Journal of
Classical
Analysis
Volume 3, Number 1 (2013), 69–83 doi:10.7153/jca-03-06

ON RATIONAL APPROXIMATION OF FUNCTIONS IN REARRANGEMENT INVARIANT SPACES

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Abstract. Some direct theorems for polynomial and rational approximation of functions in the complex plane are proved.

1. Introduction and main results

Let $(\mathscr{R},\mu)$ be a non-atomic σ-finite measure space, i.e., a measure space with non-atomic σ-finite measure μ given on a σ-algebra of subsets of $\mathscr{R}$. We shall denote by $\mathscr{M}$ the set of all μ-measurable complex valued functions on $\mathscr{R}$, and $\mathscr{M}^+$ will be the subset of functions from $\mathscr{M}$ whose values lie in $[0,\infty]$. The characteristic function of a μ-measurable set $E\subset\mathscr{R}$ will be denoted by χ_E.

A mapping $\rho:\mathscr{M}^+\to[0,\infty]$ is called a function norm if it satisfies the following properties for all functions $f,g,f_n\in\mathscr{M}^+\,(n\in\mathbb{N})$, for all constants $a\geqslant 0$, and for all μ-measurable set $E\subset\mathscr{R}$:

(1) $\rho(f)=0\Leftrightarrow f=0\ \mu-a.e.,\ \rho(af)=a\rho(f),\ \rho(f+g)\leqslant\rho(f)+\rho(g),$

(2) $0\leqslant g\leqslant f\ \ \mu-a.e.\Rightarrow\rho(g)\leqslant\rho(f),$

(3) $0\leqslant f_n\uparrow f\ \mu-a.e.\Rightarrow\rho(f_n)\uparrow\rho(f),$

(4) $\mu(E)<\infty\Rightarrow\rho(\chi_E)<\infty,$

(5) $\mu(E)<\infty\Rightarrow\int_E fd\mu\leqslant C_E\rho(f)$, where C_E is a constant depending on E and ρ, but does not depend on f.

If ρ is a function norm, its associate function norm ρ is defined by

$$\rho'(g):=\sup\left\{\int_{\mathscr{R}}fgd\mu : f\in\mathscr{M}^+,\ \rho(f)\leqslant 1\right\} \tag{1}$$

for $g\in\mathscr{M}^+$. If ρ is a function norm, then ρ' is itself a function norm [2, pp. 8–9].

Let ρ be a function norm. We denote by $X=X(\rho)$ the linear space of all functions $f\in\mathscr{M}$ for which $\rho(|f|)<\infty$. The space X is called a Banach function space. If we define the norm of $f\in X$ by

$$\|f\|_X:=\rho(|f|)$$

Mathematics subject classification (2010): 30E10, 41A20, 41A25, 46E30.
Keywords and phrases: Carleson curve, Cauchy singular operator, rearrangement invariant space.

Paper JCA-03-06

then X will be a Banach space [2, pp. 6–7]. It follows by property (5) that if the measure space $(\mathscr{R},\mu)$ is finite, i.e., if $\mu(\mathscr{R})<\infty$, then $X\subset L_1(\mathscr{R},\mu)$.

Let ρ be a function norm and ρ' be its associate function norm. The Banach function space determined by the function ρ' is called the associate space of X and will be denoted by X'.

If $f\in X$ and $g\in X'$, then the Hölder inequality

$$\int_{\mathscr{R}}|fg|\,d\mu\leqslant\|f\|_X\,\|g\|_{X'}$$

holds [2, Ch. 1, p. 9, Theorem 2.4].

Every Banach function space X coincides [2, Theorem 2.7, pp. 10–12] with its second associate space $X''=(X')'$ and $\|f\|_X=\|g\|_{X''}$ for all $f\in X$. So, by (1) we have

$$\|f\|_X=\sup\left\{\int_{\mathscr{R}}|fg|\,d\mu:g\in X',\|g\|_{X'}\leqslant 1\right\}\tag{2}$$

and

$$\|g\|_{X'}=\sup\left\{\int_{\mathscr{R}}|fg|\,d\mu:f\in X,\ \|f\|_X\leqslant 1\right\}.\tag{3}$$

Let $\mathscr{M}_0$ and $\mathscr{M}_0^+$ be classes of $\mu-a.e.$ finite functions in $\mathscr{M}$ and $\mathscr{M}^+$, respectively. The distribution function μ_f of $f\in\mathscr{M}_0$ is defined by

$$\mu_f(\lambda):=\mu\{x\in\mathscr{R}:|f(x)|>\lambda\}$$

for $\lambda\geqslant 0$. Two functions $f,g\in\mathscr{M}_0$ are said to be equimeasurable if $\mu_f(\lambda)=\mu_g(\lambda)$ for all $\lambda\geqslant 0$.

DEFINITION 1. [2, p. 59] If $\rho(f)=\rho(g)$ for every pair of equimeasurable functions $f,g\in M_0^+$, the function norm ρ is called a rearrangement invariant function norm. In this case, the Banach function space generated by ρ is called a rearrangement invariant function space.

Let $f\in\mathscr{M}_0$. The function f^* defined by

$$f^*(t):=\inf\left\{\lambda:\mu_f(\lambda)\leqslant t\right\},\quad t\geqslant 0$$

is called the decreasing rearrangement of the function f.

Let X be a rearrangement-invariant space over a non-atomic finite measure space $(\mathscr{R},\mu)$. By the Luxemburg representation theorem [2, pp. 62–64], there is a (not necessarily unique) rearrangement-invariant function norm $\overline{\rho}$ over $(\mathbb{R}_+,m)$ such that

$$\rho(f)=\overline{\rho}(f^*),\quad f\in\mathscr{M}_0^+,$$

where m is the Lebesgue measure on $\mathbb{R}_+:=[0,\infty)$.

The rearrangement invariant space over $(\mathbb{R}_+,m)$ generated by $\overline{\rho}$ is denoted by $\overline{X}$.

Let us consider the dilation operator E_x, $x>0$ defined on $\mathscr{M}_0(\mathbb{R}_+,m)$ by

$$(E_xf)(t):=\begin{cases} f(xt), & xt\in[0,\mu(\mathscr{R})]\\ 0, & xt\notin[0,\mu(\mathscr{R})]\end{cases},\quad t>0.$$

It is known that $E_{1/x} \in B\left(\overline{X}\right)$ for each $x > 0$, where $B\left(\overline{X}\right)$ is the Banach algebra of bounded linear operators on $\overline{X}$ ([2, p. 165]). Let $h_X(x)$ denote the operator norm of $E_{1/x}$, i.e., $h_X(x) := \|E_{1/x}\|_{B(\overline{X})}$.

The numbers α_X and β_X defined by

$$\alpha_X = \sup_{0<x<1} \frac{\log h_X(x)}{\log x}, \quad \beta_X = \inf_{1<x<\infty} \frac{\log h_X(x)}{\log x},$$

are called the lower and upper indices Boyd of X, respectively. It is known that [2, p. 149] the Boyd indices satisfy

$$0 \leqslant \alpha_X \leqslant \beta_X \leqslant 1.$$

The Boyd indices are said to be nontrivial if $0 < \alpha_X \leqslant \beta_X < 1$.

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$ and let $G := Int\Gamma$, $G^- := Ext\Gamma$. Without loss of generality we suppose that $0 \in G$. Further, let

$$\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}, \quad \mathbb{U} := Int\mathbb{T}, \quad \mathbb{U}^- := Ext\mathbb{T}.$$

We denote by φ and φ_1 the conformal mappings of G^- and G onto $\mathbb{U}^-$ normalized by the conditions

$$\varphi(\infty) = \infty, \quad \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0$$

and

$$\varphi_1(0) = \infty, \quad \lim_{z\to 0} z\varphi_1(z) > 0,$$

respectively, and let ψ and ψ_1 be the inverse mappings of φ and φ_1.

For $z \in \Gamma$ and $\varepsilon > 0$, we denote by $\Gamma(z,\varepsilon)$ the portion of Γ in the open disk of radius ε centered at z, i. e.,

$$\Gamma(z,\varepsilon) := \{t \in \Gamma : |t - z| < \varepsilon\}.$$

Further, let $|\Gamma(z,\varepsilon)|$ denotes the length of $\Gamma(z,\varepsilon)$.

DEFINITION 2. Γ is called a Carleson curve if the condition

$$\sup_{z\in\Gamma} \sup_{\varepsilon>0} \frac{1}{\varepsilon} |\Gamma(z,\varepsilon)| < \infty$$

holds.

The class of Carleson curves is a wide class of curves. For example, analytic curves, Lavrentiev curves and Dini-Smooth curves are Carleson curves.

We assume that the rectifiable Jordan curve Γ is equipped with the arclength measure and in this case we denote any rearrangement invariant space over Γ by $X(\Gamma)$.

For $\varsigma \in \Gamma$ we define three points $\varsigma_\theta \in \Gamma$, $\varsigma_{1\theta} \in \Gamma$ and $\varsigma_{2\theta} \in \Gamma$ as

$$\varsigma_\theta := \psi[\varphi(\varsigma) e^{i\theta}], \quad \varsigma_{1\theta} := \psi_1[\varphi_1(\varsigma) e^{i\theta}], \quad \varsigma_{2\theta} := \psi_2[\varphi_2(\varsigma) e^{i\theta}],$$

where $\theta \in [-\pi, \pi]$ and the function φ_2 $(\varphi_2(0) = 0)$ maps the domain G conformally onto $\mathbb{U}$.

Let $X(\Gamma)$ be a rearrangement invariant space over Γ. For $f \in X(\Gamma)$ we define the shifts T_θ, $T_{1\theta}$ and $T_{2\theta}$ as

$$T_\theta f(\varsigma) := \frac{f(\varsigma_\theta)}{\varphi'(\varsigma_\theta)}\varphi'(\varsigma), \tag{4}$$

$$T_{1\theta} f(\varsigma) := \frac{f(\varsigma_{1\theta})}{\varphi_1'(\varsigma_{1\theta})}\varphi_1'(\varsigma)\, e^{2i\theta} \tag{5}$$

and

$$T_{2\theta} f(\varsigma) := \frac{f(\varsigma_{2\theta})}{\varphi_2'(\varsigma_{2\theta})}\varphi_2'(\varsigma) \tag{6}$$

for $\varsigma \in \Gamma$.

For example, if $\Gamma \equiv \mathbb{T}$, then $T_\theta f(w) = f(we^{i\theta})$, $T_{1\theta} f(w) = f(we^{-i\theta})$, $T_{2\theta} f(w) = f(we^{i\theta})$ and hence $T_\theta f(w)$, $T_{1\theta} f(w)$, $T_{2\theta} f(w) \in X(\Gamma)$ as soon as $f \in X(\Gamma)$ and $X(\Gamma)$ has non-trivial Boyd indices. Moreover, if $X(\Gamma)$ has non-trivial Boyd indices and

$$0 < c_1 \leqslant |\varphi'(z)| \leqslant c_2 < \infty,$$

$$0 < c_3 \leqslant |\varphi_1'(z)| \leqslant c_4 < \infty$$

or

$$0 < c_5 \leqslant |\varphi_2'(z)| \leqslant c_6 < \infty,$$

for $z \in \Gamma$ and with the constants c_1, c_2, c_3, c_4, c_5 and c_6 which are independent of z, then it easy to verify that the space $X(\Gamma)$ is invariant with respect to the shifts $T_\theta f$, $T_{1\theta} f$, $T_{2\theta} f$. Starting from this, we define the functions $\omega_X^{(2)}(.,f)$, $\omega_{1X}^{(2)}(.,f)$, $\omega_{2X}^{(2)}(.,f)$ and $\Omega_X^{(2)}(.,f)$ for $\delta \geqslant 0$ as

$$\omega_X^{(2)}(\delta,f) := \sup_{|\theta|\leqslant\delta} \left\|T_\theta f + T_{(-\theta)} f - 2f\right\|_{X(\Gamma)}$$

$$\omega_{1X}^{(2)}(\delta,f) := \sup_{|\theta|\leqslant\delta} \left\|T_{1\theta} f + T_{1(-\theta)} f - 2f\right\|_{X(\Gamma)}$$

$$\omega_{2X}^{(2)}(\delta,f) := \sup_{|\theta|\leqslant\delta} \left\|T_{2\theta} f + T_{2(-\theta)} f - 2f\right\|_{X(\Gamma)}$$

$$\Omega_X^{(2)}(\delta,f) := \omega_X^{(2)}(\delta,f) + \omega_{1X}^{(2)}(\delta,f).$$

Let $\omega(\delta)$ be a non-negative, continuous, non-decreasing real function such that $\omega(0) = 0, \omega(\delta) > 0$ for $\delta > 0$, and $\omega(n\delta) \leqslant c_7 n\omega(\delta)$ for every natural number n and with some constant $c_7 > 0$. Similarly, let another $\omega^*(\delta)$ have all properties of $\omega(\delta)$.

Let $L_p(\Gamma)$ and $E_p(G)$ $(1 \leqslant p < \infty)$ be the Lebesgue space of measurable complex valued functions on Γ and the Smirnov class of analytic functions in G, respectively. Since Γ is rectifiable, we have $\varphi' \in E_1(G^-)$, $\varphi_1' \in E_1(G)$ and ψ', $\psi_1' \in E_1(\mathbb{U}^-)$ which imply that the functions φ' and φ_1' admit the nontangential limits *a. e.* on Γ

belong to $L_1(\Gamma)$, and ψ' and ψ_1' admit the nontangential limits *a. e.* on $\mathbb{T}$ belong to $L_1(\mathbb{T})$ [9, pp. 419–453].

Now we denote by $E_X(G)$ the class of functions $f \in E_1(G)$ for which the boundary function f belongs to $X(\Gamma)$. Similarly, the class $E_X(G^-)$ can be defined. Obviously, the class $E_X(G)$ is wider than Smirnov classes $E_p(G)$ and as well as the Smirnov-Orlicz classes $E_M(G)$ given in [13] (see also [6], [7]).

We define the classes of functions $X^\omega(\Gamma), X^\omega(\Gamma)^*$, $E_X^\omega(G)$ and $E_X^\omega(G^-)$ as

$$X^\omega(\Gamma) := \left\{ f \in X(\Gamma) : \Omega_X^{(2)}(\delta, f) \leqslant c_8 \omega(\delta) \right\},$$

$$X^\omega(\Gamma)^* := \left\{ f \in X(\Gamma) : \omega_X^{(2)}(\delta, f) \leqslant c_9 \omega(\delta) \text{ and } \omega_{2X}^{(2)}(\delta, f) \leqslant c_{10} \omega^*(\delta) \right\},$$

$$E_X^\omega(G) := \left\{ f \in E_X(G) : \omega_X^{(2)}(\delta, f)^* \leqslant c_{11} \omega(\delta) \right\},$$

$$E_X^\omega\left(G^-\right) := \left\{ f \in E_X\left(G^-\right) : \omega_{1X}^{(2)}(\delta, f) \leqslant c_{12} \omega(\delta) \right\},$$

where the constants c_8, c_9, c_{10}, c_{11}, and c_{12} are independent of f and δ.

It is clear that if $f \in X^\omega(\Gamma)$, then $T_\theta f \in X(\Gamma)$ and $T_{1\theta} f \in X(\Gamma)$. If $f \in X^\omega(\Gamma)^*$, then $T_\theta f \in X(\Gamma)$ and $T_{2\theta} f \in X(\Gamma)$. Similarly, if $f \in E_X^\omega(G)$, then $T_\theta f \in X(\Gamma)$, and $T_{1\theta} f \in X(\Gamma)$ for $f \in E_X^\omega(G^-)$.

For $a > 0$ and $b > 0$, we will use the expression $a \preceq b$ (order inequality) if $a \leqslant cb$. The expression $a \asymp b$ means that $a \preceq b$ and $b \preceq a$ simultaneously. Through the paper, c_i, $i = 1, 2, \ldots$ will denote the positive constants which are not important for the questions involve in the paper and can be different in each occurrence.

Our main results are given in the following theorems.

THEOREM 1. *Let Γ be a Carleson curve, $X(\Gamma)$ be a rearrangement invariant space with non-trivial Boyd indices, and $f \in X^\omega(\Gamma)$. Then, for each natural number n, there exists a rational function $R_n(z, f)$ such that*

$$\|f - R_n(., f)\|_{X(\Gamma)} \leqslant c_{13} \omega(1/n), \tag{7}$$

with a constant c_{13}, which is independent of n.

COROLLARY 1. *If $f \in E_X^\omega(G)$, then for each natural number n, there exists an algebraic polynomial $P_n(z, f)$ of degree $\leqslant n$ such that*

$$\|f - P_n(., f)\|_{X(\Gamma)} \leqslant c_{14} \omega(1/n) \tag{8}$$

holds with a constant c_{14}, which is independent of n.

COROLLARY 2. *If $f \in E_X^\omega(G^-)$, then for each natural number n, there exists a polynomial $\tilde{Q}_n(1/z, f)$ of $1/z$ such that*

$$\left\|f - \tilde{Q}_n(., f)\right\|_{X(\Gamma)} \leqslant c_{15} \omega(1/n) \tag{9}$$

holds with a constant c_{15}, which is independent of n.

Similar results were obtained in [8].

THEOREM 2. *Let Γ be a Carleson curve and $f\in X^{\omega}(\Gamma)^{*}$ and*

$$\tilde{f}(z):=\frac{1}{\pi i}\int_{\Gamma}\frac{f(\varsigma)}{\varsigma-z}d\varsigma.$$

Then the following properties are satisfied for the function

$$F_{AB}(z):=Af(z)+B\tilde{f}(z).$$

(1) *If $A=B$, then for each natural number n, there exists an algebraic polynomial $P_n(z,f)$ of degree $\leqslant n$ such that*

$$\|F_{AB}-P_n(.,f)\|_{X(\Gamma)}\leqslant c_{16}[2A]\,\omega(1/n).$$

(2) *If $A\neq B$, then for each natural number n, there exists a rational polynomial $R_n(z,f)$ of degree $\leqslant n$ such that*

$$\|F_{AB}-R_n(.,f)\|_{X(\Gamma)}\leqslant c_{17}\left[(A+B)\,\omega(1/n)+(B-A)\,\omega^{*}(1/n)\right].$$

The analogue of Theorem 2 was proved in [11] for Orlicz spaces $L_M(\Gamma)$.

2. Auxiliary results

Let Γ be a rectifiable Jordan curve and $f\in L_1(\Gamma)$. Then the functions f^{+} and f^{-} defined by

$$f^{+}(z):=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\varsigma)}{\varsigma-z}d\varsigma,\quad z\in G \tag{10}$$

and

$$f^{-}(z):=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\varsigma)}{\varsigma-z}d\varsigma,\quad z\in G^{-} \tag{11}$$

are analytic in G and G^{-}, respectively, and $f^{-}(\infty)=0$.

The Cauchy singular integral of $f\in L_1(\Gamma)$ at $z\in\Gamma$ is defined by

$$S_{\Gamma}f(z):=\lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_{\Gamma\setminus\Gamma(z,\varepsilon)}\frac{f(\varsigma)}{\varsigma-z}d\varsigma.$$

It is known that this limit exists for almost every $z\in\Gamma$ ([3, pp. 117–144]).

The functions f^{+} and f^{-} have nontangential limits *a.e.* on Γ, and the formulae

$$f^{+}(z)=S_{\Gamma}f(z)+\frac{1}{2}f(z),\quad f^{-}(z)=S_{\Gamma}f(z)-\frac{1}{2}f(z) \tag{12}$$

holds *a.e.* on Γ ([9, p. 431]), and hence

$$f=f-f^{-} \tag{13}$$

a.e. on Γ.

For $f\in L_1(\Gamma)$, we associate the function $S_{\Gamma}f$ taking the value $S_{\Gamma}f(z)$ exists *a.e.* on Γ. The linear operator S_{Γ} defined in such way is called the Cauchy singular operator.

A necessary and sufficient condition for the boundedness of Cauchy singular operator in rearrangement invariant spaces was given in [12].

THEOREM 3. *Let Γ be a rectifiable Jordan curve and $X(\Gamma)$ be a rearrangement invariant space with non-trivial Boyd indices. Then the inequality*

$$\|S_{\Gamma}(f)\|_{X(\Gamma)} \leqslant c_{18} \|f\|_{X(\Gamma)}, \quad f \in X(\Gamma) \tag{14}$$

holds if and only if Γ is a Carleson curve.

Let k be a nonnegative integer. Then the function $\varphi'(z)\varphi^k(z)$ has a pole of order k at ∞. Hence there exists a polynomial $B_k(z)$ of degree k and a function $E_k(z)$ analytic in G^- such that $E_k(\infty) = 0$ and

$$\varphi^k(z)\varphi'(z) = B_k(z) + E_k(z)$$

holds for every $z \in G^-$.

The polynomials $B_k(z)$ $(k = 0, 1, 2, \ldots)$ are called the Faber polynomals of the second kind for G and satisfy the expansion

$$\frac{1}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{B_k(z)}{w^{k+1}} \tag{15}$$

for $z \in G$ and $w \in \mathbb{U}^-$ [16, p. 95].

Now let's consider the function $[\varphi_1(z)]^{k-2}\varphi_1'(z)$. This function is analytic in $G \setminus \{0\}$ and has a pole of order k at the point 0. If we denote its principal part at 0 by $\tilde{B}_k(1/z)$, then there exists an analytic function $\tilde{E}_k(z)$ in G such that

$$[\varphi_1(z)]^{k-2}\varphi_1'(z) = \tilde{B}_k(1/z) + \tilde{E}_k(z)$$

holds for every $z \in G \setminus \{0\}$ and for the principal parts $\tilde{B}_k(1/z)$ the expansion

$$\frac{w^{-2}}{\psi_1(w) - z} = \sum_{k=0}^{\infty} -\frac{\tilde{B}_k(1/z)}{w^{k+1}}, \quad z \in G,\ w \in \mathbb{U}^- \tag{16}$$

holds ([4]).

3. Proofs of main results

Let $f \in X(\Gamma)$. Since $X(\Gamma) \subset L_1(\Gamma)$, we get $f \in L_1(\Gamma)$. Hence the functions

$$f_0(w) := f(\psi(w))\psi'(w)$$

and

$$f_1(w) := f(\psi_1(w))\psi_1'(w)w^2$$

are integrable on $\mathbb{T}$. We can associate the series

$$f_0(w) \sim \sum_{k=-\infty}^{\infty} a_k w^k \tag{17}$$

and

$$f_1(w) \sim \sum_{k=-\infty}^{\infty} \tilde{a}_k w^k \tag{18}$$

for $w \in \mathbb{T}$.

Let

$$K_n(\theta) = \sum_{m=-n}^{n} \lambda_m^{(n)} e^{im\theta}$$

be an even, nonnegative trigonometric polynomial satisfying the conditions

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\theta)\, d\theta = 1, \tag{19}$$

$$\int_0^{\pi} \theta K_n(\theta)\, d\theta \leqslant c_9/n, \tag{20}$$

for every natural number n and with a constant $c_9 > 0$. In special case, the Jackson kernel

$$J_n(\theta) = \frac{3\sin^4(n\theta/2)}{n(2n^2+1)\sin^4(\theta/2)}$$

satisfies these conditions ([5, p. 203]).

Let us consider the integral

$$I(\theta, z) := \frac{1}{2\pi i} \int_{\Gamma} \left[\frac{f(\varsigma_{-\theta})}{\varphi'(\varsigma_{-\theta})} + \frac{f(\varsigma_{\theta})}{\varphi'(\varsigma_{\theta})} \right] \frac{\varphi'(\varsigma)}{\varsigma - z} d\varsigma, \quad z \in G.$$

Using the change of variables $\varsigma = \psi(e^{it})$, we obtain

$$I(\theta, z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f_0\left(e^{i(t-\theta)}\right) + f_0\left(e^{i(t+\theta)}\right) \right] \frac{e^{it}}{\psi(e^{it}) - z} d\varsigma.$$

Since by (17)

$$f_0(e^{it}) \sim \sum_{k=-\infty}^{\infty} a_k e^{ikt}$$

and by (15)

$$\frac{e^{it}}{\psi(e^{it}) - z} \sim \sum_{k=0}^{\infty} \frac{B_k(z)}{e^{ikt}}$$

we can associate to $I(\theta, z)$ the series expansion

$$I(\theta, z) \sim \sum_{k=0}^{\infty} a_k B_k(z) \left[e^{-ik\theta} + e^{ik\theta} \right], \quad z \in G \text{ ([1, pp. 74–75]).}$$

Since $I(\theta, z) \in L_1([-\pi, \pi])$ and $K_n(\theta)$ is of bounded variation, by the generalized Parseval identity [1, pp. 225–228] and (19), we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\theta) I(\theta, z)\, d\theta = \sum_{k=0}^{n} \left(\mu_k^{(n)} + \lambda_k^{(n)} \right) a_k B_k(z).$$

By definition of $I(\theta,z)$ we obtain

$$\frac{1}{4\pi^2 i}\int_{-\pi}^{\pi} K_n(\theta)\left\{\int_\Gamma \left[\frac{f(\varsigma_{-\theta})}{\varphi'(\varsigma_{-\theta})}+\frac{f(\varsigma_\theta)}{\varphi'(\varsigma_\theta)}\right]\frac{\varphi'(\varsigma)}{\varsigma-z}d\varsigma\right\}d\theta=\sum_{k=0}^{n}\left(\mu_k^{(n)}+\lambda_k^{(n)}\right)a_k B_k(z)$$

for $z\in G$.

Hence, we conclude that

$$P_n(z,f):=\frac{1}{4\pi^2 i}\int_{0}^{\pi} K_n(\theta)\left\{\int_\Gamma \left[\frac{f(\varsigma_{-\theta})}{\varphi'(\varsigma_{-\theta})}+\frac{f(\varsigma_\theta)}{\varphi'(\varsigma_\theta)}\right]\frac{\varphi'(\varsigma)}{\varsigma-z}d\varsigma\right\}d\theta, \quad z\in G$$

is an algebraic polynomial of degree n.

Now, we consider the integral

$$I_1(\theta,z):=\frac{1}{2\pi i}\int_\Gamma \left[\frac{f(\varsigma_{-1\theta})e^{-2i\theta}}{\varphi_1'(\varsigma_{-1\theta})}+\frac{f(\varsigma_{1\theta})e^{2i\theta}}{\varphi_1'(\varsigma_{1\theta})}\right]\frac{\varphi_1'(\varsigma)}{\varsigma-z}d\varsigma, \quad z\in G^-.$$

Substituting $\varsigma=\psi_1\left(e^{it}\right)$ here, we obtain

$$I_1(\theta,z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f_1\left(e^{i(t-\theta)}\right)+f_1\left(e^{i(t+\theta)}\right)\right]\frac{e^{-it}}{\psi_1(e^{it})-z}dt.$$

Then, (16) and (18), the function $I_1(\theta,z)$ has [1, pp. 74–75] Fourier expansion

$$I_1(\theta,z)\sim-\sum_{k=0}^{\infty}\tilde{a}_k\tilde{B}_k(1/z)\left[e^{-ik\theta}+e^{ik\theta}\right], \quad z\in G^-.$$

Since $I_1(\theta,z)$ is integrable and the kernel $K_n(\theta)$ is of bounded variation, generalized Parseval identity ([1, pp. 225–228]) yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(\theta)I_1(\theta,z)d\theta=-\sum_{k=0}^{n}\left(\tilde{\mu}_k^{(n)}+\tilde{\lambda}_k^{(n)}\right)\tilde{a}_k\tilde{B}_k(1/z), z\in G^-,$$

for $z\in G^-$. Taking into account definition of $I_1(\theta,z)$, for $z\in G^-$ is seen that

$$Q_n(1/z,f):=\frac{1}{4\pi^2 i}\int_0^{\pi}K_n(\theta)\left\{\int_\Gamma \left[\frac{f(\varsigma_{-1\theta})e^{-2i\theta}}{\varphi_1'(\varsigma_{-1\theta})}+\frac{f(\varsigma_{1\theta})e^{2i\theta}}{\varphi_1'(\varsigma_{1\theta})}\right]\frac{\varphi_1'(\varsigma)}{\varsigma-z}d\varsigma\right\}d\theta$$

a polynomial of degree n of $1/z$.

On the other hand, by (4) and (5), we obtain

$$P_n(z,f)=\frac{1}{4\pi^2 i}\int_0^{\pi}K_n(\theta)\left\{\int_\Gamma\frac{F_\theta(\varsigma)d\varsigma}{\varsigma-z}\right\}d\theta, \quad z\in G$$

and

$$Q_n(1/z,f) = \frac{1}{4\pi^2 i}\int_0^{\pi} K_n(\theta)\left\{\int_{\Gamma}\frac{F_{1\theta}(\varsigma)\,d\varsigma}{\varsigma - z}\right\}d\theta, \quad z\in G^-,$$

where

$$F_\theta(\varsigma) := T_{(-\theta)}f(\varsigma) + T_\theta f(\varsigma)$$

and

$$F_{1\theta}(\varsigma) := T_{1(-\theta)}f(\varsigma) + T_{1\theta}f(\varsigma).$$

Taking into account (10) and (11), we get

$$P_n(z,f) = \frac{1}{2\pi}\int_0^{\pi} K_n(\theta)(F_\theta)^+(z)\,d\theta, \quad z\in G \tag{21}$$

and

$$Q_n(1/z,f) = \frac{1}{2\pi}\int_0^{\pi} K_n(\theta)(F_{1\theta})^-(z)\,d\theta, \quad z\in G^-. \tag{22}$$

Proof of Theorem 1. Let $f\in X^{\omega}(\Gamma)$. Let us take $z'\in G$. By (19) we have

$$f^+(z') = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^+(z')K_n(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{\pi} 2f^+(z')K_n(\theta)\,d\theta,$$

which together with (21) implies that

$$f^+(z') - P_n(z',f) = \frac{1}{2\pi}\int_0^{\pi} K_n(\theta)\left\{2f^+(z') - (F_\theta)^+(z')\right\}d\theta.$$

Limiting $z'\to z\in\Gamma$, along all nontangential paths inside Γ, by (12) we have

$$\begin{aligned} f^+(z) - P_n(z,f) = {} & \frac{1}{4\pi}\int_0^{\pi} K_n(\theta)\left[(2f - F_\theta)(z)\right]d\theta \\ & + \frac{1}{2\pi}\int_0^{\pi} K_n(\theta)\left[S_\Gamma(2f - F_\theta)(z)\right]d\theta \end{aligned} \tag{23}$$

for almost all $z\in\Gamma$.

Taking the supremum over all functions $g\in X'(\Gamma)$ with $\|g\|_{X'(\Gamma)}\leqslant 1$ in the relation (23), we get

$$\begin{aligned} \|f^+ - P_n(.,f)\|_{X(\Gamma)} &= \sup\int_\Gamma \left|f^+(z) - P_n(z,f)\right||g(z)||dz| \\ &\leqslant \sup\int_\Gamma \left|\frac{1}{4\pi}\int_0^{\pi} K_n(\theta)\left[(2f - F_\theta)(z)\right]d\theta\right||g(z)||dz| \\ &\quad + \sup\int_\Gamma \left|\frac{1}{2\pi}\int_0^{\pi} K_n(\theta)\left[S_\Gamma(2f - F_\theta)(z)\right]d\theta\right||g(z)||dz| \\ &\leqslant \sup\int_\Gamma \left\{\frac{1}{4\pi}\int_0^{\pi} K_n(\theta)\left|(2f - F_\theta)(z)\right|d\theta\right\}|g(z)||dz| \\ &\quad + \sup\int_\Gamma \left\{\frac{1}{2\pi}\int_0^{\pi} K_n(\theta)\left|S_\Gamma(2f - F_\theta)(z)\right|d\theta\right\}|g(z)||dz|. \end{aligned}$$

By Fubini theorem and (14) we obtain

$$\begin{aligned}\left\|f^{+}-P_n(.,f)\right\|_{X(\Gamma)} &\leqslant \frac{1}{4\pi}\int_0^{\pi} K_n(\theta)\left\{\sup\int_{\Gamma}|(2f-F_{\theta})(z)|\,|g(z)|\,|dz|\right\}d\theta \\ &\quad+\frac{1}{2\pi}\int_0^{\pi} K_n(\theta)\left\{\sup\int_{\Gamma}|S_{\Gamma}(2f-F_{\theta})(z)|\,|g(z)|\,|dz|\right\}d\theta \\ &\leqslant \frac{1}{4\pi}\int_0^{\pi} K_n(\theta)\left[\|2f-F_{\theta}\|_{X(\Gamma)}\right]d\theta \\ &\quad+\frac{1}{2\pi}\int_0^{\pi} K_n(\theta)\left[\|S_{\Gamma}(2f-F_{\theta})\|_{X(\Gamma)}\right]d\theta \\ &\leqslant c_{19}\int_0^{\pi} K_n(\theta)\left[\|F_{\theta}-2f\|_{X(\Gamma)}+\|F_{\theta}-2f\|_{X(\Gamma)}\right]d\theta\end{aligned}$$

and then by definition of $\omega_X^{(2)}(.,f)$ we have

$$\left\|f^{+}-P_n(.,f)\right\|_{X(\Gamma)} \leqslant c_{20}\int_0^{\pi} K_n(\theta)\,\omega_X^{(2)}(\theta,f)d\theta. \tag{24}$$

Similarly, for $z' \in G^{-}$ we obtain

$$f^{-}(z')-Q_n(1/z',f)=\frac{1}{2\pi}\int_0^{\pi} K_n(\theta)\left\{2f^{-}(z')-(F_{1\theta})^{-}(z')\right\}d\theta$$

Here letting $z' \to z \in \Gamma$, along all nontangential paths outside Γ by (12) we get

$$\begin{aligned}f^{-}(z)-Q_n(1/z,f) &= \frac{1}{2\pi}\int_0^{\pi} K_n(\theta)\left[S_{\Gamma}(2f-F_{1\theta})(z)\right]d\theta \\ &\quad+\frac{1}{4\pi}\int_0^{\pi} K_n(\theta)\left[(F_{1\theta}-2f)(z)\right]d\theta\end{aligned}$$

for almost all $z \in \Gamma$. Therefore,

$$\left\|f^{-}-Q_n(.,f)\right\|_{X(\Gamma)} \leqslant c_{21}\int_0^{\pi} K_n(\theta)\left[\|F_{\theta}-2f\|_{X(\Gamma)}\right]d\theta$$

and by definition of $\omega_{1X}^{(2)}(.,f)$ we obtain

$$\left\|f^{-}-Q_n(.,f)\right\|_{X(\Gamma)} \leqslant c_{22}\int_0^{\pi} K_n(\theta)\,\omega_{1X}^{(2)}(\theta,f)d\theta. \tag{25}$$

If we set

$$R_n(z,f) := P_n(z,f)-Q_n(1/z,f),$$

then by (13), (24), (25) and by definition of $\Omega_X^{(2)}(.,f)$ we get

$$\begin{aligned}\|f-R_n(.,f)\|_{X(\Gamma)} &\leqslant \left\|f^+-P_n(.,f)\right\|_{X(\Gamma)}+\left\|f^--Q_n(.,f)\right\|_{X(\Gamma)}\\ &\leqslant c_{23}\int_0^\pi K_n(\theta)\,\Omega_X^{(2)}(\theta,f)\,d\theta\\ &\leqslant c_{24}\int_0^\pi K_n(\theta)\,\omega(\theta)\,d\theta\\ &= c_{24}\int_0^\pi K_n(\theta)\,\omega(n\theta/n)\,d\theta\\ &\leqslant c_{25}\,\omega(1/n)\int_0^\pi K_n(\theta)(n\theta+1)\,d\theta.\end{aligned}$$

This relation, (19) and (20) give (7). □

Proof of Corollary 1. Let $f\in E_X^{\omega}(G)$. Let's take $z'\in G^-$. Since then $f\in E_X(G)\subset E_1(G)$ we have by the Cauchy theorem

$$f^-\left(z'\right)=\frac{1}{2\pi i}\int_\Gamma\frac{f(\varsigma)}{\varsigma-z}d\varsigma=0.$$

Thus $f^-(z')=0$ for almost all $z\in\Gamma$ and hence $f=f^+$ *a.e.* on Γ. Now, by (24), by the definition of $\omega_X^{(2)}(.,f)$, and by properties (19) and (20) of the kernel $K_n(\theta)$, we have

$$\begin{aligned}\|f-P_n(.,f)\|_{L_M(\Gamma)} &\leqslant c_{26}\int_0^\pi K_n(\theta)\,\omega_X^{(2)}(\theta,f)d\theta\leqslant c_{27}\int_0^\pi K_n(\theta)\,\omega(\theta)\,d\theta\\ &\leqslant c_{28}\,\omega(1/n),\end{aligned}$$

and hence (8) is proved. □

Proof of Corollary 2. Let $f\in E_X^{\omega}(G^-)$ and $z'\in G$. Then by the Cauchy formula we have

$$f^+\left(z'\right)=\frac{1}{2\pi i}\int_\Gamma\frac{f(\varsigma)}{\varsigma-z}d\varsigma=f(\infty).$$

Hence $f^+(z')=f(\infty)$ *a.e.* on Γ and by (12) we have $f=f(\infty)-f^-$ *a.e.* on Γ. Now, setting

$$\tilde{Q}_n(1/z,f):=f(\infty)-Q_n(1/z,f)$$

and applying the relation (25) we conclude that

$$\begin{aligned}\left\|f-\tilde{Q}_n(.,f)\right\|_{X(\Gamma)} &\leqslant c_{29}\int_0^\pi K_n(\theta)\,\omega_{1X}^{(2)}(\theta,f)d\theta\leqslant c_{30}\int_0^\pi K_n(\theta)\,\omega(\theta)\,d\theta\\ &\leqslant c_{31}\,\omega(1/n),\end{aligned}$$

and the proof is completed. □

Proof of Theorem 2. Let Γ be a Carleson curve and $f \in X^{\omega}(\Gamma)^{*}$. Since $X^{\omega}(\Gamma)^{*} \subset X(\Gamma) \subset L_1(\Gamma)$ ([12]) we obtain $f \in L_1(\Gamma)$. Cauchy's singular integral $S_{\Gamma} f(z)$ exists a.e. on Γ. Hence, by Privalov's theorem Cauchy type integrals f^{+} and $f^{-}(z)$ have non-tangential limits a.e. on Γ([9, p. 431]).

Therefore, we have

$$F_{AB}(t) = Af(t) + B\tilde{f}(t) = A\left[f^{+}(t) - f^{-}(t)\right] + B\left[f^{+}(t) + f^{-}(t)\right]$$
$$= (A+B)f^{+}(t) + (B-A)f^{-}(t) \tag{26}$$

a.e. on Γ. By this relation, for the approximation of the function $F_{AB}(t)$, it is sufficient to approximate the functions f^{+} and f^{-} which are analytic inside and outside of the curve Γ, respectively.

From the property of the kernel $K_n(\theta)$, the function f^{+} can be written as

$$f^{+}(t) = \frac{1}{\pi}\int_0^{\pi} K_n(\theta) S_{\Gamma}(t)\, d\theta + \frac{1}{4\pi}\int_0^{\pi} K_n(\theta) f(t)\, d\theta.$$

By Corollary 1, we get

$$\left\|f^{+} - P_n(.,f)\right\|_{X(\Gamma)} \leqslant c_{32}\,\omega(1/n).$$

Then we obtain

$$\|(A+B)f - (A+B)P_n(.,f)\|_{X(\Gamma)} \leqslant c_{33}(A+B)\,\omega(1/n). \tag{27}$$

To complete the proof, we investigate the approximation of the function $f^{-}(z)$. We denote by Γ' the image of Γ under the transformation $z \to \frac{1}{z'}$ and set $f_1(z') = f\left(\frac{1}{z'}\right)$.

We have $f^{-}(z) = f_1^{+}(z)$ where the function f_1^{+} is analytic in $Int\Gamma'$. Then by the relation (24), there exist an algebraic polynomial $\tilde{P}_n(z', f_1)$ such that

$$\left\|f_1^{+}(z') - \tilde{P}_n(z', f_1)\right\|_{X(\Gamma)} \leqslant c_{34}\int_0^{\pi} K_n(\theta)\,\omega_X^{(2)}(\theta, f_1)\, d\theta.$$

By the definition of $\omega_X^{(2)}(\theta, f_1)$, (19) and (20) we get

$$\left\|f_1^{+}(z') - \tilde{P}_n(z', f_1)\right\|_{X(\Gamma)} \leqslant c_{35}\int_0^{\pi} K_n(\theta)\,\omega_X^{(2)}(\theta, f_1)\, d\theta \leqslant c_{37}\,\omega_X^{(2)}(1/n, f_1). \tag{28}$$

Further, since $f_1^{+}(t') = f^{-}(t)$ and the point $z = 0$ is in $Int\Gamma$, we obtain

$$\left\|f_1^{+}(t') - \tilde{P}_n(t', f_1)\right\|_{X(\Gamma')} \asymp \left\|f^{-}(t) - \tilde{P}_n(1/t, f)\right\|_{X(\Gamma)}. \tag{29}$$

Using the same method in Teorem of [15] we can prove that

$$\omega_X^{(2)}(1/n, f_1) \asymp \omega_{2X}^{(2)}(1/n, f). \tag{30}$$

Since (26), (29) and (30) we obtain

$$\left\|f^{-}(t)-\tilde{P}_n(1/t,f)\right\|_{X(\Gamma)} \leqslant c_{38}\,\omega_{2X}^{(2)}(1/n,f). \tag{31}$$

Then, by (31) we can write

$$\left\|(B-A)f^{-}(t)-(B-A)\tilde{P}_n(1/t,f)\right\|_{X(\Gamma)} \leqslant c_{39}(B-A)\,\omega_{2X}^{(2)}(1/n,f) \tag{32}$$

$$\leqslant c_{40}(B-A)\,\omega^{*}(1/n). \tag{33}$$

Hence from relation (13), by virtue of (27) and (32), we get

$$\begin{aligned}
&\left\|F_{AB}(t)-\left[(A+B)P_n(t,f)+(B-A)\tilde{P}_n(1/t,f)\right]\right\|_{X(\Gamma)}\\
&\leqslant \left\|(A+B)f^{+}(t)-(A+B)P_n(t,f)\right\|_{X(\Gamma)}\\
&\quad+\left\|(B-A)f^{-}(t)-(B-A)\tilde{P}_n(1/t,f)\right\|_{X(\Gamma)}\\
&\leqslant c_{41}(A+B)\,\omega(1/n)+c_{15}(B-A)\,\omega^{*}(1/n).
\end{aligned}$$

Setting

$$R_n(t,f)=(A+B)P_n(t,f)+(B-A)\tilde{P}_n(1/t,f),$$

we have

$$\|F_{AB}(t)-R_n(t,f)\|_{X(\Gamma)} \leqslant c_{42}(A+B)\,\omega(1/n)+c_{15}(B-A)\,\omega^{*}(1/n).$$

Now we study the following cases:

1. Let $A=B$; then

$$\|\,F_{AB}(t)-2AP_n(t,f)\|_{X(\Gamma)} \leqslant c_{43}\,[2A]\,\omega(1/n).$$

and in this case we have polynomial approximation of the function $F_{AB}(t)$. In special case, if $A=\frac{1}{2}$, $B=\frac{1}{2}$ we find

$$F_{AB}(t)=\frac{1}{2}f(t)+\frac{1}{2}\tilde{f}(t)=f^{+}(t)$$

and thus we have

$$\left\|f^{+}-P_n(t,f)\right\|_{X(\Gamma)} \leqslant c_{44}\,\omega(1/n).$$

2. Let $A\neq B$; then

$$\|F_{AB}(z)-R_n(z,f)\|_{X(\Gamma)} \leqslant c_{45}(A+B)\,\omega(1/n)+c_{46}(B-A)\,\omega^{*}(1/n).$$

If $A=1$, $B=0$ we get

$$F_{AB}(z)=f(z),\quad R_n(z,f)=P_n(z,f)+\tilde{P}_n(1/z,f)=R_{n_1}(z,f). \tag{34}$$

By (34) we obtain

$$\|f(z)-R_{n_1}(z,f)\|_{X(\Gamma)} \leqslant c_{47}\,\omega(1/n)+c_{48}\,\omega^{*}(1/n).$$

Let $A = -\frac{1}{2}$, $B = \frac{1}{2}$. Thus

$$F_{AB}(t) = -\frac{1}{2} f(t) + \frac{1}{2} \tilde{f}(t) = f^{-}(t) \tag{35}$$

and by (35) we have

$$\left\| f^{+} - R_{n_2}(z, f) \right\|_{X(\Gamma)} \leqslant c_{49} \omega^{*}(1/n).$$

where

$$R_{n_2}(z, f) = \tilde{P}_n(1/z, f).$$

Thus the proof of Theorem 2 is completed. □

REFERENCES

[1] N. K. BARY, *A Treatise on Trigonometric Series*, Volume I, Pergamon Press, 1964.
[2] C. BENNETT AND R. SHARPLEY, *Interpolation of operators*, Academic Press, New York, 1988.
[3] A. BÖTTCHER, YU. I. KARLOVICH, *Carleson curves, Muckenhoupt weights and Toeplitz operators*, Birkhauser, 1997.
[4] A. CAVUS AND D. M. ISRAFILOV, *Approximation by Faber-Laurent rational functions in the mean of functions of class $L_p(\Gamma)$ with $1 < p < \infty$*, Approx. Th. Appl. **11** (1995), 105–118.
[5] R. A. DEVORE AND G. G. LORENTZ, *Constructive Approximation*, Springer, Berlin, 1993.
[6] A. GUVEN AND D. M. ISRAFILOV, *Polynomial approximation in Smirnov-Orlicz classes*, Comput Methods Funct. Theory 2 (2002), 509–517.
[7] A. GUVEN AND D. M. ISRAFILOV, *Rational approximation in Orlicz spaces on Carleson curves*, Bull. Belg. Math. Soc. Simon Stevin 12 (2005), no. 2, 223–234.
[8] A. GUVEN AND D. M. ISRAFILOV, *Approximation in rearrangement invariant spaces on Carleson curves*, East J. Approx. 12 (2006), no. 4, 381–395.
[9] G. M. GOLUZIN, *Geometric Theory of Functions of a Complex Variable*, Translation of Mathematical Monographs, Vol. 26, Providence, RI: AMS, 1968.
[10] D. M. ISRAFILOV, R. AKGUN, *Approximation by polynomials and rational functions in weighted rearrangement invariant spaces*, J. Math. Anal. Appl. 346 (2008), no. 2, 489–500.
[11] S. Z. JAFAROV, *Approximation by polynomials and rational functions in Orlicz spaces*, J. Comput. Anal. Appl. 13 (2011), no. 5, 953–962.
[12] A. YU. KARLOVICH, *Singular integral operators with piecewise continuous coefficients in reflexive rearrangement-invariant spaces*, Integr. Equat. Oper. Theory **32** (1998), 436–481.
[13] V. KOKILASHVILI, *A direct theorem on mean approximation of analytic functions by polynomials*, Soviet Math. Dokl. **10** (1969), 411–414.
[14] DZH. I. MAMEDKHANOV, *Approximation in complex plane and singular operators with a Cauchy kernel*, Dissertation Doct. Phys-math. nauk. The University of Tbilisi, 1984 (in Russian).
[15] J. I. MAMEDKHANOV AND I. B. DADASHOVA, *Rational Approximation on Closed Curves*, Appl. Math. **2** (2012), no. 3, 90–93.
[16] P. K. SUETIN, *Series of Faber Polynomials*, Gordon and Breach Science Publishers, Amsterdam, 1998.

(Received April 4, 2013)

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