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ON THE QUASI MONOTONE AND GENERALIZED POWER INCREASING SEQUENCES AND THEIR NEW APPLICATIONS

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Abstract. In this paper, we prove a general theorem dealing with $\mid C,\alpha,\gamma,\beta;\sigma\mid_k$ summability factors by using a general class of power increasing sequences instead of an almost increasing sequence. This theorem also includes several known and new results.

1. Introduction

A sequence (B_n) is said to be δ-quasi-monotone, if $B_n \to 0$, $B_n > 0$ ultimately and $\Delta B_n \geqslant -\delta_n$, where $\Delta B_n = B_n - B_{n+1}$ and $\delta = (\delta_n)$ is a sequence of positive numbers (see [2]). A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leqslant b_n \leqslant Nc_n$ (see [1]. A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence, if there exists a constant $K = K(X,f)$ such that $Kf_nX_n \geqslant f_mX_m$ for all $n \geqslant m \geqslant 1$, where $f = (f_n) = \{n^\eta(\log n)^\kappa,\ \kappa \geqslant 0,\ 0 < \eta < 1\}$ (see [11]). If we take κ=0, then we get a quasi-η-power increasing sequence. It is also known that every almost increasing sequence is a quasi-η-power increasing sequence for any nonnegative η, but the converse is not true for $\eta > 0$ (see [10]). Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α,β), with $\alpha+\beta > -1$, of the sequence (na_n), that is (see [6])

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1 \quad and \quad A_{-n}^{\alpha+\beta} = 0 \quad for \quad n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $\mid C,\alpha,\gamma,\beta;\sigma\mid_k$, $k \geqslant 1$, $\sigma \geqslant 0$ and γ is a real number, if (see [4])

$$\sum_{n=1}^{\infty} n^{\gamma(\sigma k+k-1)-k} \mid t_n^{\alpha,\beta} \mid^k < \infty. \tag{3}$$

If we take $\gamma = 1$ and $\sigma = 0$, then $\mid C,\alpha,\gamma,\beta;\sigma\mid_k$ summability reduces to $\mid C,\alpha,\beta\mid_k$ summability (see [7]). If we take $\beta = 0$, then we have $\mid C,\alpha,\gamma;\sigma\mid_k$ summability (see

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[12]). Also if we take $\gamma=1$, $\beta=0$ and $\sigma=0$, then we get $\mid C,\alpha\mid_k$ summability (see [8]). Furthermore if we take $\gamma=1$ and $\beta=0$, then we get $\mid C,\alpha;\sigma\mid_k$ summability (see [9]). In [5], Bor and Özarslan have proved the following theorem dealing with $\mid C,\alpha,\gamma,\beta;\sigma\mid_k$ summability factors.

THEOREM A. *Let (X_n) be an almost increasing sequence such that $\mid \Delta X_n \mid = O(\frac{X_n}{n})$ and let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\sum nX_n\delta_n < \infty$, $\sum B_nX_n$ is convergent and $\mid \Delta\lambda_n \mid \leqslant \mid B_n \mid$ for all n. If the sequence $(\theta_n^{\alpha,\beta})$ defined by*

$$\theta_n^{\alpha,\beta} = \begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha=1,\ \beta>-1 \\ \max_{1\leqslant v\leqslant n}\left|t_v^{\alpha,\beta}\right|, & 0<\alpha<1,\ \beta>-1 \end{cases} \tag{4}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\gamma(\sigma k+k-1)-k}(\theta_n^{\alpha,\beta})^k = O(X_m) \quad as \quad m\to\infty, \tag{5}$$

then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha,\gamma,\beta;\sigma\mid_k$, $k\geqslant 1$, $0\leqslant\sigma<\alpha\leqslant 1$, $(\alpha+\beta+1)k - \gamma(\sigma k+k-1>1$ and γ is a real number.

2. The main result

The aim of this paper is to extend Theorem A by using a quasi-f-power increasing sequence instead of an almost increasing sequence.

Now we shall prove the following general theorem.

THEOREM. *Let (X_n) be a quasi-f-power increasing sequence and let $\lambda_n \to 0$ as $n\to\infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\Delta B_n \leqslant \delta_n$, $\sum nX_n\delta_n<\infty$, $\sum B_nX_n$ is convergent and $\mid \Delta\lambda_n \mid \leqslant \mid B_n \mid$ for all n. If the condition (5) is satisfied, then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha,\gamma,\beta;\sigma\mid_k$, $k\geqslant 1$, $0\leqslant\sigma<\alpha\leqslant 1$, $(\alpha+\beta+1)k-\gamma(\sigma k+k-1)>1$ and γ is a real number.*

We need the following lemmas for the proof of our theorem.

LEMMA 1. [3] *If $0<\alpha\leqslant 1$, $\beta>-1$ and $1\leqslant v\leqslant n$, then*

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1}A_p^{\beta}a_p \mid \leqslant \max_{1\leqslant m\leqslant v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1}A_p^{\beta}a_p \mid. \tag{6}$$

LEMMA 2. *Under the conditions regarding (λ_n) and (X_n) of the theorem, we have*

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n\to\infty. \tag{7}$$

Proof. Since $\lambda_n \to 0 \quad as \quad n \to \infty$, we have that

$$| \lambda_n | X_n = X_n | \sum_{v=n}^{\infty} \Delta\lambda_v | \leqslant X_n \sum_{v=n}^{\infty} | \Delta\lambda_v | \leqslant \sum_{v=n}^{\infty} X_v | \Delta\lambda_v | \leqslant \sum_{v=1}^{\infty} X_v | B_v | < \infty.$$

This completes the proof of Lemma 2. □

LEMMA 3. *Let (X_n) be a quasi-f-power increasing sequence. If (B_n) is a δ-quasi-monotone sequence with $\Delta B_n \leqslant \delta_n$ and $\sum n\delta_n X_n < \infty$, then*

$$\sum_{n=1}^{\infty} nX_n | \Delta B_n | < \infty, \tag{8}$$

$$nB_nX_n = O(1) \quad as \quad n \to \infty. \tag{9}$$

Proof. Since (X_n) is a positive sequence and $|\Delta B_n| \leqslant \delta_n$, we have that

$$\sum_{v=1}^{\infty} vX_v|\Delta B_v| \leqslant \sum_{v=1}^{\infty} vX_v\delta_v < \infty.$$

Also, since $(n^{\eta}(\log n)^{\kappa}X_n)$ is non-decreasing and $B_n \to 0$, we have that

$$\begin{aligned}
nX_nB_n &= n^{1-\eta}(\log n)^{-\kappa}n^{\eta}(\log n)^{\kappa}X_n|\sum_{v=n}^{\infty}\Delta B_v| \\
&\leqslant n^{1-\eta}(\log n)^{-\kappa}n^{\eta}(\log n)^{\kappa}X_n\sum_{v=n}^{\infty}|\Delta B_v| \\
&\leqslant n^{1-\eta}(\log n)^{-\kappa}\sum_{v=n}^{\infty}v^{\eta}(\log v)^{\kappa}X_v|\Delta B_v| \\
&\leqslant \sum_{v=n}^{\infty}v^{1-\eta}(\log v)^{-\kappa}v^{\eta}(\log v)^{\kappa}X_v|\Delta B_v| \\
&= \sum_{v=n}^{\infty}vX_v|\Delta B_v| = O(1).
\end{aligned}$$

This completes the proof of Lemma 3. □

3. Proof of the theorem

Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean of the sequence $(na_n\lambda_n)$. Then, by (1), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\lambda_v.$$

Applying Abel's transformation first and then using Lemma 1, we have that

$$
\begin{aligned}
T_n^{\alpha,\beta} &= \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \\
\mid T_n^{\alpha,\beta} \mid &\leqslant \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^{\alpha+\beta}} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \mid \\
&\leqslant \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \theta_n^{\alpha,\beta} \\
&= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}.
\end{aligned}
$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$
\sum_{n=1}^{\infty} n^{\gamma(\sigma k+k-1)-k} \mid T_{n,r}^{\alpha,\beta} \mid^k < \infty, \quad for \quad r = 1,2.
$$

Whenever $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$
\begin{aligned}
\sum_{n=2}^{m+1} n^{\gamma(\sigma k+k-1)-k} \mid T_{n,1}^{\alpha,\beta} \mid^k &\leqslant \sum_{n=2}^{m+1} n^{\gamma(\sigma k+k-1)-k} \mid \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \Delta\lambda_v \mid^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{(\alpha+\beta+1)k-\gamma(\sigma k+k-1)}} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} v^{\beta k} \mid B_v \mid (\theta_v^{\alpha,\beta})^k \right\} \\
&\quad \times \left\{ \sum_{v=1}^{n-1} \mid B_v \mid \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \mid B_v \mid (\theta_v^{\alpha,\beta})^k \sum_{n=v+1}^{m+1} \frac{1}{n^{(\alpha+\beta+1)k-\gamma(\sigma k+k-1)}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \mid B_v \mid (\theta_v^{\alpha,\beta})^k \int_v^{\infty} \frac{dx}{x^{(\alpha+\beta+1)k-\gamma(\sigma k+k-1)}} \\
&= O(1) \sum_{v=1}^{m} v \mid B_v \mid v^{\gamma(\sigma k+k-1)-k} (\theta_v^{\alpha,\beta})^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v \mid B_v \mid) \sum_{p=1}^{v} p^{\gamma(\sigma k+k-1)-k} (\theta_p^{\alpha,\beta})^k \\
&\quad + O(1) m \mid B_m \mid \sum_{v=1}^{m} v^{\gamma(\sigma k+k-1)-k} (\theta_v^{\alpha,\beta})^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta(v \mid B_v \mid)| X_v + O(1) m \mid B_m \mid X_m
\end{aligned}
$$

$$= O(1)\sum_{v=1}^{m-1} \mid (v+1)\Delta \mid B_v \mid - \mid B_v \mid\mid X_v + O(1)m\mid B_m \mid X_m$$
$$= O(1)\sum_{v=1}^{m-1} v \mid \Delta B_v \mid X_v + O(1)\sum_{v=1}^{m-1} \mid B_v \mid X_v$$
$$+O(1)m\mid B_m \mid X_m = O(1) \quad \text{as} \quad m\to\infty,$$

in view of hypotheses of the theorem and Lemma 3. Similarly, we have that

$$\sum_{n=1}^{m} n^{\gamma(\sigma k+k-1)-k} \mid T_{n,2}^{\alpha,\beta} \mid^k = O(1)\sum_{n=1}^{m} |\lambda_n| n^{\gamma(\sigma k+k-1)-k}(\theta_n^{\alpha,\beta})^k$$
$$= O(1)\sum_{n=1}^{m-1}\Delta(|\lambda_n|)\sum_{v=1}^{n} v^{\gamma(\sigma k+k-1)-k}(\theta_v^{\alpha,\beta})^k$$
$$+O(1)|\lambda_m|\sum_{v=1}^{m} v^{\gamma(\sigma k+k-1)-k}(\theta_v^{\alpha,\beta})^k$$
$$= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n + O(1)|\lambda_m|X_m$$
$$= O(1)\sum_{n=1}^{m-1} \mid B_n \mid X_n + O(1)|\lambda_m|X_m$$
$$= O(1) \quad as \quad m\to\infty,$$

by virtue of hypotheses of the theorem and Lemma 2. This completes the proof of the theorem. If we take $\beta = 0$ and $\gamma = 1$, then we get a new result for $\mid C,\alpha;\sigma \mid_k$ summability. Also, if we take $\gamma = 1$, then we have a new result for $\mid C,\alpha,\beta;\sigma \mid_k$. Furthermore, if we take $\gamma = 1$, $\beta = 0$, $\alpha = 1$ and $\sigma = 0$, then we obtain a result for $\mid C,1 \mid_k$ summability factors. If we take (X_n) as an almost increasing sequence such that $\mid \Delta X_n \mid = O(\frac{X_n}{n})$, then we get Theorem A, in this case condition '$\Delta B_n \leqslant \delta_n$' is not needed. Finally, if take (X_n) as a quasi-η-power increasing sequence, then we have a new result.

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