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ASYMPTOTIC FORMULAE ASSOCIATED WITH THE WALLIS POWER FUNCTION AND DIGAMMA FUNCTION

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Abstract. Let s,t be two given real numbers, $s \neq t$. We determine the coefficients $c_j(s,t)$ such that

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \sim \exp\left(\psi\left(x+\sum_{j=0}^{\infty} c_j(s,t)x^{-j}\right)\right)$$

as $x \to \infty$, where $\psi(x) = \Gamma'(x)/\Gamma(x)$ denotes the digamma function. Also, the analysis of the coefficients in the asymptotic expansion of the composition $\exp(\psi(x+s))$ is given in details.

1. Introduction

Euler's gamma function:

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,\mathrm{d}t \quad (\Re(z) > 0)$$

is one of the most important functions in mathematical analysis and its applications in various diverse areas. The logarithmic derivative of the gamma function:

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\Gamma(z) = \int_1^z \psi(t)\,\mathrm{d}t$$

is known as the psi (or digamma) function.

In 1959 W. Gautschi [10] presented the remarkable inequality:

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < \exp\left((1-s)\psi(n+1)\right) \tag{1}$$

for $0 < s < 1$ and $n \in \mathbb{N} := \{1,2,3,\ldots\}$. In 1983 D. Kershaw [12] gave the following closer bounds:

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s}, \tag{2}$$

$$\exp[(1-s)\psi(x+\sqrt{s})] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right] \tag{3}$$

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for real $x>0$ and $0<s<1$. In 2005 D. Kershaw [13] proved the following inequality:

$$\psi(x+\sqrt{st})<\frac{1}{t-s}\ln\frac{\Gamma(x+t)}{\Gamma(x+s)}<\psi\left(x+\frac{s+t}{2}\right) \tag{4}$$

for $x\geqslant 0$ and $0<s\leqslant t$, but the better bounds was already proved in [7].

$$\psi(x+I_\psi(s,t))<\frac{1}{t-s}\ln\frac{\Gamma(x+t)}{\Gamma(x+s)}<\psi\left(x+\frac{s+t}{2}\right) \tag{5}$$

where

$$I_\psi(s,t)=\psi^{-1}\left(\frac{1}{t-s}\int_s^t\psi(u)du\right)$$

is integral ψ-mean of s and t, ψ^{-1} denotes the inverse function of ψ. Namely, in [8] it is proved that

$$\sqrt{st}\leqslant\frac{t-s}{\ln t-\ln s}\leqslant I_\psi(s,t).$$

Since

$$I_\psi(x+s,x+t)-x\to\frac{s+t}{2}\quad\text{as}\quad x\to\infty.$$

this implies that

$$\frac{1}{t-s}\ln\frac{\Gamma(x+t)}{\Gamma(x+s)}\sim\psi\left(x+\frac{s+t}{2}\right)\quad\text{as}\quad x\to\infty. \tag{6}$$

The main intention of this paper is to extend this formula and obtain full asymptotical expansion of the form

$$\frac{1}{t-s}\ln\frac{\Gamma(x+t)}{\Gamma(x+s)}\sim\psi\left(x+\sum_{j=0}^{\infty}c_j(s,t)x^{-j}\right).$$

The inequalities (2) to (3) have attracted much interest of many mathematicians and have motivated a large number of research papers involving various generalizations and improvements, see [15] and an overview in [16] and the references cited therein. It was shown in [7] that the function

$$z(x)=\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}-x$$

is convex and decreasing on $(-r,\infty)$ for $|t-s|<1$, and concave and increasing on the same interval for $|t-s|>1$, where s,t are given real numbers and $r=\min(s,t)$. See also [5, 17] for alternative proofs. This implies the following result: For all $x>0$, the inequalities

$$x+\frac{s+t-1}{2}<\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}<x+\left[\frac{\Gamma(t)}{\Gamma(s)}\right]^{1/(t-s)} \tag{7}$$

holds for $|t-s|<1$, and with reversed sign for $|t-s|>1$.

Recently, Burić and Elezović [2, Theorem 2.1] gave the following complete asymptotic expansion for the Wallis power function

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \sim \sum_{n=0}^{\infty} P_n(t,s)x^{-n+1}, \tag{8}$$

where $P_n(t,s)$ are polynomials of order n defined by

$$\begin{aligned} &P_0(t,s)=1,\\ &P_n(t,s)=\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\frac{B_{k+1}(t)-B_{k+1}(s)}{(k+1)(t-s)}P_{n-k}(t,s) \qquad (n\in\mathbb{N}). \end{aligned} \tag{9}$$

Here $B_k(t)$ stands for the Bernoulli polynomials defined by the following generating function:

$$\frac{xe^{tx}}{e^x-1}=\sum_{k=0}^{\infty}B_k(t)\frac{x^k}{k!}. \tag{10}$$

Polynomials $P_n(t,s)$ have complicated form so by the change of variables

$$\alpha=\frac{s+t-1}{2},\quad \beta_1=\frac{t-s+1}{2},\quad \beta_2=\frac{-t+s+1}{2},$$
$$\beta=\beta_1\beta_2=\frac{1-(t-s)^2}{4},$$

authors in [2, Theorem 5.1] presented the following expansion for (8):

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \sim x+\sum_{n=0}^{\infty}Q_{n+1}(\alpha,\beta)\frac{1}{x^n}, \tag{11}$$

where $Q_n(\alpha,\beta)$ is a polynomial obtained from $P_n(t,s)$ and has much more natural form than $P_n(t,s)$. Moreover, the authors gave an efficient recurrence formula for determining the coefficients $Q_n(\alpha,\beta)$ and finally derived closed form for polynomials $Q_n(\alpha,\beta)$.

In this paper we continue the analysis of such asymptotic expansion, with special attention to the connection with digamma function.

The asymptotic expansion

$$f(x)\sim a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\dots \qquad \text{as } x\to\infty \tag{12}$$

is called *asymptotic power series*.

The formal manipulations with asymptotic series in the paper are justified by properties of asymptotic power series; see [9, § 1.6]. It is known that two such expansions can be added or multiplied, and also divided provided that leading coefficient of denominator is different from zero. Also, asymptotic power series may be substituted in finite linear combinations, in polynomials, and in asymptotic power series. Coefficients of the new expansion are obtained by formal substitution and rearrangement of terms.

The following two lemmas will be explicitly used in the sequell.

LEMMA 1. ([9, p. 20]) *If the function g has expansion into power series*

$$g(x) = \sum_{n=0}^{\infty} c_n x^n, \qquad \text{as } x \to 0$$

and $f(x)$ has asymptotic expansion (12) with leading coefficient $a_0 = 0$, then $g(f(x))$ has asymptotic expansion whose coefficients can be obtained by formal substitution and rearrangement of terms.

LEMMA 2. ([9, p. 21]) *If the function f in (12) is differentiable and if f' possesses an asymptotic power series expansion, then*

$$f'(x) \sim -\frac{a_1}{x^2} - \frac{2a_2}{x^3} - \frac{3a_3}{x^4} - \ldots, \qquad \text{as } x \to \infty. \tag{13}$$

2. Functional transformations of an asymptotic series

In this section we will prove some useful technical lemmas.

LEMMA 3. *Let A be a function with asymptotic expansion*

$$A(x) \sim \sum_{n=1}^{\infty} a_n x^{-n}.$$

Then the composition $B(x) = \exp(A(x))$ has asymptotic expansion of the following form

$$B(x) \sim \sum_{n=0}^{\infty} b_n x^{-n}$$

where $b_0 = 1$ and

$$b_n = \frac{1}{n} \sum_{k=1}^{n} k a_k b_{n-k}, \qquad n \geqslant 1. \tag{14}$$

Proof. The existence of the asymptotic expansion of the function B follows from Lemma 1. Suppose for the moment that A is differentiable and that A' has asymptotic expansion. Differentiating equation $B(x) = e^{A(x)}$ we get

$$B'(x) = A'(x)e^{A(x)} = A'(x)B(x)$$

so B' also has asymptotic expansion. So we can write

$$\begin{aligned} B'(x) &\sim \left(\sum_{n=1}^{\infty} a_n (-n) x^{-n-1} \right) \left(\sum_{n=0}^{\infty} b_n x^{-n} \right) \\ &\sim \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-k) a_k b_{n-k} \right) x^{-n-1}, \end{aligned}$$

but also

$$B'(x) \sim \sum_{n=0}^{\infty} b_n(-n)x^{-n-1},$$

which implies that for $n \geqslant 1$

$$b_n = \frac{1}{n}\sum_{k=1}^{n} k a_k b_{n-k}.$$

Let us show that assumption about existence of asymptotic expansion of the function A is not necessary. Choose N arbitrary large and define

$$\tilde{A}(x) = \sum_{n=1}^{N} a_n x^{-n},$$
$$\tilde{B}(x) = \exp(\tilde{A}(x)).$$

Then the first N coefficients of the asymptotic expansion of the function $\tilde{B}$ coincides with the coefficients of function B. But $\tilde{A}$ is differentiable and $\tilde{A}'$ has asymptotic expansion. From the equation $\tilde{B}'(x) = \tilde{A}'(x)\tilde{B}(x)$ one gets the same connection (14) between coefficients for all indices $< N$. This finishes the proof. □

Applying this procedure to the coefficients of the expression (23) one get (25). Here is also a dual result, the proof is similar:

LEMMA 4. *Let $c_0 \neq 0$ and*

$$C(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}$$

be a given asymptotical expansion. Then the composition $A(x) = \ln(C(x))$ has asymptotic expansion of the following form

$$A(x) \sim \sum_{n=1}^{\infty} a_n x^{-n}$$

where

$$a_n = \frac{c_n}{c_0} - \frac{1}{nc_0}\sum_{k=1}^{n-1} k a_k c_{n-k}, \qquad n \geqslant 1. \tag{15}$$

Since $C(x) = \exp(A(x))$, it is easy to extract (15) from equation (14).

In the sequel we shall need also the following transform, see [4, Lemma 3.2] for the similar statement. This lemma has its origin in Euler's work, see [11] for historical treatment in the case of Taylor series.

LEMMA 5. *Let g be a function with asymptotical expansion (as $x \to \infty$):*

$$g(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}, \qquad (c_0 \neq 0).$$

Then for all real r *it holds*

$$[g(x)]^r \sim \sum_{n=0}^{\infty} P_n(r)x^{-n}$$

where

$$P_0(r) = c_0^r,$$
$$P_n(r) = \frac{1}{nc_0}\sum_{k=1}^{n}[k(1+r)-n]c_kP_{n-k}(r). \tag{16}$$

Proof. Denote $f(x) = [g(x)]^r$. This can be written as $f(x) = \exp[r\ln(g(x))]$ and existence of asymptotic expansion follows from lemmas above. If g is differentiable and g' has an asymptotic expansion, then for $r \neq 0$ it holds

$$-rf(x)g'(x) = f'(x)g(x).$$

Hence

$$-r\left(\sum_{k=0}^{\infty} P_k(r)x^{-k}\right)\left(\sum_{j=0}^{\infty}(-j)c_jx^{-j-1}\right) \sim \left(\sum_{k=0}^{\infty}(-k)P_k(r)x^{-k-1}\right)\left(\sum_{j=0}^{\infty}c_jx^{-j}\right),$$

$$r\sum_{k=0}^{n} kc_kP_{n-k}(r) = \sum_{k=0}^{n}(-n+k)c_kP_{n-k}(r),$$

and (16) follows. By direct inspection one can see that (16) remains valid for $r = 0$.

The rest of the proof follows using the same technique as in Lemma 3. □

Using this lemma we can obtain another formula for the coefficients in Lemma 3.

COROLLARY 6. *Let* A *be a function with asymptotic expansion*

$$A(x) \sim \sum_{n=1}^{\infty} a_nx^{-n}.$$

Then the composition $B(x) = \exp(A(x))$ *has asymptotic expansion of the following form*

$$B(x) \sim \sum_{n=0}^{\infty} b_nx^{-n}$$

where $b_0 = 1$ *and*

$$b_n = \sum_{k=0}^{n-1}\frac{1}{(n-k)!}P_k(n-k)x^{-n} \tag{17}$$

where (P_n) *is defined as in (16), using* $c_n = a_{n+1}$.

Proof. We can write

$$B(x) \sim \sum_{k=0}^{\infty}\frac{1}{k!}x^{-k}\left(\sum_{n=0}^{\infty}a_{n+1}x^{-n}\right)^k \tag{18}$$

$$\sim \sum_{k=0}^{\infty} \frac{x^{-k}}{k!} \sum_{n=0}^{\infty} P_n(k) x^{-n} \tag{19}$$

$$\sim \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} P_n(k) x^{-n-k} \tag{20}$$

$$\sim \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!} P_k(n-k) x^{-n}. \tag{21}$$

For $k = n$ we have $P_n(0) = 0$, and (17) follows. □

Remark 1. In fact, we can give explicit representation of the coefficients b_n,

$$\exp\left(\sum_{n=1}^{\infty} \frac{a_n}{x^n}\right) \sim 1 + \sum_{n=1}^{\infty} \frac{b_n}{x^n},$$

with the coefficients b_n given by

$$b_n = \sum_{k_1+2k_2+\ldots+nk_n=n} \frac{a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}}{k_1! k_2! \cdots k_n!}$$

summed over all nonnegative integers k_n satisfying the equation

$$k_1 + 2k_2 + \ldots + nk_n = n.$$

The representation using recursive algorithm is better for numerical evaluations.

3. Exponential of digamma function

The psi function behaves like a logarithm, a very simple relation

$$\ln(x + \tfrac{1}{2}) < \psi(x+1) < \ln(x + e^{-\gamma})$$

is proved in [7]. Of course, this behaviour can be also seen from the asymptotic expansion [1, p. 259]:

$$\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots \quad \text{as} \quad x \to \infty, \tag{22}$$

Therefore, one may expect that the function

$$H(x) := e^{\psi(x)}$$

is close to identity. In this sections we give the detailed analysis of asymptotic expansion of the function $H(x+s)$ and analyse properties of related polynomials.

Asymptotic expansion of the function $e^{\psi(x+t)}$ can be obtained from the known expansion [14, p. 33]

$$\psi(x+s) \sim \ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} B_k(s)}{k} x^{-k}. \tag{23}$$

The following result is direct application of Lemma 3.

THEOREM 7. *It holds:*

$$e^{\psi(x+s)} \sim \sum_{n=0}^{\infty} S_n(s)x^{-n+1}, \tag{24}$$

where $S_0 = 1$ *and*

$$S_n(s) = \frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}B_k(s)S_{n-k}(s), \qquad n \geqslant 1. \tag{25}$$

From (25) it seems that S_n is a polynomial of degree n, but this is not the case. There is the collapses of the degree for $n = 2$.

THEOREM 8. *Polynomial* S_n *is of degree* $n-2$ *for* $n \geqslant 2$ *and it satisfies*

$$S_n(s+t) = \sum_{k=2}^{n}(-1)^{n-k}\binom{n-2}{n-k}S_k(s)t^{n-k} \tag{26}$$

Proof. We have

$$\begin{aligned}
H(x+t+s) = e^{\psi(x+t+s)} &\sim \sum_{n=0}^{\infty} S_n(s)(x+t)^{-n+1} \\
&\sim S_0(s)(x+t) + S_1(s) + \sum_{n=2}^{\infty} S_n(s)\sum_{k=0}^{\infty}\binom{-n+1}{k}t^k x^{-n+1-k} \\
&\sim S_0(s)(x+t) + S_1(s) + \sum_{n=2}^{\infty}\sum_{k=0}^{\infty} S_n(s)(-1)^k\binom{n+k-2}{k}t^k x^{-(n+k)+1} \\
&\sim S_0(s)(x+t) + S_1(s) + \sum_{n=2}^{\infty}\sum_{k=2}^{n} S_k(s)(-1)^{n-k}\binom{n-2}{n-k}t^{n-k}x^{-n+1}.
\end{aligned}$$

The manipulations with asymptotic series is justified by Lemma 1. This equals

$$H(x+t+s) \sim \sum_{n=0}^{\infty} S_n(s+t)x^{-n+1}.$$

Comparing coefficients of x^{-n+1} we obtain:

$$\begin{aligned}
&S_0(s+t) = S_0(s), \\
&S_1(s+t) = tS_0(s) + S_1(s), \\
&S_n(s+t) = \sum_{k=2}^{n} S_k(s)(-1)^{n-k}\binom{n-2}{n-k}t^{n-k}, \quad \text{for } n \geqslant 2.
\end{aligned}$$

From (25) easily follows $S_2(0) \neq 0$ (in fact, $S_2(0) = 1/24$) which means that S_n is polynomial of degree $n-2$, for $n \geqslant 2$.

□

Therefore, the leading coefficient of S_n, $n \geqslant 2$ is $(-1)^n S_2(0) = (-1)^n/24$. The first few polynomials (S_n) are

$$\begin{aligned}
S_0 &= 1,\\
S_1 &= s - \frac{1}{2},\\
S_2 &= \frac{1}{24},\\
S_3 &= -\frac{s}{24} + \frac{1}{48},\\
S_4 &= \frac{s^2}{24} - \frac{s}{24} + \frac{23}{5760},\\
S_5 &= -\frac{s^3}{24} + \frac{s^2}{16} - \frac{23s}{1920} - \frac{17}{3840},\\
S_6 &= \frac{s^4}{24} - \frac{s^3}{12} + \frac{23s^2}{960} + \frac{17s}{960} - \frac{10099}{2903040},\\
&\vdots
\end{aligned}$$

Thus, the function $\exp(\psi(x))$ has the following asymptotic expansion:

$$e^{\psi(x)} \sim x - \frac{1}{2} + \frac{1}{24x} + \frac{1}{48x^2} - \frac{23}{5760x^3} - \frac{17}{3840x^4} + \frac{10099}{2903040x^5} + \ldots \tag{27}$$

This function has a simpler expansion in the terms of $x - \frac{1}{2}$, which contains only odd powers. Equivalently, we can write this expansion by taking $s = \frac{1}{2}$:

$$e^{\psi(x+\frac{1}{2})} \sim x + \frac{1}{24x} - \frac{37}{5760x^3} + \frac{10313}{2903040x^5} + \ldots \tag{28}$$

If we write polynomial $S_n(s)$ as a function of variable α, then the expressions will be somewhat simpler. Let $R_n(\alpha) = S_n(s)$, where $\alpha = s - \frac{1}{2}$ and $\beta = \frac{1}{4}$. Therefore, R_n can be calculated as $Q_n(\alpha, \frac{1}{4})$ by formulas from [2] The first few polynomials from this expansion are:

$$\begin{aligned}
R_0 &= 1,\\
R_1 &= \alpha,\\
R_2 &= \frac{1}{24},\\
R_3 &= -\frac{\alpha}{24},\\
R_4 &= \frac{\alpha^2}{24} - \frac{37}{5760},\\
R_5 &= -\frac{\alpha^3}{24} + \frac{37\alpha}{1920},
\end{aligned}$$

$$R_6 = \frac{\alpha^4}{24} - \frac{37\alpha^2}{960} + \frac{10313}{2903040}.$$

THEOREM 9. *Properties of polynomials* (R_n) *are:*
(1) Explicit formulae

$$R_n(\alpha+h) = \sum_{k=2}^{n} (-1)^{n-k} \binom{n-2}{n-k} R_k(h)\alpha^{n-k}, \qquad n \geqslant 2, \tag{29}$$

and

$$R_n(\alpha) = \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{n-2k} \binom{n-2}{n-2k} R_{2k}(0)\alpha^{n-2k}, \qquad n \geqslant 2. \tag{30}$$

Hence

$$R_{2n+1}(0) = 0, \qquad n \geqslant 1 \tag{31}$$

(2) Appell's property:

$$R_n'(\alpha) = -(n-2)R_{n-1}(\alpha). \tag{32}$$

Proof. The equation (29) follows from (26) since $R_n(\alpha) = S_n(\alpha + \frac{1}{2})$.

Let us prove (31). We start with the known asymptotic expansion [14]

$$\psi(x+\tfrac{1}{2}) \sim \ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} B_{k+1}(\frac{1}{2})}{k+1} x^{-k+1}.$$

Since

$$B_{k+1}(\tfrac{1}{2}) = -(1-2^k)B_{k+1}$$

this coefficient is equal to 0 for all even k. Hence, asymptotic expansion of the function $H(x+\frac{1}{2})$ is of the form

$$H(x+\tfrac{1}{2}) \sim x \cdot \exp(H_1(x^2)) = x \cdot H_2(x^2)$$

for some series H_1 and H_2, so it contains only odd powers. Hence, (31) holds true. Putting $h = 0$ in (29), it follows (30).

It remains to prove Appell's property. From

$$H(x+t) = H(x+\tfrac{1}{2}+\alpha) = e^{\psi(x+\frac{1}{2}+\alpha)} \sim \sum_{n=0}^{\infty} R_n(\alpha)x^{-n+1},$$

it holds

$$\frac{\partial H(x+\frac{1}{2}+\alpha)}{\partial \alpha} = \frac{\partial H(x+\frac{1}{2}+\alpha)}{\partial x}.$$

Therefore

$$\sum_{n=0}^{\infty} R_n'(\alpha)x^{-n+1} = \sum_{n=0}^{\infty} R_n(\alpha)(-n+1)x^{-n}$$

and (32) easily follows. $\square$

The property (32) can be called Appell's property, since (32) implies that

$$V_n(\alpha) = (-1)^n R_{n+2}(\alpha)$$

are Appell polynomials.

The most important case is $s = \frac{1}{2}$. Then $\alpha = 0$ and the following expansion holds true:

$$e^{\psi(x+\frac{1}{2})} \sim x + \frac{1}{24x} - \frac{37}{5760x^3} + \frac{10313}{2903040x^5} - \frac{5509121}{1393459200x^7} + \dots \tag{33}$$

4. The main result

THEOREM 10. *Let us denote*

$$F(x,t,s) = \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} = \exp(\psi(G(x))) \tag{34}$$

Then the function G has the following asymptotic expansion

$$G(x) \sim \sum_{k=0}^{\infty} c_k(t,s) x^{-k+1} \tag{35}$$

where

$$\begin{aligned} c_0(t,s) &= 1, \\ c_n(t,s) &= \frac{1}{n}\sum_{k=1}^{n-1} k a_k c_{n-k}(t,s) + \sum_{k=1}^{n} \frac{B_k(1)}{k} b_{n-k}(k) \\ &\qquad + \frac{B_{n+1}(1-t) - B_{n+1}(1-s)}{n(n+1)(t-s)}. \end{aligned} \tag{36}$$

Here $B_n(t)$ are Bernoulli polynomials, $(b_n(k))$, are defined by (16)*, and (a_k) are defined by* (15)

Proof. Exponential and digamma functions are strictly increasing, therefore the inverse of $x \mapsto \exp(\psi(x))$ is well defined. This is sufficient for the existence of the function G. The existence of its asymptotic expansion will be clear from the constructive proof which follows.

Let us denote c_n instead of $c_n(t,s)$. We want to determine coefficients (c_n) from the following equation:

$$\psi\left(x \sum_{j=0}^{\infty} c_j x^{-j}\right) = \log F(x,t,s).$$

The following two expansions will be used, see [14, p. 32 and p. 33]

$$\log F(x,t,s) \sim \ln x + \sum_{n=1}^{\infty} \frac{B_{n+1}(1-t) - B_{n+1}(1-s)}{n(n+1)(t-s)} \frac{1}{x^n},$$

$$\psi(x) \sim \ln x - \sum_{n=1}^{\infty} \frac{B_n(1)}{n} \frac{1}{x^n}.$$

Then we obtain

$$\begin{aligned} &\ln x + \ln\left(\sum_{k=0}^{\infty} c_k x^{-k}\right) - \sum_{k=1}^{\infty} \frac{B_k(1)}{k}\left(x \sum_{j=0}^{\infty} c_j x^{-j}\right)^{-k} \\ &\quad \sim \ln x + \sum_{n=1}^{\infty} \frac{B_{n+1}(1-t) - B_{n+1}(1-s)}{n(n+1)(t-s)} \frac{1}{x^n}. \end{aligned} \tag{37}$$

Extracting the coefficients of the power x^0, it follows from here that $\ln c_0 = 0$, and hence $c_0 = 1$. Using (15) and (16), the left side of this equation can be written, after removing the term $\ln x$, as

$$\begin{aligned} &\sum_{n=1}^{\infty} a_n x^{-n} - \sum_{k=1}^{\infty} \frac{B_k(1)}{k} x^{-k} \sum_{j=0}^{\infty} b_j(k) x^{-j} \\ &\quad \sim \sum_{n=1}^{\infty} \left(a_n - \sum_{k=1}^{n} \frac{B_k(1)}{k} b_{n-k}(k)\right) x^{-n}. \end{aligned} \tag{38}$$

The coefficient c_n which should be determined from here is hidden in the calculation of a_n. Using (15) we can write

$$a_n = c_n - \frac{1}{n} \sum_{k=1}^{n-1} k a_k c_{n-k}. \tag{39}$$

Linking together (37), (38) and (39) immediately follows (36), which proves the theorem. □

Using this procedure the following coefficients can be derived:

$$\begin{aligned} c_0 &= 1, \\ c_1 &= \frac{1}{2}(t+s), \\ c_2 &= -\frac{1}{24}(t-s)^2, \\ c_3 &= \frac{1}{48}(t-s)^2(s+t-1), \\ c_4 &= -\frac{1}{5760}(t-s)^2\Big[73(t^2+s^2) + 94ts - 120(t+s) + 20\Big], \\ c_5 &= \frac{1}{3840}(t-s)^2\Big[33(t^3+s^3) + 47ts(t+s) \\ &\qquad - 73(t^2+s^2) - 94ts + 20(t+s) + 20)\Big], \\ c_6 &= \frac{1}{2903040}(t-s)^2\Big[18125(t^4+s^4) + 27292(t^3s+ts^3) + 30126t^2s^2 \\ &\qquad - 49896(t^3+s^3) - 71064(t^2s+s^2t) + 19404(t^2+s^2) \end{aligned}$$

$$+21672ts+30240(t+s)-10248\Big]$$

We shall now express coefficients c_n in terms of variables α and β. Denote

$$\overline{c}_n(\alpha,\beta)=c(t,s).$$

In relation (36) we substitute the only term which depends on t and s:

$$\frac{B_{n+1}(1-t)-B_{n+1}(1-s)}{(n+1)(t-s)}$$

with $(-1)^{n+1}\nabla_n(\alpha,\beta)$, see [2]. After simplyfication, the first few coefficients can be written as:

$$\begin{aligned}
\overline{c}_0&=1,\\
\overline{c}_1&=\frac{1}{2}+\alpha,\\
\overline{c}_2&=\frac{1}{24}(-1+4\beta),\\
\overline{c}_3&=\frac{1}{24}\alpha(1-4\beta),\\
\overline{c}_4&=\frac{1}{5760}\left(-27+240\alpha^2-52\beta\right)(-1+4\beta),\\
\overline{c}_5&=-\frac{1}{1920}\alpha\left(-27+80\alpha^2-52\beta\right)(-1+4\beta),\\
\overline{c}_6&=-\frac{1}{2903040}\big[7625+120960\alpha^4-3024\alpha^2(27+52\beta)\\
&\qquad+88\beta(185+134\beta)\big](-1+4\beta).
\end{aligned}$$

Polynomials $(\overline{c}_n)$ has analogous properties as those given for polynomials (S_n) and (R_n). The proof is similar, so we omit it.

THEOREM 11. *It holds*
(1) Explicit formulae

$$\overline{c}_n(\alpha,\beta)=\sum_{k=2}^{n}(-1)^{n-k}\binom{n-2}{n-k}\overline{c}_k(0,\beta)\alpha^{n-k},\quad \textit{for } n\geqslant 2. \tag{40}$$

(2) Appell property

$$\frac{\partial\overline{c}_n(\alpha,\beta)}{\partial\alpha}=-(n-2)\overline{c}_{n-1}(\alpha,\beta),\qquad n\geqslant 2. \tag{41}$$

Let us denote

$$d_n(\beta)=\overline{c}_n(0,\beta).$$

THEOREM 12. *Function G from (34) has the following asymptotic expansion*

$$G(x) \sim (x+\alpha) + \tfrac{1}{2} + \sum_{k=1}^{\infty} d_{2k}(\beta)(x+\alpha)^{-2k+1}, \tag{42}$$

where

$$\begin{aligned} d_0(\beta) &= 1, \\ d_n(\beta) &= \frac{1}{n}\sum_{k=1}^{n-1} k a_k d_{n-k}(\beta) + \sum_{k=1}^{n-1} \frac{B_k(1)}{k} b_{n-k}(k) \\ &\quad + \frac{(-1)^{n+1}}{n(n+1)} \sum_{k=0}^{n} \binom{n+1}{k+1} B_{n-k} T_k(\beta), \qquad n \geqslant 1, \end{aligned} \tag{43}$$

where $(b_n(j))$ are defined by (16) (with d_k instead of c_k) and (T_k) are polynomials in β defined by

$$\begin{aligned} T_0(\beta) &= T_1(\beta) = 1, \\ T_n(\beta) &= T_{n-1}(\beta) - \beta T_{n-2}(\beta). \end{aligned} \tag{44}$$

Proof. The formula

$$G(x) \sim \sum_{k=0}^{\infty} d_k(\beta)(x+\alpha)^{-k+1}$$

and the calculation of the coefficients given in (43) follow immediately from the previous theorem, using representation of the Bernoulli quotient through internal variables α and β given in [2]. The only fact which has to be proved is: $d_{2n+1}(\beta) = 0$.

The starting expansion will be

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \sim \sum_{n=0}^{\infty} A_{2n}(\beta)(x+\alpha)^{-2n+1},$$

or the equivalent form given in [14, p. 34]. Take here $t = \alpha + \beta_1$, $s = \alpha + \beta_2$. Then for $\alpha = 0$ it follows that asymptotic expansion of the function F contains only odd powers and has the form

$$F(x, \beta_1, \beta_2) = x F_1(x^2)$$

for some series F_1. On the other hand, this expansion can be written in the form

$$x F_1(x^2) = H(G(x)) \tag{45}$$

where $H(x) = \exp(\psi(x))$. From (33), in the case $\alpha = 0$ the asymptotic expansion of H has the form

$$H(x) \sim x + \sum_{n=1}^{\infty} R_{2n}(\beta) x^{-2n-1}.$$

Suppose that in the asymptotic expansion of the function G

$$G(x) \sim \sum_{n=0}^{\infty} d_n(\beta) x^{-n+1}$$

it holds $d_{2k+1} \neq 0$ for some $k \geqslant 1$. Then there exist a minimal k with this property. In the equation (45) the right side will contain the member $d_{2k+1}x^{-2k}$ (and the exponent $2k$ is minimal with this property) which does not exists on the left side of this equation. Thus, it must be $d_{2k+1}(\beta) = 0$ for all $k \geqslant 1$ and the theorem is proved. □

The first few coefficients d_n are:

$$\begin{aligned}
d_0 &= 1, \\
d_1 &= \frac{1}{2}, \\
d_2 &= \frac{1}{24}(-1+4\beta), \\
d_4 &= \frac{1}{5760}(27-56\beta-208\beta^2), \\
d_6 &= \frac{1}{2903040}(-7625+14220\beta+53328\beta^2+47168\beta^3),
\end{aligned}$$

In particular, for $s = \frac{1}{4}$ and $t = \frac{3}{4}$ it holds $\alpha = 0$ and $\beta = \frac{3}{16}$, so we obtain

$$\left[\frac{\Gamma(x+\frac{3}{4})}{\Gamma(x+\frac{1}{4})}\right]^2 \sim \exp\left(\psi\left[x+\frac{1}{2}-\frac{1}{96x}+\frac{49}{30720x^3}-\frac{177473}{185794560x^5}+\dots\right]\right). \tag{46}$$

As a consequence of obtained expansion, the following hypothesis about lower bound in Kershow second inequality can be posed:

HYPOTHESIS. It holds

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} > \exp\left(\psi(x+\tfrac{1}{2}(t+s)-\tfrac{1}{24}(t-s)^2x^{-1})\right).$$

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