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# A CLASS OF HARMONIC STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS ASSOCIATED WITH THE SRIVASTAVA–WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTION

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*Abstract.* Making use of Srivastava-Wright operator we introduced a new class of complex-valued harmonic functions with respect to symmetric points which are orientation preserving, univalent and starlike. We obtain coefficient conditions, extreme points, distortion bounds, convex combination.

## 1. Introduction

Denote by $\mathscr{H}$ the family of functions

$$f = h + \overline{g}\,, \tag{1.1}$$

which are analytic univalent and sense-preserving in the unit disc $U = \{z : |z| < 1\}$. So that $f$ is normalized by $f(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathscr{H}$, we may express the analytic functions $h$ and $g$ in the forms

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k\,,\ |b_1| < 1\,. \tag{1.2}$$

where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $\mathscr{H}$ is that $|h^{'}(z)| > |g^{'}(z)|$ in $\mathscr{H}$ (see [3]).

Hence

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k},\ |b_1| < 1. \tag{1.3}$$

We denote $\overline{\mathscr{H}}$ the subclass of $\mathscr{H}$ consists of harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k},\ |b_1| < 1. \tag{1.4}$$

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Let the Hadamard product (or convolution) of two power series $\Phi(z) = z + \sum_{k=2}^{\infty} \phi_k z^k$ and $\Psi(z) = z + \sum_{k=2}^{\infty} \psi_k z^k$ be defined by

$$(\Phi * \Psi)(z) = z + \sum_{k=2}^{\infty} \phi_k \psi_k z^k = (\Psi * \Phi)(z).$$

Let $\alpha_1, A_1, ..., \alpha_q, A_q$ and $\beta_1, B_1, ..., \beta_s, B_s$ $(q, s \in \mathbb{N})$ be positive and real parameters such that

$$1 + \sum_{j=1}^{s} B_j - \sum_{j=1}^{q} A_j \geqslant 0.$$

The Srivastava-Wright generalized hypergeometric function [18] (see also [4–5], [17], [12], [13] and [14])

$${}_q\Psi_s[(\alpha_1, A_1), ..., (\alpha_q, A_q); (\beta_1, B_1), ..., (\beta_s, B_s); z] = {}_q\Psi_s\left[(\alpha_i, A_i)_q; (\beta_i, B_i)_s; z\right]$$

is defined by

$${}_q\Psi_s\left[(\alpha_i, A_i)_q; (\beta_i, B_i)_s; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{q} \Gamma(\alpha_i + nA_i)}{\prod_{i=1}^{s} \Gamma(\beta_i + nB_i)} . \frac{z^n}{n!} \quad (z \in U).$$

If $A_i = 1$ $(i = 1, ..., q)$ and $B_i = 1$ $(i = 1, ..., s)$, we have the relationship:

$$\Omega_q \Psi_s\left[(\alpha_i, 1)_q; (\beta_i, 1)_s; z\right] = {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z),$$

where ${}_qF_s$ is the generalized hypergeometric function (see for details [5,6]) and

$$\Omega = \frac{\prod_{i=1}^{s} \Gamma(\beta_i)}{\prod_{i=1}^{q} \Gamma(\alpha_i)}. \tag{1.5}$$

In [4] Dziok and Riana extended the linear operator by using Srivastava-Wright generalized hypergeometric function (see also [11] and [16]). First we define a function ${}_q\Phi_s\left[(\alpha_i, A_i)_q; (\beta_i, B_i)_s; z\right]$ by

$${}_q\Phi_s\left[(\alpha_i, A_i)_q; (\beta_i, B_i)_s; z\right] = \Omega z \ {}_q\Psi_s\left[(\alpha_i, A_i)_q; (\beta_i, B_i)_s; z\right]$$

and consider the following linear operator

$$\theta_{q,s}\left[(\alpha_i, A_i)_q; (\beta_i, B_i)_s\right] : S_H \to S_H,$$

defined by the convolution

$$\theta_{p,q,s}\left[(\alpha_i,A_i)_q;(\beta_i,B_i)_s\right]f(z)={}_q\Phi_s\left[(\alpha_i,A_i)_q;(\beta_i,B_i)_s;z\right]*f(z).$$

We observe that, for a function $f(z)$ of the form $(1.1)$, we have

$$\theta_{q,s}\left[(\alpha_i,A_i)_q;(\beta_i,B_i)_s\right]f(z)=z+\sum_{k=2}^{\infty}\Omega\sigma_k(\alpha_1)a_kz^k, \tag{1.6}$$

where $\Omega$ is given by $(1.5)$ and $\sigma_k(\alpha_1)$ is defined by

$$\sigma_k(\alpha_1)=\frac{\Gamma(\alpha_1+A_1(k-1))\ldots\Gamma(\alpha_q+A_q(k-1))}{\Gamma(\beta_1+B_1(k-1))\ldots\Gamma(\beta_s+B_s(k-1))(k-1)!}. \tag{1.7}$$

If, for convenience, we write

$$\theta_{q,s}[\alpha_1,A_1,B_1]f(z)=\theta_{q,s}[(\alpha_1,A_1),\ldots,(\alpha_q,A_q);(\beta_1,B_1),\ldots,(\beta_s,B_s)]f(z),$$

then one can easily verify from the definition $(1.6)$ that (see [10] and [16]):

$$zA_1(\theta_{q,s}[\alpha_1,A_1,B_1]f(z))'=\alpha_1\theta_{q,s}[\alpha_1+1,A_1,B_1]f(z)-(\alpha_1-A_1)\theta_{q,s}[\alpha_1,A_1,B_1]f(z). \tag{1.8}$$

Applying the Srivastava-Wright operator to the harmonic functions $f=h+\overline{g}$ given by (1.1) we get

$$\theta_{q,s}[\alpha_1,A_1,B_1]f(z)=\theta_{q,s}[\alpha_1,A_1,B_1]h(z)+\overline{\theta_{q,s}[\alpha_1,A_1,B_1]g(z)} \tag{1.9}$$

Motivated by Jahangiri et al. [7–9] and Ahuja and Jahangiri [1], we define a new subclass $HS_{s^*}([\alpha_1,A_1,B_1],\gamma)$ of $\mathscr{H}$ that are starlike with respect symmetric points.

DEFINITION 1. For $0\leqslant\gamma<1$ and $z=re^{i\theta}\in U$, we let $\mathscr{H}S_{s^*}([\alpha_1,A_1,B_1],\gamma)$ a subclass of $\mathscr{H}$ of the form $f=h+\overline{g}$ be given by (1.3) and satisfying the analytic criteria

$$\operatorname{Re}\left\{\frac{2z\left(\theta_{q,s}[\alpha_1,A_1,B_1]f(z)\right)'}{z'\left[\theta_{q,s}[\alpha_1,A_1,B_1]f(z)-\theta_{q,s}[\alpha_1,A_1,B_1]f(-z)\right]}\right\}>\gamma, \tag{1.10}$$

where $\theta_{q,s}[\alpha_1,A_1,B_1]f(z)$ is defined by $(1.9)$ and $z'=\frac{\partial}{\partial\theta}\left(z=re^{i\theta}\right)$. We also let $\overline{\mathscr{H}S_{s^*}}([\alpha_1,A_1,B_1],\gamma)=\mathscr{H}S_{s^*}([\alpha_1,A_1,B_1],\gamma)\cap\overline{\mathscr{H}}$. The family $\mathscr{H}S_{s^*}([\alpha_1,A_1,B_1],\gamma)$ is of special interest because for suitable choices of $q$, $s$, $[A_1]$, $[B_1]$ and $[\alpha_1]$, we note that

(i) If $A_i=1$ $(i=1,\ldots,q)$ and $B_j=1$ $(j=1,\ldots,s)$, we have $\mathscr{H}S_{s^*}([\alpha_1,1,1],\gamma)=\mathscr{H}S_{s^*}([\alpha_1],\gamma)$, which was studied by Murugusundaramoorthy et al. $[11]$;

(ii) If $f(-z)=-f(z)$, $A_i=1$ $(i=1,\ldots,q)$ and $B_j=1$ $(j=1,\ldots,s)$, we have $\mathscr{H}S_{s^*}([\alpha_1,1,1],\gamma)=S^*_{\mathscr{H}}(\alpha_1,\gamma)$, which was studied by Al-Kharsani and AL-Khal $[2]$.

REMARK 1. If the co-analytic part of $f=h+\overline{g}$ is zero, $\alpha_i=A_i=1$ $(i=1,\ldots,q)$ and $\beta_j=B_j=1$ $(j=1,\ldots,s)$ then $\mathscr{H}S_{s^*}([(1,1),\gamma])$ turns out to be the class $S_s^*(\gamma)$ of starlike functions with respect to symmetric points which was introduced by Sakaguchi [15].

In this paper, we have obtained the coefficient conditions for the classes $\mathcal{HS}_{s^*}([\alpha_1,A_1,B_1],\gamma)$ and $\overline{\mathcal{HS}_{s^*}}([\alpha_1,A_1,B_1],\gamma)$. Further a representation theorem, inclusion properties and distortion bounds for the class $\overline{\mathcal{HS}_{s^*}}([\alpha_1,A_1,B_1],\gamma)$ are also established.

## 2. Coefficient characterization

Unless otherwise mentioned, we assume throughout this paper that $q,s\in\mathbb{N}$, $a_1=1$, $\alpha_1,A_1,\dots,\alpha_q,A_q$, $\beta_1,B_1,\dots,\beta_s,B_s\in\mathbb{R}^+$ and $0\leqslant\gamma<1$ . We begin with a sufficient condition for functions in $\mathcal{HS}_{s^*}([\alpha_1,A_1,B_1],\gamma)$.

THEOREM 1. *Let $f=h+\overline{g}$ be given by (1.3). Furthermore, let*

$$\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|a_k|+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|b_k|\leqslant 1 \tag{2.1}$$

*where $\Omega$ and $\sigma_k(\alpha_1)$ are defined by (1.5) and (1.7), respectively. Then $f$ is sense-preserving, harmonic univalent in $U$ and $f\in\mathcal{HS}_{s^*}([\alpha_1,A_1,B_1],\gamma)$.*

*Proof.* According the condition (1.10), we only need to show that if (2.1) holds, then

$$\operatorname{Re}\left\{\frac{2z\left(\theta_{q,s}[\alpha_1,A_1,B_1]f(z)\right)'}{z'\left[\theta_{q,s}[\alpha_1,A_1,B_1]f(z)-\theta_{q,s}[\alpha_1,A_1,B_1]f(-z)\right]}\right\}=\operatorname{Re}\frac{A(z)}{B(z)}>\gamma$$

where

$$A(z)=2z\left(\theta_{q,s}[\alpha_1,A_1,B_1]f(z)\right)'=2z'\left[z+\sum_{k=2}^{\infty}k\Omega\sigma_k(\alpha_1)a_kz^k-\sum_{k=1}^{\infty}k\Omega\sigma_k(\alpha_1)\overline{b_k\,z^k}\right]$$

and

$$\begin{aligned}B(z)&=z'\left[\theta_{q,s}[\alpha_1,A_1,B_1]f(z)-\theta_{q,s}[\alpha_1,A_1,B_1]f(-z)\right]\\&=z'\left[2z+\sum_{k=2}^{\infty}\left[1-(-1)^k\right]\Omega\sigma_k(\alpha_1)a_kz^k+\sum_{k=1}^{\infty}\left[1-(-1)^k\right]\Omega\sigma_k(\alpha_1)\overline{b_k\,z^k}\right]\end{aligned}$$

Using the fact that Re $\{w(z)\}>\gamma$ if and only if $|1-\gamma+w|>|1+\gamma-w|$, it suffices to show that

$$|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|>0. \tag{2.2}$$

Substituting for $A(z)$ and $B(z)$ in $(2.2)$ and by using $(2.1)$, we obtain

$$\begin{aligned}
&\Bigg|2(2-\gamma)z+\sum_{k=2}^{\infty}\left[2k+(1-\gamma)(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)\,a_k z^k \\
&\quad -\sum_{k=1}^{\infty}\left[2k-(1-\gamma)(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)\,\overline{b_k z^k}\Bigg| \\
&\quad -\Bigg|-2\gamma z+\sum_{k=2}^{\infty}\left[2k-(1+\gamma)(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)\,a_k z^k \\
&\quad -\sum_{k=1}^{\infty}\left[2k+(1+\gamma)(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)\,\overline{b_k z^k}\Bigg| \\
&\geqslant 4(1-\gamma)|z|-2\sum_{k=2}^{\infty}\left[2k-\gamma(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)\,|a_k|\,|z|^k \\
&\quad -2\sum_{k=1}^{\infty}\left[2k+\gamma(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)\,|b_k|\,|z|^k \\
&=4(1-\gamma)|z|\Bigg[1-\sum_{k=2}^{\infty}\frac{\left[2k-\gamma(1-(-1)^k)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)\,|a_k|\,|z|^{k-1} \\
&\quad -\sum_{k=1}^{\infty}\frac{\left[2k+\gamma(1-(-1)^k)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)\,|b_k|\,|z|^{k-1}\Bigg]. \\
&\geqslant 4(1-\gamma)\Bigg[1-\sum_{k=2}^{\infty}\frac{\left[2k-\gamma(1-(-1)^k)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)\,|a_k| \\
&\quad -\sum_{k=1}^{\infty}\frac{\left[2k+\gamma(1-(-1)^k)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)\,|b_k|\Bigg]\geqslant 0.
\end{aligned}$$

This last expression is non-negative by $(2.1)$.

The harmonic univalent functions

$$\begin{aligned}
f(z)=z&+\sum_{k=2}^{\infty}\frac{2(1-\gamma)}{\left[2k-\gamma(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)}X_k z^k \\
&+\sum_{k=1}^{\infty}\frac{2(1-\gamma)}{\left[2k-\gamma(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)}\overline{Y_k}\,\overline{z}^k,
\end{aligned}\tag{2.3}$$

where $\sum_{k=2}^{\infty}|X_k|+\sum_{k=1}^{\infty}|Y_k|=1$, show that the coefficient bound given by $(2.1)$ is sharp.

The functions of the form $(2.3)$ are in $\mathscr{H}S_{s^*}([\alpha_1,A_1,B_1],\gamma)$ because

$$\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|a_k|+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|b_k|$$
$$=\sum_{k=2}^{\infty}|X_k|+\sum_{k=1}^{\infty}|Y_k|=1.$$

This completes the proof of Theorem 1. □

In the following theorem, it is shown that the condition $(2.1)$ is also necessary for functions $f(z)$ are of the form $(1.4)$.

THEOREM 2. *Let* $f=h+\overline{g}$ *be given by* $(1.4)$. *Then* $f\in\overline{\mathscr{H}S_{s^*}}([\alpha_1,A_1,B_1],\gamma)$ *if and only if*

$$\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|a_k|+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|b_k|\leqslant 1 \tag{2.4}$$

*where* $\Omega$ *and* $\sigma_k(\alpha_1)$ *are defined by* $(1.5)$ *and* $(1.7)$, *respectively.*

*Proof.* Since $\overline{\mathscr{H}S_{s^*}}([\alpha_1,A_1,B_1],\gamma)\subset\mathscr{H}S_{s^*}([\alpha_1,A_1,B_1],\gamma)$, we only need to prove the "only if" part of the theorem. To this end, for functions $f(z)$ of the form $(1.4)$, we notice that the condition

$$\operatorname{Re}\left\{\frac{2z\,\left(\theta_{q,s}[\alpha_1,A_1,B_1]f(z)\right)'}{z'\,\left[\theta_{q,s}[\alpha_1,A_1,B_1]f(z)-\theta_{q,s}[\alpha_1,A_1,B_1]f(-z)\right]}\right\}>\gamma$$

is equivalent to

$$\operatorname{Re}\left\{\frac{2(1-\gamma)-\sum_{k=2}^{\infty}\left[2k-\gamma(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)a_kz^{k-1}-\frac{\overline{z}}{z}\sum_{k=1}^{\infty}[2k+\gamma(1-(-1)^k)]\Omega\sigma_k(\alpha_1)b_k\overline{z}^{k-1}}{2-\sum_{k=2}^{\infty}(1-(-1)^k)\Omega\sigma_k(\alpha_1)a_kz^{k-1}+\frac{\overline{z}}{z}\sum_{k=1}^{\infty}(1-(-1)^k)\Omega\sigma_k(\alpha_1)b_k\overline{z}^{k-1}}\right\}>0 \tag{2.5}$$

The above required condition $(2.5)$ must hold for all values of $z$ in $U$. Upon choosing the values of $z$ on the positive real axis where $0\leqslant z=r<1$, we must have

$$\left\{\frac{2(1-\gamma)-\sum_{k=2}^{\infty}\left[2k-\gamma(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)a_kr^{k-1}-\sum_{k=1}^{\infty}[2k+\gamma(1-(-1)^k)]\Omega\sigma_k(\alpha_1)b_kr^{k-1}}{2-\sum_{k=2}^{\infty}(1-(-1)^k)\Omega\sigma_k(\alpha_1)a_kr^{k-1}+\sum_{k=1}^{\infty}(1-(-1)^k)\Omega\sigma_k(\alpha_1)b_kr^{k-1}}\right\}>0. \tag{2.6}$$

If the condition $(2.4)$ does not hold, then the numerator in $(2.6)$ is negative for $r$ sufficiently close to $1$. Hence there exists $z_0=r_0$ in $(0,1)$ for which the quotient in $(2.6)$ is negative. This contradicts the required condition for $f(z)\in\overline{\mathscr{H}S_{s^*}}([\alpha_1,A_1,B_1],\gamma)$ and so the proof of Theorem 2 is completed. □

## 3. Extreme points and distortion theorem

Our next theorem is on the extreme points of convex hulls of $\overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ denoted by $clco\ \overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$.

THEOREM 3. *A function* $f_k(z) \in clco\overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ *if and only if* $f_k(z)$ *can be expressed in the form*

$$f_k(z) = \sum_{k=1}^{\infty} [X_k h_k(z) + Y_k g_k(z)], \tag{3.1}$$

*where* $h_1(z) = z$*,*

$$h_k(z) = z - \frac{2(1-\gamma)}{\left[2k - \gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)} z^k \quad (k \geqslant 2),$$

*and*

$$g_k(z) = z + \frac{2(1-\gamma)}{\left[2k + \gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)} \overline{z}^k \quad (k \geqslant 1),$$

$$X_k \geqslant 0, \qquad Y_k \geqslant 0, \qquad \sum_{k=1}^{\infty} (X_k + Y_k) = 1.$$

*In particular, the extreme points of* $\overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ *are* $\{h_k\}$ *and* $\{g_k\}$.

*Proof.* For functions $f_k(z)$ of the form (3.1), we have

$$f_k(z) = z - \sum_{k=2}^{\infty} \frac{2(1-\gamma)}{\left[2k-\gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)} X_k z^k + \sum_{k=1}^{\infty} \frac{2(1-\gamma)}{\left[2k+\gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)} Y_k \overline{z}^k.$$

Then by Theorem 2

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{\left[2k - \gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} \Omega\sigma_k(\alpha_1)|a_k| + \sum_{k=1}^{\infty} \frac{\left[2k + \gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} \Omega\sigma_k(\alpha_1)|b_k| \\
&= \sum_{k=2}^{\infty} \frac{\left[2k - \gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} \Omega\sigma_k(\alpha_1) \frac{2(1-\gamma)}{\left[2k - \gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)} X_k \\
&\quad + \sum_{k=1}^{\infty} \frac{\left[2k + \gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} \Omega\sigma_k(\alpha_1) \left( \frac{2(1-\gamma)}{\left[2k + \gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)} Y_k \right) \\
&= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leqslant 1.
\end{aligned}$$

and so $f_k \in \overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$.

Conversely, if $f_k \in clco\, \overline{\mathscr{H}S_{s^*}}\left([\alpha_1, A_1, B_1], \gamma\right)$. Setting

$$X_k = \frac{\left[2k - \gamma\left(1 - (-1)^k\right)\right]}{2(1-\gamma)} \Omega\sigma_k(\alpha_1)|a_k| \quad (k \geqslant 2),$$

and

$$Y_k = \frac{\left[2k + \gamma\left(1 - (-1)^k\right)\right]}{2(1-\gamma)} \Omega\sigma_k(\alpha_1)|b_k| \quad (k \geqslant 1).$$

We obtain $f_k(z) = \sum\limits_{k=1}^{\infty} [X_k h_k(z) + Y_k g_k(z)]$ as required. $\square$

THEOREM 4. *Let the functions* $f(z)$ *defined by (1.4) be in the class* $\overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$. *Then for* $|z| = r < 1$, *we have*

$$(1 - |b_1|)r - \tfrac{1}{\Omega\sigma_2(\alpha_1)}\left\{\tfrac{1-\gamma}{2} - \tfrac{1+\gamma}{2}|b_1|\right\} r^2 \leqslant |f(z)|$$
$$\leqslant (1 + |b_1|)r + \tfrac{1}{\Omega\sigma_2(\alpha_1)}\left\{\tfrac{1-\gamma}{2} - \tfrac{1+\gamma}{2}|b_1|\right\} r^2.$$

*The result is sharp.*

*Proof.* We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $f(z) \in \overline{\mathscr{H}S_{s^*}}\left([\alpha_1, A_1, B_1], \gamma\right)$. Taking the absolute value of $f$ we have

$$\begin{aligned}
|f(z)| &\leqslant (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k \leqslant (1+|b_1|)r + r^2 \sum_{k=2}^{\infty}(|a_k| + |b_k|) \\
&\leqslant (1+|b_1|)r + \tfrac{(1-\gamma)}{\Omega\sigma_2(\alpha_1)} \sum_{k=2}^{\infty} \frac{\Omega\sigma_k(\alpha_1)}{1-\gamma}(|a_k| + |b_k|)r^2 \\
&= (1+|b_1|)r + \tfrac{(1-\gamma)r^2}{\Omega\sigma_2(\alpha_1)} \sum_{k=2}^{\infty} \left\{ \tfrac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{4(1-\gamma)}|a_k| + \tfrac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{4(1-\gamma)}|b_k| \right\} \Omega\sigma_k(\alpha_1) \\
&= (1+|b_1|)r + \frac{(1-\gamma)r^2}{2\Omega\sigma_2(\alpha_1)} \sum_{k=2}^{\infty} \left\{ \tfrac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}|a_k| + \tfrac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}|b_k| \right\} \Omega\sigma_k(\alpha_1) \\
&\leqslant (1+|b_1|)r + \frac{(1-\gamma)r^2}{2\Omega\sigma_2(\alpha_1)}\left(1 - \frac{1+\gamma}{1-\gamma}|b_1|\right) \\
&= (1+|b_1|)r + \frac{1}{\Omega\sigma_2(\alpha_1)}\left[\frac{1-\gamma}{2} - \frac{1+\gamma}{2}|b_1|\right] r^2.
\end{aligned}$$

The bounds given in Theorem 4 for functions $f = h + \overline{g}$ of form (1.4) also hold for functions of the form (1.2) if the coefficient condition (2.1) is satisfied. The upper bound given for $f \in \overline{\mathscr{H}S_{s^*}}\left([\alpha_1, A_1, B_1], \gamma\right)$ is sharp and the equality occurs for the functions

$$f(z) = z + \overline{b_1 z} + \frac{1}{\Omega\sigma_2(\alpha_1)}\left[\frac{1-\gamma}{2} - \frac{1+\gamma}{2}b_1\right]\overline{z}^2,$$

show that the bounds given in Theorem 4 are sharp. This completes the proof of Theorem 4. $\square$

## 4. Convolution and convex combination

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k \overline{z}^k, \; |b_1| < 1 \tag{4.1}$$

and

$$G(z) = z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k \overline{z}^k \; (A_k \geqslant 0; B_k \geqslant 0) \tag{4.2}$$

we define the convolution of two harmonic functions $f$ and $G$ as

$$(f * G)(z) = f(z) * G(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} b_k B_k \overline{z}^k. \tag{4.3}$$

Using this definition, we show that the class $\overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ is closed under convolution.

THEOREM 5. *For $0 \leqslant \mu \leqslant \gamma < 1$, let $f \in \overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ and $G \in \overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \mu)$. Then $f * G \in \overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma) \subset \overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \mu)$.*

*Proof.* Let the function $f(z)$ defined by $(4.1)$ be in the class $\overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ and let the function $G(z)$ defined by $(4.2)$ be in the class $\overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \mu)$. Then the convolution $f * G$ is given by $(4.3)$. We wish to show that the coefficients of $f * G$ satisfy the required condition given in Theorem 2. For $G \in \overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \mu)$ we note that $0 \leqslant A_k \leqslant 1$ and $0 \leqslant B_k \leqslant 1$. Now, for the convolution function $f * G$ we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \left[2k - \gamma\left(1 - (-1)^k\right)\right] \Omega\sigma_k(\alpha_1) |a_k| A_k + \sum_{k=1}^{\infty} \left[2k + \gamma\left(1 - (-1)^k\right)\right] \Omega\sigma_k(\alpha_1) |b_k| B_k \\
&\leqslant \sum_{k=2}^{\infty} \left[2k - \gamma\left(1 - (-1)^k\right)\right] \Omega\sigma_k(\alpha_1) |a_k| + \sum_{k=1}^{\infty} \left[2k + \gamma\left(1 - (-1)^k\right)\right] \Omega\sigma_k(\alpha_1) |b_k| \\
&\leqslant 2(1 - \gamma),
\end{aligned}$$

since $0 \leqslant \mu \leqslant \gamma < 1$ and $f \in \overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$. Therefore $f * G \in \overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma) \subset \overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \mu)$, since the above inequality bounded by $2(1-\gamma)$ while $2(1-\gamma) \leqslant 2(1-\mu)$. $\square$

Now, we show that the class $\overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ is closed under convex combinations of its members.

THEOREM 6. *The class $\overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ is closed under convex combination.*

*Proof.* For $i = 1, 2, \ldots$, let $f_i \in \overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$, where $f_i$ is given by

$$f_i(z) = z - \sum_{k=2}^{\infty} \left|a_{k_i}\right| z^k + \sum_{k=1}^{\infty} \left|b_{k_i}\right| \overline{z}^k, \; (a_{k_i} \geqslant 0\,;\, b_{k_i} \geqslant 0\,;\, z \in U).$$

Then by using Theorem 2, we have

$$\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^{k}\right)\right]}{2(1-\gamma)}\Omega\sigma_{k}(\alpha_{1})\left|a_{k_{i}}\right|+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^{k}\right)\right]}{2(1-\gamma)}\Omega\sigma_{k}(\alpha_{1})\left|b_{k_{i}}\right|\leqslant 1. \tag{4.4}$$

For $\sum\limits_{i=1}^{\infty} t_i = 1, 0 \leqslant t_i \leqslant 1$, the convex combination of $f_i$ may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i \left|a_{k_i}\right|\right) z^k + \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i \left|b_{k_i}\right|\right)\overline{z}^k. \tag{4.5}$$

Then, by using $(4.4)$, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^{k}\right)\right]}{2(1-\gamma)}\Omega\sigma_{k}(\alpha_{1})\left(\sum_{i=1}^{\infty} t_i\left|a_{k_i}\right|\right)+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^{k}\right)\right]}{2(1-\gamma)}\Omega\sigma_{k}(\alpha_{1})\left(\sum_{i=1}^{\infty} t_i\left|b_{k_i}\right|\right)\\
&=\sum_{i=1}^{\infty} t_i\left[\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^{k}\right)\right]}{2(1-\gamma)}\Omega\sigma_{k}(\alpha_{1})\left|a_{k_i}\right|+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^{k}\right)\right]}{2(1-\gamma)}\Omega\sigma_{k}(\alpha_{1})\left|b_{k_i}\right|\right]\\
&\leqslant\sum_{i=1}^{\infty} t_i = 1,
\end{aligned}$$

this is the necessary and sufficient condition given by (2.4) and so $\sum\limits_{i=1}^{\infty} t_i f_i(z) \in \overline{\mathscr{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$. This completes the proof of Theorem 6. □

REMARK 2. Putting $A_i = 1 \ (i = 1, ..., q)$ and $B_j = 1 \ (j = 1, ..., s)$ in our results we obtain the results obtained by Murugusundaramoorthy et al. $[11]$.

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