Journal of Classical Analysis
Volume 5, Number 1 (2014), 43–60 doi:10.7153/jca-05-04

AN APPROXIMATION PROBLEM WITH A NEW CHARACTERISTIC AND A PROBLEM ON MEAN APPROXIMATION ON ARCS IN A COMPLEX PLANE

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Abstract. In the paper classical approximation theorems are investigated on the curves in a complex domain. In particular, direct and inverse theorems are cited on the curves Γ in a complex domain in the metric $L_p(\Gamma)$. The obtained results remain new on the segment $[-1,1]$ as well. Furthermore, a new characteristic by which the author obtained the analogies of Markov-Bernstein type and also S. M. Nikolskiy type classical estimates is also considered.

In future, it is presupposed to get the analogies of classic theorems of Jackson, Jackson-Bernstein, Bernstein and Nikolskiy-Timan-Dzyadyk by these characteristics.

1. Introduction

Before we reduce a new characteristic of approximation in a complex plane, and reduce the basic theorem proved in the present paper that combines three basic classic theorems on approximation on an arc in a complex plane in an integral metric, we remind basic approximation theorems and related problems.

1. JACKSON'S THEOREM. *If* $f \in Lip_{[-1,1]}1$*, then*

$$E_n(f,[-1,1]) = \inf_{P_n} \|f - P_n\|_{C[-1,1]} \leqslant \frac{const}{n}.$$

2. JACKSON-BERNSTEIN THEOREM. *In order that*

$$f(x) \in Lip_{[0,2\pi]}\alpha \quad (0<\alpha<1) \Leftrightarrow \quad E_n(f;[0,2\pi]) \leqslant \frac{const}{n^\alpha}.$$

3. BERNSTEIN THEOREM. *In order that* $f_0(\theta) \in Lip_{[0,2\pi]}\alpha$ $(0<\alpha<1)$,

$$(f_0(\theta) = f(\cos\theta) = f(x), \quad 0 \leqslant \theta \leqslant 2\pi, \quad -1 \leqslant x \leqslant 1) \Leftrightarrow E_n(f;[-1,1]) \leqslant \frac{const}{n^\alpha}.$$

Mathematics subject classification (2010): 30E10, 41A10, 41A17.
Keywords and phrases: Polynomial approximation, classical approximation theorems, constructive characteristics, quasiconformal curves.

Paper JCA-05-04

4. NIKOLSKIY-TIMAN-DZYADYK THEOREM. *In order that*

$$f(x) \in Lip_{[-1,1]}\alpha \quad (0 < \alpha < 1) \Leftrightarrow \exists P_n \quad \forall x \in [-1,1],$$

$$|f(x) - P_n(x)| \leqslant const \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \right)^{\alpha}.$$

In the case when $\alpha = 1$, in all these theorems the A.Zygmund (Z) class is considered.

The first problem related with these theorems belongs to J. L. Walsh [23] and it is related with Jackson-Bernstein theorem that in a complex domain seems as follows.

5. J. L. WALSH, J. H. CURTISS, W. E. SEWELL THEOREM. $f \in H^{\alpha}(\Gamma) \cap A(G)$, $0 < \alpha < 1$ $(\Gamma = \partial G)$, Γ *is a closed analytic curve* $\Leftrightarrow$

$$\rho_n(f,\Gamma) = \inf_{P_n} \|f - P_n\|_{C(\Gamma)} \leqslant \frac{const}{n^{\alpha}}.$$

THE J. L. WALSH PROBLEM (1959). *What necessary and sufficient conditions should satisfy the closed curve* Γ *for the analogy of Jackson-Bernstein theorem be valid on it.*

The following problem was formulated by D. Newman [17] and it is related with the Jackson theorem.

THE D. NEWMAN PROBLEM (1974). *What necessary and sufficient conditions should satisfy the arc* Γ *in a complex plane for the Jackson theorem be valid on it.*

Furthermore, D. Newman formulated a conjecture and proved some theorems related with this theorem.

Similar to these problems we formulated [9–12] the problems for the Bernstein theorem and Nikolskiy-Timan-Dzyadyk theorem.

Notice that all these problems are actual in the integral metric $L_p(\Gamma)$ as well.

For reducing several results related with these problems we give some definitions and notations:

2. Definitions and notations

Let Ω be an arbitrary simply connected domain of a complex domain, containing the point $z = \infty$; $\bar{B}$ be a continuum being a complement to $\Omega : d_0 \stackrel{df}{=} diamB > 0$; $\Gamma = \partial\Omega = \partial B$ be their common boundary; $w = \varphi(z)$ $(w = \tilde{\varphi}(z))$ be a function that conformarlly and univalently maps Ω onto exterior (interior of $\bar{B}$) of a unit circle γ_0 with normalization:

$$\varphi(\infty) = \infty, \quad \lim_{z \to \infty} \frac{\varphi(z)}{z} > 0; \quad z = \psi(w) = \varphi^{-1}(w);$$

$\Gamma_{1+\sigma} \overset{df}{=} \{t : |\varphi(t)| = 1+\sigma > 1\}$ is a level line of the continuum $\bar{B}$; $d(z,\sigma) \overset{df}{=} \inf_{t\in\Gamma_{1+\sigma}} |z-t|$, for $z \in \Gamma$; $\tilde{d}(t,\sigma) \overset{df}{=} \inf_{z\in\Gamma} |z-t|$, for $t \in \Gamma_{1+\sigma}$.

Assume that Γ is a Jordan rectifiable curve of length ℓ, diameter d $\left(d = \sup_{t,\tau\in\Gamma} |t-\tau|\right)$ given by the equation $t = t(s)$ $(0 < s \leqslant \ell)$ in arc coordinates.

DEFINITION 1. $\Gamma \in C'$ (smooth curve) if $t'(s) \in C([0,\ell])$.

DEFINITION 2. $\Gamma \in \Lambda$, if $t'(s) \in Lip_{[0,1]}\alpha$.

DEFINITION 3. $\Gamma \in D$, if $\forall \alpha > 0$ $\int_0^a \frac{w(t',\theta)}{\theta} d\theta < +\infty$.

DEFINITION 4. $\Gamma \in A$, if $\forall \alpha > 0$ $\int_0^a \frac{w(t',\theta)}{\theta} \ln \theta d\theta < +\infty$.

Denote $\Gamma_\sigma(t) = \{\tau \in \Gamma : |t-\tau| \leqslant \delta\}$ $(0 \leqslant \delta \leqslant d)$, $\theta_t(\delta) = mes\Gamma_\delta(t)$ (Lebesgue measure), $\theta(\delta) = \sup_{t\in\Gamma} \theta_t(\delta)$. Obviously, we have: $\theta(\delta) \geqslant \delta$.

DEFINITION 5. The curve Γ belongs to the class S_θ (curves in S_θ are called Ahlfors-regular), if there exists a constant $C(\Gamma) \geqslant 1$ such that $\theta(\delta) \leqslant C(\Gamma)\delta$.

Notice that to the present time the class of curves S_θ introduced by V. V. Salayev [21] (using of Ahlfors condition [19]) is the most general visible class of curves on which the Plemel'-Privalov theorem is valid.

DEFINITION 6. The curve Γ being an image of a circle for some K-quasiconformal mapping of a plane onto itself is said to be a K-quasiconformal curve ($\Gamma \in A_k$).

DEFINITION 7. They say that the curve Γ belongs to the class of K-Lavrentiev curves [19] if whatever were the points z_1, z_2 on it, the smallest of the arcs $\ell(z_1,z_2)$ that connects them has the same order length that the length of the chord that connects these points, i.e. $\exists k(\Gamma) = k > 0$ such that $\ell(z_1,z_2) \leqslant k \cdot |z_1 - z_2|$ for $\forall z_1, z_2 \in \Gamma$.

DEFINITION 8. The class of curves on which the Riesz theorem is valid, i.e.

$$\forall f \in L_p(\Gamma) \quad (p > 1) \Rightarrow |S_\Gamma f|_{L_p(\Gamma)} \leqslant C(p) \cdot |f|_{L_p(\Gamma)},$$

where $(S_\Gamma f)(t) = S_\Gamma f = \frac{1}{\pi i} \int_\Gamma \frac{f(z)}{z-t} dz$, is said to be a class of the Riesz curves $(\Gamma \in R)$.

Notice that after long laborious investigations it became known that $S_\theta = R$ [6].

Obviously, the imbedding $\wedge \subset A \subset D \subset K \subset A_k$, $\wedge \subset A \subset D \subset K \subset S_\theta \equiv R$ is valid for the Jordan curves. The class of K-quasiconformal curves A_k may contain unrectifiable curves as well.

Investigations related with the J. L. Walsh problem were carried out by W. E. Sewell, J. H. Curtiss, S. Ya. Alper, A. I. Markushevich, S. N. Mergelyan and others [11]. In particular, if by U we denote a class of curves on which the analogy of the Jackson-Bernstein theorem is valid, then it follows from the Sewell-Curtiss paper that $\wedge \subset U$ and from the S. Ya. Alper's paper that $A \subset U$. The class D was studied in the paper of A. I. Markushevich and S. N. Mergelyan, but the imbedding $D \subset U$ was proved by us. Furthermore, we stated the following conjecture:

If a class of closed Jordan curves for which $\varphi(z)$, $\tilde{\varphi}(z) \in H^1(\gamma_0)$ (H^1 is a Hölder class of order 1) is denoted by M, then $M = U$.

For the D. Newman problem:

If we denote by J a class of arcs Γ in a complex domain on which the Jackson theorem is valid, then D. Newman himself showed that $\wedge \subset J$. Furthermore, he formulated the conjecture: $C' = J$? and put a question if there will be the angle $[0,i] \cup [0,1] \in J$?

We and irrespective of us, V. K. Dzyadyk showed that the angle $[0,i] \cup [0,1] \bar{\in} J$ [4, 12].

Furthermore, we showed that D. Newman conjecture is invalid in two directions: i.e. $C' \bar{\subset} J$ and $J \bar{\subset} C'$.

J. M. Anderson, V. K. Dzyadyk, F. D. Lesley and others worked on the D. Newman problem, but the problem was solved by us and V. V. Maimeskul [11–13]. Exactly, we proved:

For $\Gamma \in J \Leftrightarrow \psi \in H^1(\gamma_0)$.

The investigation related with theorem 4 [4] in a complex plane began by V. K. Dzyadyk. During long years he and his followers carried out investigations around the problem related with Theorem 4.

The works of V. I. Belyi [2] is the powerful incitement in this direction.

Very general results on problems related with Theorems 3 and 4 in the metric C were obtained by V. V. Andrievskii [1, 18].

In the present report in the first place we study D. Newman problem and the problem connected with theorem 3 and 4 in the metric $L_p(\Gamma)$. In the second place, instead of the characteristic $\frac{1}{n}$, $d\left(z, \frac{1}{n}\right)$ we introduce the characteristic

$$\delta_n(z) = \left(\int\limits_{\Gamma_{1+\frac{1}{n}}} \frac{|dt|}{|t-z|^2} \right)^{-1}, z \in \Gamma.$$

This characteristic may allow us to study the approximation problems on the most general class of curves, namely, in arbitrary rectifiable Jordan curves both in the metric C and in $L_p(\Gamma)$.

We have already used this characteristic in the papers [14] and [15], where the analogies of classic inequalities of Markov-Bernstein type and S. M. Nikolskiy type were proved.

Notice that we [16] and E. P. Dolzhenko with V. I. Danchenko [7] proved that for

$$\delta_n(z) \asymp d\left(z, \frac{1}{n}\right)$$

it is necessary and sufficient that $\Gamma \in S_\theta$. This fact allows us to obtain the results on arbitrary rectifiable curves in terms of $\delta_n(z)$, and on the classes of curves S_θ to replace this characteristic by $\delta_n\left(z, \frac{1}{n}\right)$, if it is necessary.

Furthermore, as is known, the Nikolskiy-Timan-Dzyadyk theorem in the metric $L_p([-1,1])$ has no its analogy with the characteristic $d(x,1/n)$. In our opinion, the characteristic $\delta_n(x)$ may solve this problem.

Finally, we cite the basic result of the present paper related to D. Newman problem and the with the problem connected with theorem 3 and 4 in the metric $L_p(\Gamma)$.

3. The main result

THEOREM 1. *Let $\Gamma \in S_\theta$ and $f \in L_p(\Gamma)$ $(p>2)$ with smoothness modulus*

$$\omega_p^{(2)}(f,\delta) = \sup_{|h|\leqslant\delta} \| f(z_h) - 2f(z) + f(z_h) \|_{L_p(\Gamma)},$$

where $z_h = \psi(\varphi(z)\, e^{ih})$.

Then, for each natural n there exists a polynomial $P_n(z)$ of degree $\leqslant n$ such that

$$\inf_{P_n} \| f - P_n \|_{L_p(\Gamma)} \leqslant const\, \omega_p^{(2)}\left(f, \frac{1}{n}\right).$$

Proof. Without loss of generality, we'll consider that the points -2 and 2 are the ends of the arc Γ. Then, it is obvious that the function $z=\tau+\tau^{-1}$ maps the arc Γ in the plane z onto the closed curve C in the plane τ, moreover, univalently the exterior of Γ onto the interior of C (or interior of C). And it is easy to be convinced that in the case $\Gamma \in S_\theta$ $(\Gamma \in R)$ and $C \in S_\theta$ $(C \in R)$. Furthermore, it is easy to see that $f \in L_p(\Gamma)$, $f(\tau+\tau^{-1})(1-\tau^{-2})^{1/p} \in L_p(C)$. Hence, it directly follows that

$$f(\tau+\tau^{-1}) = f_*(\tau) \in L_p(C,\vartheta) \quad \left(\vartheta = \left|1-\tau^{-2}\right|^{1/p}\right).$$

Then, by Privalov's lemma [18] it follows that the Cauchy type integral

$$\Phi(t) = \frac{1}{2\pi i} \int_C \frac{f_*(z)}{z-t} dz, \quad t \bar{\in} C,$$

has almost everywhere on C definite angular values equal

$$\Phi^{\pm}(t) = \pm\frac{1}{2} f_*(t) + \frac{1}{2\pi i} \int_C \frac{f_*(z)}{z-t} dz.$$

Hence, we directly have

$$f_*(t) = \Phi^+(t) - \Phi^-(t). \qquad (*)$$

This relation shows that in order to approximate the function $f(t)$ given only on the curve C, in the metric of the space $L_p(C,\vartheta)$ it is enough to approximate in the same space the functions Φ^+ and Φ^- analytic interior and exterior of the given curve C. At first prove the following result.

Let $f_* \in L_p(C,\vartheta)$, then $\vartheta(t) = (1-t^2)^{1/p}$, $p > 2$. Then for each n there exists such a polynomial P_n of degree n that[1]

$$\|\Phi^+ - P_n\|_{L_p(C,\vartheta)} \preceq \omega^{(2)}_{\Gamma,\vartheta}\left(f, \frac{1}{n}\right). \tag{1}$$

In order to prove this statement we cite the following reasoning.

By the fact that $f_* \in L_p(C,\vartheta)$, it directly follows that $F(z_h) \in L_1(C)$ $(F(z_h) = f_*(z_h) + f_*(z_{-h}))$.

Indeed,

$$\begin{aligned}\int_C |F(z_h)||dz| &= \int_C \frac{|f_*(z_h)+f_*(z_{-h})-2f_*(z)||1-z^{-2}|^{1/p}|dz|}{|1-z^{-2}|^{1/p}} \\ &\leqslant \int_C \frac{|f_*(z_h)+f_*(z_{-h})-2f_*(z)||1-z^{-2}|^{1/p}}{|1-z^{-2}|^{1/p}}|dz| \\ &\quad +2\int_C \frac{|f_*(z)||1-z^{-2}|^{1/p}}{|1-z^{-2}|^{1/p}}|dz| \leqslant \|f_*(z_h)+f_*(z_{-h})-2f_*(z)\|_{L_p(C,\vartheta)} \\ &\quad \times \left(\int_C \frac{|dz|}{|1-z^{-2}|^{q/p}}\right)^{\frac{1}{q}} + 2\|f_*\|_{L_p(C,\vartheta)} \cdot \left(\int_C \frac{|dz|}{|1-z^{-2}|^{1/p}}\right)^{\frac{q}{p}} \\ &< +\infty \quad \left(\frac{1}{p}+\frac{1}{q}=1\right).\end{aligned}$$

Belonging of $F(z_h)$ to the space $L_1(C)$ and $C \in S_\theta$ $(C \in R)$ allows to state that the singular integral

$$\int_C \frac{f_*(z_h)+f_*(z_{-h})}{z-t}dz.$$

exists almost everywhere on C in the sense of the principal value. The last statement enables to approximate the function Φ^+ by Jackson-Dzyadyk polynomials [4] that for $t \in C$ are representable in the form

$$P(t) = \frac{1}{4\pi^2 i}\int_0^\pi K_n(h)dh \int_C \frac{f_*(z_h)+f_*(z_{-h})}{z-t}dz + \frac{1}{4\pi}\int_0^\pi K_n(h)[f(t_h)+f(t_{-h})]dh,$$

where $K_n(h)$ is a kernel representing a trigonometric polynomial of at most n power and satisfying the conditions

$$\int_{-\pi}^{\pi} K_n(t)dt = 1 \quad (n = 0,1,\ldots) \tag{a}$$

[1]In the cases when in the inequalities of type $\alpha \leqslant A_\beta$ the absolute constant A has not essential value, we'll write $\alpha \preceq \beta$. The relation $\alpha \asymp \beta$ means $A_1\beta \leqslant \alpha \leqslant A_2\beta$.

$$\int_{-\pi}^{\pi} |K_n(t)|dt \leqslant C_1 \quad (n=0,1,...) \tag{b}$$

$$\int_{-\pi}^{\pi} |t|^k |K_n(t)||dt| \leqslant C_2(n+1)^{-k} \quad (n=0,1,...) \quad (k>0) \tag{c}$$

where C_1 and C_2 are positive constants.

Notice that, in particular, the Jackson kernels [22] satisfy these conditions.

Furthermore, from the fulfilment of conditions (c) we directly get:

$$\int_{-\pi}^{\pi} \left(|t|+\frac{1}{n}\right)^{-k} |K_n(t)|dt \leqslant C_3(k)n^{-k} \quad (n=1,2,...) \tag{d}$$

where $C_3(k)$ is a positive constant dependent only on k.

Taking into account relation (a) we can represent the function Φ^+ in the following form:

$$\Phi^+(t) = \frac{1}{4\pi^2 i}\int_0^{\pi} K_n(h)dh \int_C \frac{2f_*(z)dz}{z-t} + \frac{2}{4\pi}\int_0^{\pi} K_n(h)f(z)dh.$$

Now, using the representations Φ^+ and P_n, we have

$$\begin{aligned}\|\Phi^+ - P_n\|_{L_p(C,|1-t^{-2}|^{1/p})} = \Bigg\| &\frac{1}{4\pi^2 i}\int_0^{\pi} \frac{f_*(z_h)+f_*(z_{-h})-2f(z)}{z-t}dz \\ &+\frac{1}{4\pi}\int_0^{\pi} K_n(h)[f_*(z_h)+f_*(z_{-h})-2f_*(z)]dh\Bigg\|_{L_p(C,|1-t^{-2}|^{1/p})}.\end{aligned}$$

Hence, by the Minkovsky inequality we get:

$$\begin{aligned}\|\Phi^+ - P_n\|_{L_p(C,\vartheta)} \preceq &\left\|\int_0^{\pi} K_n(h)dh \int_C \frac{f_*(z_h)+f_*(z_{-h})-2f_*(z)}{z-t}dz\right\|_{L_p(C,\vartheta)} \\ &+\left\|\int_0^{\pi} K_n(h)[f_*(t_h)+f_*(t_{-h})-2f_*(t)]dh\right\|_{L_p(C,\vartheta)}.\end{aligned}$$

Now, applying Minkovsky's generalized inequality, we'll have

$$\begin{aligned}\|\Phi^+ - P_n\|_{L_p(C,\vartheta)} \preceq &\int_0^{\pi} K_n(h)dh\|\int_C \frac{f_*(z_h)+f_*(z_{-h})-2f_*(z)}{z-t}dz\|_{L_p(C,\vartheta)} \\ &+\int_0^{\pi} K_n(h)dh\|f_*(z_h)+f_*(z_{-h})-2f_*(z)dz\|_{L_p(C,\vartheta)} \\ \leqslant &\int_0^{\pi} K_n(h)A(h)dh + \int_0^{\pi} K_n(h)\omega_{p,\vartheta}^{(2)}(f_*,h)dh,\end{aligned}$$

where

$$A(h) = \left(\int_C \left|\int_C \frac{f_*(z_h)+f_*(z_{-h})-2f_*(z)}{z-t}dz\right|^p |1-t^{-2}||dt|\right)^{1/p}.$$

Further, we'll need B. V. Khvedelidze's following statement ([8, p. 79–81]).

Let $\Gamma \in R$ $(\Gamma \in S_\theta)$. Then, the singular operator $S_\Gamma f$ is bounded in the space $L_p(\Gamma, b)$, where

$$b(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}, \quad t_k \in \Gamma, \quad -\frac{1}{p} < \alpha_k < \frac{1}{q} \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

In other words, for $\forall \varphi \in L_p(\Gamma, b)$ it holds the relation:

$$\left(\int_\Gamma \left| \frac{b(t)}{\pi i} \int_\Gamma \frac{f(z)}{z - t} dz \right|^p |dt| \right)^{1/p} \preceq \left(\int_\Gamma |b(t)\varphi(t)|^p |dt| \right)^{1/p}.$$

Using this statement and taking into account that in the case $p \in (2, \infty)$, $\frac{1}{p} < \frac{1}{q}$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, we'll have:

$$A(h) = \left(\int_C |f_*(z_h) + f_*(z_{-h}) - 2f_*(z)|^p \, |1 - z^{-2}| |dt| \right)^{1/p}.$$

We get:

$$\left\| \Phi^+ - P_n \right\|_{L_p(C,\vartheta)} \leqslant \int_0^\pi K_n(h) \omega_{p,\vartheta}^{(2)}(f_*, h) dh.$$

Further, using the known classic method of approximation theory, we'll have:

$$\left\| \Phi^+ - P_n \right\|_{L_p(C,\vartheta)} \preceq \omega_{p,\vartheta}^{(2)} \left(f_*, \frac{1}{n} \right)_C.$$

Now, approximate the function $\Phi^-(t)$ by means of the polynomial P_n of power n with respect to t^{-1}. And prove that under the conditions of our theorem, it holds the following relation

$$\left\| \Phi^- - \tilde{P}_n \right\|_{L_p(C,\vartheta)} \preceq \tilde{\omega}_{p,\vartheta}^{(2)} \left(f_*, \frac{1}{n} \right), \tag{2}$$

where

$$\tilde{\omega}_{p,\vartheta}^{(2)} \left(f_*, \frac{1}{n} \right) = \sup_{|h| \leqslant \frac{1}{n}} \|f_*(\tilde{z}_h) + f_*(\tilde{z}_{-h}) - 2f_*(z)\|_{L_p(\Gamma,\vartheta)},$$

$\tilde{z}_s = \tilde{\psi}(\tilde{\varphi}(z)e^{is})$, $\tilde{\psi}$, $\tilde{\varphi}$ are the functions mapping the interior of a unit circle onto the interior of the curve C and vice versa.

To prove relation (2) we map the plane t onto the plane $\tilde{t}$ be means of the function $\tilde{t} = \frac{1}{t}$.

Thereby, it is easy to show that the contour $C \in R$ $(C \in S_\theta)$ is mapped into some contour $\tilde{C} \in R$ $(\tilde{C} \in S_\theta)$, and the function $f_*(t) = f_*(\tilde{t}^{-1}) = f(\tilde{t})$. Moreover, $f(\tilde{t}) \in$

$L_p(\tilde{C},\vartheta)$. At the same time, notice that for $t \bar{\in} \bar{C}^{+}$, the function $\Phi(t)$ on the plane $\tilde{t}$ takes the form:

$$\Phi(t) = \frac{1}{2\pi i}\int\limits_{C} \frac{f_*(z)}{z-t}dz = \frac{1}{2\pi i}\int\limits_{\tilde{C}} \frac{f(\tilde{z})}{\tilde{z}-\tilde{t}}d\tilde{z} = \tilde{\Phi}(\tilde{t}).$$

Thereby, it is obvious that in the plane t the function $\Phi^{-}(t)$ will correspond to the function $\Phi^{+}(t)$ in the plane $\tilde{t}$. Whence by (1) we'll have

$$\left\|\tilde{\Phi}(\tilde{t}) - \tilde{P}_n(\tilde{t})\right\|_{L_p(\tilde{C},\vartheta)} \preceq \omega^{(2)}_{p,\vartheta}(\tilde{f},\frac{1}{n}). \tag{3}$$

Further, taking into account $\tilde{\Phi}^{+}(\tilde{t}) = \Phi^{-}(t)$, we find

$$\begin{aligned} \left\|\tilde{\Phi}^{+}(\tilde{t}) - \tilde{P}_n(\tilde{t})\right\|_{L_p(\tilde{C},\vartheta)} &= \left(\int\limits_{C} \left|\tilde{\Phi}^{+}(\tilde{t}) - \tilde{P}_n(\tilde{t})\right|^{p} |1-\tilde{t}^{-2}||d\tilde{t}|\right)^{1/p} \\ &= \left(\int\limits_{C} \left|\Phi^{-1}(t) - \tilde{P}_n(t^{-1})\right|^{p} |1-t^{-2}||dt|\right)^{1/p}. \end{aligned} \tag{4}$$

To complete the proof of relation (2) we must show that

$$\omega^{(2)}_{p,\vartheta}(\tilde{f},\delta) = \tilde{\omega}^{(2)}_{p,\vartheta}(f_*,\delta). \tag{5}$$

For that it suffices to show

$$\begin{aligned} &\|f(\tilde{t}_h) + f(\tilde{t}_{-h}) - 2f(\tilde{t})\|_{L_p(\tilde{C},\vartheta)} \\ &= \left\|f_*(t^*_h) + f_*(t^*_{-h}) - 2f_*(t^*)\right\|_{L_p(C,\vartheta)} \end{aligned} \tag{6}$$

where $\tilde{t}_s = \psi(\varphi(\tilde{t})e^{is})$, $\tilde{t} \in \tilde{C}$, $\tilde{t}^*_s = \psi(\tilde{\varphi}(t)e^{is})$, $t \in C$.

In order to prove this relation, we must find the relation between the functions ψ, φ mapping the exterior of C onto $|\omega| > 1$ and inversely, and between the functions $\tilde{\psi}$, $\tilde{\varphi}$ mapping the interior of C onto $|\omega| < 1$ and inversely.

Obviously, this relation will be determined by the following formulas:

$$\varphi\tilde{(t)} = \tilde{\varphi}^{-1}\tilde{(}t^{-1}), \quad t = \tilde{t}^{-1}$$

$$\psi(u) = \tilde{\psi}^{-1}(u^{-1}).$$

Putting these formula in (6), and taking into account that $\tilde{t}_s^{-1} = t_s$ follows from $\tilde{t}^{-1} = t$, we get validity of relation (6), and validity of relation (5). Hence, the validity of relation (2) will follow by relation (3) and (4).

Now, using relations (1) and (2), we carry out approximation of the functions $f_* \in L_p(C, |1-z^{-2}|^{1/p})$ by means of the rational functions of the form

$$R_n(z) = \sum_{k=-n}^{n} a_k(z-b)^k,$$

where b is a point lying strictly interior to C. Without loss of generality, we'll consider that $b = 0$

So, if for $R_n(z)$ we take

$$R_n(z) = \tilde{P}_n(z) + P_n\left(\frac{1}{z}\right),$$

then by (1) and (2) among these rational functions there exists such a function that

$$\|f_n - R_n\|_{L_p(C,\vartheta)} \preceq \omega_{p,\vartheta}^{(2)}\left(f_*, \frac{1}{n}\right)_C + \tilde{\omega}_{p,\vartheta}^{(2)}\left(f_*, \frac{1}{n}\right)_C. \tag{7}$$

Now, in order to prove the theorem itself, we remind that from the fact $f \in L_p(\Gamma)$ it follows that

$$f(\tau + \tau^{-1})(1 - \tau^{-2})^{-1/p} \in L_p(C),$$

i.e.

$$f(\tau + \tau^{-1}) = f_*(\tau) \in L_p(C) \quad (\vartheta = |1 - \tau^{-2}|^{1/p}).$$

Now by (7) we have:

$$\|f_n - R_n\|_{L_p(C,\vartheta)} \preceq \omega_{p,\vartheta}^{(2)}\left(f_*, \frac{1}{n}\right)_C + \tilde{\omega}_{p,\vartheta}^{(2)}\left(f_*, \frac{1}{n}\right)_C. \tag{7'}$$

Proof the next relations

$$\omega_{p,\vartheta}^{(2)}(f_*, \delta)_C = \omega_p^{(2)}(f, \delta)_\Gamma, \tag{8}$$

$$\tilde{\omega}_{p,\vartheta}^{(2)}(f_*, \delta)_C = \omega_p^{(2)}(f, \delta)_\Gamma. \tag{9}$$

First, we prove relation (8). Obviously, it is enough to show the relation

$$\|f_*(t_h) + f_*(t_{-h}) - 2f_*(t)\|_{L_p(C,\vartheta)} = \|f(z_h) + f(z_{-h}) - 2f(z)\|_{L_p(\Gamma)}, \tag{8'}$$

where $t_s = \psi_1(\varphi_1(t)e^{is})$, $z_s = \psi(\varphi(z)e^{is})$ and the functions ψ_1, φ_1 map the exterior of the unit circle γ_0 onto the exterior of the curve C, and vice versa, the functions ψ, φ map the exterior of a unit circle γ_0 onto the exterior of the arc Γ and vice versa.

In order to get relation $(8')$, we have to get dependence between the functions $\varphi_1(t)$, $\varphi(z)$ and also between $\psi_1(\omega)$ and
$psi(\omega)$.

Let $\tau \in C$. We can map it onto the circle γ_0 by two ways:

a) directly by means of the function $\omega = \varphi_1(\tau)$;

b) as a result of successive mappings: at first onto the plane z by means of the function $z = \tau + \tau^{-1}$, then onto the circle γ_0 of the plane ω by means of the function $\omega = \varphi(z)$.

As a result, we get:

$$\omega = \varphi(z) = \varphi(\tau + \tau^{-1}). \tag{10}$$

Under appropriate normalization of the functions $\varphi_1(z)$ and $\varphi(z)$ from (9) and (10) we'll have

$$\varphi_1(z)=\varphi(\tau+\tau^{-1}) \quad (z=(\tau+\tau^{-1}). \tag{11}$$

In the same way, we can get the following relation

$$\psi(\omega)=\psi_1(\omega)+\psi_1^{-1}(\omega). \tag{12}$$

Notice that it follows from (11) and (12) that if $z=\tau+\tau^{-1}$, then $z_h=\tau_h+\tau_h^{-1}$.

Hence, it directly follows that if

$$f_*(\tau)=f(\tau+\tau^{-1})=f(z), \quad f_*(\tau_h)=f(\tau_h+\tau_h^{-1})=f(z_h),$$

then

$$f_*(\tau_{-h})=f(z_{-h}). \tag{13}$$

Hence

$$\begin{aligned}
&\|f_*(\tau_h)+f_*(\tau_{-h})-2f_*(\tau)\|_{L_p(C,\vartheta)}\\
&=\left(\int_C |f_*(\tau_h)+f_*(\tau_{-h})-2f_*(\tau)|^p\,|1-\tau^{-1}||dz|\right)^{1/p}\\
&=\left(\int_C \left|f(\tau_h+\tau_h^{-1})-f(\tau_{-h}+\tau_{-h}^{-1})-2f(\tau+\tau^{-1})\right|^p\left|d\left(\tau+\frac{1}{\tau}\right)\right|\right)^{1/p}\\
&=\left(\int_\Gamma |f(z_h)+f(z_{-h})-2f(z)|^p\,|dz|\right)^{1/p}=\|f(z_h)+f(z_{-h})-2f(z)\|_{L_p(\Gamma)}.
\end{aligned}$$

So, we proved relation $(8')$ wherefrom relation (8) will follow in the evident form. In the same way, we prove the relation (9).

Now, using relations (8) and (9), from relation $(7')$ we find

$$\|f_*(\tau)-R_n(\tau)\|_{L_p(C,\vartheta)}\preceq\omega_p^{(2)}\left(f,\frac{1}{n}\right)_\Gamma \tag{14}$$

or

$$\left\|f\left(\tau+\frac{1}{\tau}\right)-R_n(\tau)\right\|_{L_p(C,\vartheta)}\preceq\omega_p^{(2)}\left(f,\frac{1}{n}\right)_\Gamma. \tag{15}$$

Hence changing τ in the left hand side of the last inequality by τ^{-1}, we get:

$$\|f(\tau+\tau^{-1})-R_n(\tau^{-1})\|_{L_p(C,\vartheta)}\preceq\omega_p^{(2)}\left(f,\frac{1}{n}\right)_\Gamma. \tag{16}$$

The relations (15) and (16) allow to state that

$$\left\|f(\tau+\tau^{-1})-\frac{R_n(\tau)+R_n(\tau^{-1})}{2}\right\|_{L_p(C,\vartheta)}\preceq\omega_p^{(2)}\left(f,\frac{1}{n}\right)_\Gamma. \tag{17}$$

It is easy to notice that

$$\frac{R_n(\tau)+R_n(\tau^{-1})}{2}$$

is a polynomial of power n with respect to $\tau+\tau^{-1}$. Then we can write (17) in the following form

$$\left\|f(\tau+\tau^{-1})-P_n(\tau+\tau^{-1})\right\|_{L_p(\Gamma,\vartheta)} \preceq \omega_p^{(2)}\left(f,\frac{1}{n}\right)_{\Gamma}$$

or

$$\|f(z)-P_n(z)\|_{L_p(\Gamma,\vartheta)} \preceq \omega_p^{(2)}\left(f,\frac{1}{n}\right)_{\Gamma}. \quad \square$$

From theorem 1 we can derive next Corollary in the case when Γ is a segment $[-1,1]$. Notice that in this classic case also, the result of the theorem remains new.

COROLLARY. *If* $\Gamma = x : -1 \leqslant x \leqslant 1$ *and* $f(x) \in L_p[-1,1]$, $p>2$, *then*

$$E_n^{(p)}(f;[-1,1]) = \inf_{P_n}\left(\int_{-1}^{1}|f(x)-P_n(x)|^p|dx|\right)^{1/p} \leqslant \omega_p^{(2)}\left(f,\frac{1}{n}\right)_{[-1,1]},$$

where

$$\omega_p^{(2)}(f,\delta)_{[-1,1]} = \sup_{|h|\leqslant\delta}\left(\int_0^{2\pi}|f(\cos(\theta+h))+f(\cos(\theta-h))-2f(\cos(\theta))|^p\,|sin\theta|d\theta\right)^{1/p}.$$

It is easy to get this corollary (note that the analogous result is in by [3, p. 83, Corollary 7.2.5]) if we take into account that the function $\psi(\omega)=\frac{1}{2}(\omega+\omega^{-1}$ maps the exterior of a unit circle onto to the exterior of the segment $[-1,1]$.

Now, in order to determine the constructive characteristic of the Hölder class $H^{\alpha}_{\Gamma,p}$ determined by the relation

$$\omega_p(f,\delta) = \sup_{|h|\leqslant\delta}\left(\int_{\Gamma}|f(z_h)-f(z)|^p\,|dz|\right)^{1/p} \leqslant const.\delta^{\alpha}$$

we give the approximation theorem closing the special case of theorem 1.

THEOREM 2. *Let the arc* $\Gamma \in S_\theta$ $(\Gamma \in R)$, $f \in L_p(\Gamma)$ $(p>1)$, *and for each natural* n *there exist the polynomial* $P_n(z)$ *of degree* n *such that*

$$\|f-P_n\|_{L_p(\Gamma)} \leqslant const.n^{-\alpha} \quad (0<\alpha<1). \tag{18}$$

Then $f \in H^{\alpha}_{p,\Gamma}$ $(0<\alpha<1)$, *i.e.*

$$\omega_p(f,\delta) \leqslant const\cdot\delta^{\alpha}. \tag{19}$$

Proof. Let $P_n(z)$ be a polynomial for which relation (18) is fulfilled.

Let's consider a series of polynomials

$$P_1(z)+\sum_{m=1}^{\infty}[P_{2^m}(z)-P_{2^{m-1}}(z)]=\sum_{m=0}^{\infty}U_m(z), \tag{20}$$

where

$$U_0(z)=P_1(z),\quad U_m(z)=P_{2^m}(z)-P_{2^{m-1}}(z).$$

From (18) we have

$$\|U_m\|_{L_p(\Gamma)}\leqslant 2^{-m\alpha}. \tag{21}$$

Hence and from the inequality

$$\|S_n(z)-S_m(z)\|_{L_p(\Gamma)}\preceq\sum_{k=m+1}^{n}\|U_k\|_{L_p(\Gamma)}\preceq 2^{-m\alpha},\quad (m<n),$$

where

$$S_N\preceq\sum_{m=0}^{N}U_m(z),$$

it follows that $\{S_N(z)\}$ is a fundamental sequence in the space $L_p(\Gamma)$. By the completeness of the space $L_p(\Gamma)$ the sequence $\{S_N(z)\}$ converges to the function $f(z)$ in the sense of the metric $L_p(\Gamma)$.

Furthermore, from (21) and the inequality [14]

$$\left\|d\left(z,\frac{1}{n}\right)P_n'(z)\right\|_{L_p(\Gamma)}\leqslant C(p,\Gamma)\|P_n(z)\|_{L_p(\Gamma)}$$

the following inequality is valid:

$$\|d(z,2^{-m})U_m'(z)\|_{L_p(\Gamma)}\leqslant 2^{-m\alpha}. \tag{22}$$

Now prove inequality (19). Obviously, we have

$$\begin{aligned}\|f(z_{\pm h})-f(z)\|_{L_p(\Gamma)}&\preceq\sum_{m=0}^{N_0}\|U_m(z_{\pm h})-U_m(z)\|_{L_p(\Gamma)}\\&\quad+\left\|f(z_{\pm h})-\sum_{m=0}^{N_0}U_m(z_{\pm h})\right\|_{L_p(\Gamma)}+\left\|f(z)-\sum_{m=0}^{N_0}U_m(z)\right\|_{L_p(\Gamma)}\\&=K_1+K_2+K_3.\end{aligned} \tag{23}$$

Moreover, the natural N_0 was chosen from the condition

$$2^{-(N_0-1)}\leqslant h<2^{-N_0}. \tag{24}$$

Now estimate the quantity:

$$K_1=\sum_{m=0}^{N_0}\|U_m(z_{\pm h})-U_m(z)\|_{L_p(\Gamma)}=\sum_{m=0}^{N_0}a_m(h).$$

Note that for $a_m(h)$ we have:

$$a_m(h) = \|U_m(z_{\pm h}) - U_m(z)\|_{L_p(\Gamma)}$$
$$\preceq \left\{ \int\limits_{\Gamma} \left(\int_0^h |U_m'(\psi(\varphi(z)e^{\pm ih})||\psi'(\varphi(z)e^{\pm ih})|dt \right)^p |dz| \right\}^{1/p}.$$

Hence, from the generalized Minkovsky inequality we have

$$a_m(h) \leqslant \int_0^h \left\{ \int\limits_{|\omega|=1} \left|U_m'(\psi(\varphi(z)e^{\pm it})\psi'(\varphi(z)e^{\pm it})\right|^p |dz| \right\}^{1/p} dt$$

or

$$a_m(h) = \int_0^h \left\{ \int\limits_{|\omega|=1} \left|d(\psi(\omega e^{\pm it}), 2^{-m})U_m'(\psi(\omega e^{\pm it})\right|^p \cdot Q_m^p(\omega, \pm h) \times \right.$$
$$\left. \times |\psi'(\omega e^{\pm it})||d\omega| \right\}^{1/p} dt, \tag{25}$$

where

$$Q_m^p(\omega, \pm t) = \frac{|\psi'(\omega e^{\pm it})|^p |\psi'(\omega)|}{\{d(\psi(\omega e^{\pm it}), 2^{-m})\}^p |\psi'(\omega e^{\pm it})|}. \tag{26}$$

Now estimate expression (26). Obviously, it suffices to estimate the expression

$$Q_m^p(\omega, \pm t) = Q_m^p(\omega, t).$$

Let the points z_1 and z_2 be the ends of the arc $\Gamma \in S_\theta$. Then ([4], p. 390) the following relations are valid:

$$d(z, \rho) \asymp \rho(|z - z_j|^{1/2} + \rho) \quad (j = 1, 2) \tag{27}$$

and

$$d(\psi(\omega), \rho) \asymp \rho(|\omega - \omega_i| + \rho) \quad (j = 1, 2), \tag{28}$$

where z_j $(\omega_j = \varphi(z_j))$ is the nearest from the ends of the arcs Γ to the point z for

$$|\psi'(\tau)| \asymp |\tau - \omega_j| \quad (j = 1, 2) \quad \text{and} \quad \tau \in \sigma_j \tag{29}$$

and for

$$|\psi'(\tau)| \asymp 1 \quad \text{and} \quad \tau \notin \sigma_j, \tag{30}$$

where σ_j is some fixed vicinity of the point $\omega_j = \varphi(z_j)$ lying on $|\omega| = 1$.

For estimating the expression (26), we consider two cases:

1. $\omega e^{it} \notin \sigma_j$. Obviously, in this case, if $\omega \notin \sigma_j$, then from (30)

$$|\psi'(\omega e^{it})| \asymp |\psi'(\omega)|,$$

whence

$$Q_m^p(\omega,t) \asymp 2^{mp}(|\omega e^{it}-\omega_j|+2^{-m})^{-p} \preceq 2^{mp}(\ \sigma_j)^{-p} \preceq 2^{mp}.$$

And if $\omega \in \sigma_j$, then from (29) and (30)

$$Q_m^p(\omega,t) \asymp 2^{mp}|\omega-\omega_j|(|\omega e^{it}-\omega_j|+2^{-m})^{-p} \preceq 2^{mp}(\ \sigma_j)^{-p} \preceq 2^{mp}.$$

So, if $\omega e^{it} \notin \sigma_j$, then

$$Q_m^p(\omega,t) \preceq 2^{mp}. \tag{31}$$

2. $\omega e^{it} \in \sigma_j$ $(j=1,2)$. In this case, obviously, if $|\omega-\omega_j| \geqslant 2h$, then based on $|\omega-\omega_j| \asymp |\omega e^{it}-\omega_j|$ and from relation (29) we get

$$Q_m^p(\omega,t) \asymp 2^{mp}|\omega e^{it}-\omega_j|(|\omega e^{it}-\omega_j|+2^{-m})^{-p} \preceq 2^{mp}.$$

And if $|\omega-\omega_j|<2h$, then in the case $|\omega-\omega_j| \leqslant |\omega e^{it}-\omega_j|$, we have:

$$Q_m^p(\omega,t) \asymp 2^{mp}|\omega e^{it}-\omega_j|^{p-1}|\omega-\omega_j|(|\omega e^{it}-\omega_j|+2^{-m})^{-p} \preceq 2^{mp},$$

and in the case $|\omega-\omega_j| > |\omega e^{it}-\omega_j|$, the following inequalities are valid:

$$Q_m^p(\omega,t) \asymp 2^{mp}|\omega e^{it}-\omega_j|^{p-1}|\omega-\omega_j|(|\omega e^{it}-\omega_j|+2^{-m})^{-p}$$

$$\preceq 2^{mp}|\omega-\omega_j|(|\omega e^{it}-\omega_j|+2^{-m})^{-p} \preceq 2^{mp}(2h)^p(|\omega e^{it}-\omega_j|+h)^{-p} \preceq 2^{mp}),$$

since from (24), $h<2^{-m}$ $(m=\overline{0,N})$.

So in the case when $\omega e^{it} \in \sigma_j$, we have

$$Q_m^p(\omega,t) \preceq 2^{mp}. \tag{32}$$

From (31) and (32) we get

$$Q_m^p(\omega,t) \preceq 2^{mp}), \quad \forall \omega : |\omega|=1 \tag{33}$$

Hence and from relation (25), we find

$$a_m(h) \preceq 2^m h \|d(\xi,2^{-m})U_m'(\xi)\|_{L_p(\Gamma)}. \tag{34}$$

We apply to the right hand side the estimation (22) and get:

$$a_m(h) \preceq 2^{m(1-\alpha)}h.$$

Hence, from (24), we find

$$K_1 \preceq h\sum_{m=0}^{N_0} 2^{m(1-\alpha)} \preceq h^{\alpha}. \tag{35}$$

In what follows, for the expression of K_2 we have:

$$K_2 = \left\{ \int_\Gamma |f(z_{\pm h}) - \sum_{m=0}^{N_0} U_m(z_{\pm h})|^p |dz| \right\}^{1/p}$$

$$\leqslant \sum_{m=N_0+1}^{\infty} \left\{ \int_\Gamma |U_m(z_{\pm h})|^p |dz| \right\}^{1/p}. \tag{36}$$

Let γ_j $(j=1,2)$ be a part of the curve Γ occurring inside the circle of radius $Md(z_j, 2^{-m})$ centered at the point z_j $(j=1,2)$, $Z = \bigcup_{j=1}^{2} \gamma_j$. From (36) we have

$$K_2 \preceq \sum_{m=N_0+1}^{\infty} \left\{ \int_{\Gamma \setminus Z} |U_m(z_{\pm h})|^p |dz| + \int_Z |U_m(z_{\pm h})|^p |dz| \right\}^{1/p}$$

$$= \sum_{m=N_0+1}^{\infty} (B_1(\pm h) + B_1(\pm h)). \tag{37}$$

Obviously, it suffices to consider the case $+h$. Making the substitution $z_h = \xi$ in the expression $B_1(h)$, we get

$$B_1(h) = \int_{\Gamma \setminus Z} |U_m(z_h)|^p |dz| = \int_{\Gamma \setminus Z'} |U_m(\xi)|^p \left| \frac{\varphi'(\xi)}{\varphi'(\xi_h)} \right| |dz| \tag{38}$$

where $Z' = \bigcup_{j=1}^{2} \gamma_j'$, γ_j' $(j=1,2)$ is the image of the arcs γ_j at the mapping $z_h = \xi$. It is known that ([5], p. 513) from conditions (27)–(30) we have

$$\left| \frac{\varphi'(\xi)}{\varphi'(\xi_{\pm h})} \right| \preceq 1 \quad \left(\xi = \psi(\omega) : \left| \frac{\varphi'(\xi)}{\varphi'(\xi_{\pm h})} \right| = \left| \frac{\psi'(\omega e^{\pm h})}{\psi'(\omega)} \right| \preceq 1 \right).$$

Hence and from (38) we get

$$B_1(h) \preceq \int_{\Gamma \setminus Z'} |U_m(\xi)|^p |dz|. \tag{39}$$

For estimating $B_2(h)$, using the similar results, we get

$$B_2(h) \preceq \int_\Gamma |U_m(z)|^p |dz|. \tag{40}$$

From estimations (39) and (40) we immediately get

$$K_2 \preceq \sum_{m=N_0}^{\infty} \|U_m\|_{L_p(\Gamma)}. \tag{41}$$

Whence, from (21)

$$K_2 \preceq 2^{-N_0\alpha} \leqslant h^{\alpha}.$$

We also note that by (18) it holds the following estimation

$$K_3 \preceq 2^{-N_0\alpha} \leqslant h^{\alpha}. \tag{42}$$

From estimations (35), (41) and (42) we get the validity of the statement of theorem 2.

Thus, the necessary constructive characteristic follows from theorems 1 and 2.

THEOREM 3. *In order the function $f(z)$ belonging to the class $L_p(\Gamma)$ $(p>2)$, where $\Gamma \in S_\theta$ have the best approximation*

$$\rho_n^{(\rho)}(f,\Gamma) = \inf_{P_n} \|f - P_n\|_{L_p(\Gamma)}$$

satisfying the inequality

$$\rho_n^{(\rho)}(f,\Gamma) \leqslant const.n^{-\alpha} \quad (0<\alpha<1)$$

it is sufficient $f \in H^{\alpha}_{\Gamma,p}$ $(0<\alpha<1)$.

Notice that this result remains new also in the case when Γ is a segment $[-1,1]$. In some other terms, in the segment $[-1,1]$ this problem was considered in [3, 20].

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(Received June 16, 2014)

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