\Sigma}(\gamma;\varphi)$.

For functions in the class $\mathscr{B}_{\Sigma}(\mu;\varphi)$, the following coefficient estimation holds.

THEOREM 2.8. *Let f given by (1.1) be in the class $\mathscr{B}_{\Sigma}(\mu;\varphi)$. Then*

$$|a_2| \leqslant \frac{B_1\sqrt{2B_1}}{\sqrt{\mu+1}\sqrt{\left|(\mu+2)B_1^2+2(\mu+1)(B_1-B_2)\right|}} \tag{2.30}$$

and

$$|a_3| \leqslant \begin{cases} \frac{2B_1+2|B_1-B_2|}{(\mu+2)(\mu+1)}, & \text{if } 0\leqslant\mu<1 \\ \frac{(\mu+1)B_1+2|B_1-B_2|}{(\mu+2)(\mu+1)}, & \text{if } \quad \mu\geqslant 1 \end{cases} \tag{2.31}$$

Proof. Let $f \in \mathscr{B}_\Sigma(\mu;\varphi)$. Consider the analytic functions $u,v:\mathbb{D}\to\mathbb{D}$, with $u(0)=v(0)=0$, satisfying

$$\frac{z^{1-\mu}f'(z)}{[f(z)]^{1-\mu}}=\varphi(u(z)) \text{ and } \frac{w^{1-\mu}f'(w)}{[f(w)]^{1-\mu}}=\varphi(v(w)),\ \left(g:=f^{-1}\right). \tag{2.32}$$

Since

$$\frac{z^{1-\mu}f'(z)}{[f(z)]^{1-\mu}}=1+(\mu+1)a_2\,z+\left[(\mu+2)a_3+\frac{(\mu-1)(\mu+2)}{2}a_2^2\right]z^2+\ldots$$

and

$$\frac{w^{1-\mu}f'(w)}{[f(w)]^{1-\mu}}=1-(\mu+1)a_2w+\left[-(\mu+2)a_3+\frac{(\mu+3)(\mu+2)}{2}a_2^2\right]w^2+\ldots$$

then (2.4), (2.5) and (2.32) yield

$$(\mu+1)a_2 = \frac{B_1c_1}{2}, \tag{2.33}$$

$$(\mu+2)a_3+\frac{(\mu-1)(\mu+2)}{2}a_2^2=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)B_1+\frac{1}{4}c_1^2B_2, \tag{2.34}$$

$$-(\mu+1)a_2=\frac{B_1d_1}{2} \tag{2.35}$$

and

$$-(\mu+2)a_3+\frac{(\mu+3)(\mu+2)}{2}a_2^2=\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)B_1+\frac{1}{4}d_1^2B_2. \tag{2.36}$$

Now, considering (2.33) and (2.35), we get

$$c_1=-d_1. \tag{2.37}$$

Also, from (2.34), (2.35), (2.36) and (2.37), we find that

$$a_2^2=\frac{B_1^3(c_2+d_2)}{2(\mu+1)\left[(\mu+2)B_1^2+2(\mu+1)(B_1-B_2)\right]}$$

which, in view of the inequalities $|c_2|\leqslant 2$ and $|d_2|\leqslant 2$ from Lemma 1.1, yield

$$|a_2|^2\leqslant\frac{2B_1^3}{(\mu+1)\left|(\mu+2)B_1^2+2(\mu+1)(B_1-B_2)\right|}.$$

Since $B_1>0$, the last inequality gives the desired estimate on $|a_2|$ given in (2.30).

Proceeding similarly as in the earlier proof, using (2.34), (2.35), (2.36) and (2.37) it follows that

$$a_3=\frac{(\mu+2)B_1\left[(\mu+3)c_2+(1-\mu)d_2\right]-2d_1^2(B_1-B_2)(\mu+2)}{4(\mu+1)(\mu+2)^2}.$$

Applying Lemma 1.1 for the coefficients $d_1,\ c_2$ and $d_2,$ we readily get

$$|a_3| \leqslant \frac{B_1\left[|\mu+3|+|1-\mu|\right]+4\,|B_1-B_2|}{2(\mu+1)(\mu+2)}$$

which yields the estimate (2.31). $\square$

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