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GROWTH OF THE MAXIMUM MODULUS OF POLYNOMIALS WITH PRESCRIBED ZEROS

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Abstract. If $p(z)=\sum_{j=0}^{n} a_j^j$ is a polynomial of degree n satisfying $p(z)\neq 0$ in $|z|<1$, then for $R\geqslant 1$. Ankeny and Rivlin [1] proved that $M(p,R)\leqslant\left(\frac{R^n+1}{2}\right)M(p,1)$. In this paper we obtain some results in this direction by considering polynomials of degree $n\geqslant 2$, having all its zeros on $|z|=k$, $k\leqslant 1$ which is an improvement of the result recently proved by M. S. Pukhta (2013) [*Progress in Applied Mathematics*, **6** (2), 50–58].

1. Introduction and Statement of Result

For an arbitrary entire function $p(z)$, let $M(f,r)=\max_{|z|=r}|f(z)|$. Then for a polynomial $p(z)$ of degree n, it is a simple consequence of maximum modulus principle (for reference see [4, Vol. I, p. 137, Problem III, 269]) that

$$M(p,R)\leqslant R^n M(p,1),\quad \text{for } R\geqslant 1. \tag{1.1}$$

The result is best possible and equality holds for $p(z)=\lambda z^n$, where $|\lambda|=1$, $R\geqslant 1$.

If we restrict ourselves to the class of polynomials having no zeros in $|z|<1$, then inequality (1.1) can be sharpened. In fact it was shown by Ankeny and Rivlin [1] that if $p(z)\neq 0$ in $|z|<1$, then (1.1) can be replaced by

$$M(p,R)\leqslant\left(\frac{R^n+1}{2}\right)M(p,1),\quad R\geqslant 1 \tag{1.2}$$

The result is sharp and equality holds for $p(z)=\alpha+\beta z^n$, where $|\alpha|=|\beta|$.

While trying to obtain inequality analogous to (1.2) for polynomials not vanishing in $|z|<k$, $k\leqslant 1$, K.K. Dewan and Arty Ahuja [2] proved the following result.

THEOREM A. *If $p(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros on $|z|=k$, $k\leqslant 1$, then for every positive integer s*

$$\{M(p,R)\}^s\leqslant\left(\frac{k^{n-1}(1+k)+(R^{ns}-1)}{k^{n-1}+k^n}\right)\{M(p,1)\}^s,\quad R\geqslant 1 \tag{1.3}$$

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THEOREM B. *If* $p(z)=\sum_{j=0}^{n}a_jz^j$ *is a polynomial of degree* n *having all its zeros on* $|z|=k,\ k\leqslant 1$, *then for every positive integer* s

$$\{M(p,R)\}^s\leqslant\frac{1}{k^n}\left[\frac{n|a_n|\{k^n(1+k^2)+k^2(R^{ns}-1)+|a_{n-1}|\{2k^n+R^{ns}-1\}}{2|a_{n-1}|+n|a_n|(1+k^2)}\right]\times\{M(p,1)\}^s,\quad R\geqslant 1. \tag{1.4}$$

In this paper, we not only improve Theorem A and Theorem B but also improve the results recently proved by M.S. Pukhta [7]. More precisely, we prove

THEOREM 1. *If* $p(z)=c_nz^n+\sum_{\nu=\mu}^{n}c_{n-\nu}z^{n-\nu},\ 1\leqslant\mu<n$ *is a polynomial of degree* n *having all its zeros on* $|z|=k,\ k\leqslant 1$, *then for every positive integer* s *and* $R\geqslant 1$,

$$\{M(p,R)\}^s\leqslant\frac{k^{n-2\mu+1}(1+k^\mu)+(R^{ns}-1)}{k^{n-2\mu+1}+k^{n-\mu+1}}\{M(p,1)\}^s-s|a_1|\left(\frac{R^{ns}-1}{ns}-\frac{R^{ns-2}-1}{ns-2}\right)\{M(p,1)\}^{s-1},\ \text{if } n>2 \tag{1.5}$$

and

$$\{M(p,R)\}^s\leqslant\frac{k^{n-2\mu+1}(1+k^\mu)+(R^{ns}-1)}{k^{n-2\mu+1}+k^{n-\mu+1}}\{M(p,1)\}^s-s|a_1|\left(\frac{R^{ns}-1}{ns}-\frac{R^{ns-1}-1}{ns-1}\right)\{M(p,1)\}^{s-1},\ \text{if } n=2 \tag{1.6}$$

If we choose $\mu=1$ *in Theorem 1, we get the following result recently proved by M. S. Pukhta* [7].

COROLLARY 1. *If* $p(z)=\sum_{j=0}^{n}a_jz^j$ *is a polynomial of degree* $n\geqslant 2$ *having all its zeros on* $|z|=k,\ k\leqslant 1$, *then for* $R\geqslant 1$,

$$\{M(p,R)\}^s\leqslant\frac{k^{n-1}(1+k)+(R^{ns}-1)}{k^{n-1}+k^n}\{M(p,1)\}^s-s|a_1|\left(\frac{R^{ns}-1}{ns}-\frac{R^{ns-2}-1}{ns-2}\right)\{M(p,1)\}^{s-1},\ \text{if } n>2 \tag{1.7}$$

and

$$\{M(p,R)\}^s\leqslant\frac{k^{n-1}(1+k)+(R^{ns}-1)}{k^{n-1}+k^n}\{M(p,1)\}^s-s|a_1|\left(\frac{R^{ns}-1}{ns}-\frac{R^{ns-1}-1}{ns-1}\right)\{M(p,1)\}^{s-1},\ \text{if } n=2 \tag{1.8}$$

Next we prove the following result which is a refinement of Theorem 1.

THEOREM 2. *If* $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu},\ 1 \leqslant \mu < n$ *is a polynomial of degree* n *having all its zeros on* $|z| = k,\ k \leqslant 1$, *then for every positive integer* s *and* $R \geqslant 1$,

$$\{M(p,R)\}^s \leqslant \frac{1}{k^{n-\mu+1}} \left[\frac{\begin{pmatrix} n|c_n|\{k^{n-\mu+1}(k^{\mu-1} + k^{2\mu}) + k^{2\mu}(R^{ns} - 1)\} \\ +|c_{n-\mu}|\{\mu(k^n + k^{n-\mu+1} + k^{\mu-1}(R^{ns} - 1))\} \end{pmatrix}}{\mu|c_{n-\mu}|(k^{\mu-1} + 1) + n|c_n|(k^{\mu-1} + k^{2\mu})} \right] \times \{M(p,1)\}^s - s|a_1| \left(\frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns - 2} \right) \{M(p,1)\}^{s-1}, \quad \text{if } n > 2 \tag{1.9}$$

and

$$\{M(p,R)\}^s \leqslant \frac{1}{k^{n-\mu+1}} \left[\frac{\begin{pmatrix} n|c_n|\{k^{n-\mu+1}(k^{\mu-1} + k^{2\mu}) + k^{2\mu}(R^{ns} - 1)\} \\ +|c_{n-\mu}|\{\mu(k^n + k^{n-\mu+1} + k^{\mu-1}(R^{ns} - 1))\} \end{pmatrix}}{\mu|c_{n-\mu}|(k^{\mu-1} + 1) + n|c_n|(k^{\mu-1} + k^{2\mu})} \right] \times \{M(p,1)\}^s - s|a_1| \left(\frac{R^{ns} - 1}{ns} - \frac{R^{ns-1} - 1}{ns - 1} \right) \{M(p,1)\}^{s-1}, \quad \text{if } n = 2 \tag{1.10}$$

If we choose $\mu = 1$ in Theorem 2, we get the following result.

COROLLARY 2. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* $n \geqslant 2$ *having all its zeros on* $|z| = k,\ k \leqslant 1$, *then for every* $R \geqslant 1$

$$\{M(p,R)\}^s \leqslant \frac{1}{k^n} \left[\frac{\begin{pmatrix} n|c_n|\{k^n(1 + k^2) + k^2(R^{ns} - 1)\} \\ +|c_{n-1}|\{(1 + k^n + (R^{ns} - 1))\} \end{pmatrix}}{2|c_{n-1}| + n|c_n|(1 + k^2)} \right] \{M(p,1)\}^s - s|a_1| \left(\frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns - 2} \right) \{M(p,1)\}^{s-1}, \quad \text{if } n > 2 \tag{1.11}$$

and

$$\{M(p,R)\}^s \leqslant \frac{1}{k^n} \left[\frac{\begin{pmatrix} n|c_n|\{k^n(1 + k^2) + k^2(R^{ns} - 1)\} \\ +|c_{n-1}|\{(k + k^n + k(R^{ns} - 1))\} \end{pmatrix}}{2|c_{n-\mu}| + n|c_n|(1 + k^2)} \right] \{M(p,1)\}^s - s|a_1| \left(\frac{R^{ns} - 1}{ns} - \frac{R^{ns-1} - 1}{ns - 1} \right) \{M(p,1)\}^{s-1}, \quad \text{if } n = 2 \tag{1.12}$$

2. Lemmas

For the proof of these theorems, we need the following lemmas.

LEMMA 1. *If* $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leqslant \mu < n$ *is a polynomial of degree* n *having all its zeros on* $|z| = k$, $k \leqslant 1$, *then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{k^{n-2\mu-1} + k^{n-\mu+1}} \max_{|z|=1} |p(z)|. \tag{2.1}$$

The above lemma is due to Govil [3].

LEMMA 2. *If* $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leqslant \mu < n$ *is a polynomial of degree* n *having all its zeros on* $|z| = k$, $k \leqslant 1$, *then*

$$\max |p'(z)| \leqslant \frac{n}{k^{n-\mu+1}} \left[\frac{n|c_n|k^{2\mu} + |c_{n-\mu}|k^{\mu-1}}{n|c_n|(k^{2\mu} + k^{\mu-1}) + \mu|c_{n-\mu}|(k^{\mu-1}+1)} \right] \max_{|z|=1} |p(z)| \tag{2.2}$$

The above lemma is due to Dewan and Mir [5].

LEMMA 3. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree, then for all* $R \geqslant 1$

$$\max_{|z|=R} |p(z)| \leqslant R^n M(p,1) - (R^n - R^{n-2})|p(0)|, \text{ if } n > 1 \tag{2.3}$$

and

$$\max_{|z|=R} |p(z)| \leqslant RM(p,1) - (R-1)|p(0)|, \text{ if } n = 1 \tag{2.4}$$

The above lemma is due to Frappier, Rahman and Ruscheweyh [6].

3. Proof of the theorems

Proof of Theorem 1. We first consider the case when polynomial $p(z)$ is of degree $n > 2$. Since $p(z)$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leqslant 1$, therefore, by Lemma 1, we have

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} M(p,1) \tag{3.1}$$

Now applying inequality (1.1) to the polynomial $p'(z)$ which is of degree $n-1$ and noting (3.1), it follows that for all $r \geqslant 1$ and $0 \leqslant \theta < 2\pi$

$$|p'(re^{i\theta})| \leqslant \frac{nr^{n-1}}{k^{n-2\mu+1} + k^{n-\mu+1}} M(p,1) \tag{3.2}$$

Also for each θ, $0 \leqslant \theta < \pi$ and $R \geqslant 1$, we obtain

$$\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s = \int_1^R \frac{d}{dt}\{p(te^{i\theta})\}^s dt$$
$$= \int_1^R s\{p(te^{i\theta})\}^{s-1} p'(te^{i\theta})e^{i\theta} dt.$$

This implies

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leqslant s\int_1^R |p(te^{i\theta})|^{s-1}|p'(te^{i\theta})|dt \tag{3.3}$$

Since $p(z)$ is a polynomial of of degree $n > 2$, the polynomial $p'(z)$ which is of degree $n-1 \geqslant 2$, hence applying inequality (2.3) of Lemma 3 to $p'(z)$, we have for $r \geqslant 1$ and $0 \leqslant \theta < 2\pi$

$$|p'(re^{i\theta})| \leqslant r^{n-1}M(p',1) - (r^{n-1} - r^{n-3})|p'(0)| \tag{3.4}$$

Inequality (3.4) in conjunction with inequalities (3.3) and (1.1), yields for $n > 2$ and for $R \geqslant 1$

$$\begin{aligned}
&|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \\
&\leqslant s\int_1^R (t^n M(p,1))^{s-1}[t^{n-1}M(p',1) - (t^{n-1} - t^{n-3})|p'(0)|]dt \\
&= s\int_1^R t^{ns-1}\{M(p,1)\}^{s-1}M(p',1) - (t^{ns-1} - t^{ns-3})\{M(p,1)\}^{s-1}|p'(0)|]dt \\
&= s\Bigg[\frac{R^{ns}-1}{ns}\{M(p,1)\}^{s-1}M(p',1) \\
&\qquad - \left(\frac{R^{ns}-1}{ns} - \frac{R^{ns-2}}{ns-2}\right)\{M(p,1)\}^{s-1}|p'(0)|\Bigg]
\end{aligned} \tag{3.5}$$

On applying Lemma 1 to inequality (3.5), we get for $n > 2$,

$$\begin{aligned}
|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| &\leqslant \frac{R^{ns}-1}{k^{n-2\mu+1} + k^{n-\mu+1}}\{M(p,1)\}^s \\
&\quad - s\left(\frac{R^{ns}-1}{ns} - \frac{R^{ns-2}-1}{ns-2}\right)\{M(p,1)\}^{s-1}|p'(0)|
\end{aligned}$$

This gives

$$\begin{aligned}
\{M(p,R)\}^s &\leqslant \frac{k^{n-2\mu+1}(1+k^{\mu}) + (R^{ns}-1)}{k^{n-2\mu+1} + k^{n-\mu+1}}\{M(p,1)\}^s \\
&\quad - s|a_1|\left(\frac{R^{ns}-1}{ns} - \frac{R^{ns-2}-1}{ns-2}\right)\{M(p,1)\}^{s-1}
\end{aligned}$$

This completes the proof of inequality (1.5). □

The proof of inequality (1.6) follows on the same lines as that of inequality (1.5), but instead of using inequality (2.3) of Lemma 3 we use inequality (2.4) of Lemma 3.

Proof of Theorem 2. The proof of Theorem 2 follows on the same lines as that of Theorem 1. But for the sake of completeness we give a brief outline of the proof. We first consider the case when polynomial $p(z)$ is of degree $n>2$, then the polynomial $p'(z)$ is of degree $(n-1)\geqslant 2$, hence applying inequality (2.3) of Lemma 3 to $p'(z)$, we have for $r\geqslant 1$ and $0\leqslant\theta<2\pi$

$$|p'(re^{i\theta})|\leqslant r^{n-1}M(p',1)-(r^{n-1}-r^{n-3})|p'(0)| \tag{3.6}$$

Also for each θ, $0\leqslant\theta<2\pi$ and $R\geqslant 1$, we obtain

$$\begin{aligned}\{p(Re^{i\theta})\}^s-\{p(e^{i\theta})\}^s&=\int_1^R \frac{d}{dt}\{p(te^{i\theta})\}^s dt\\&=\int_1^R s\{p(te^{i\theta})\}^{s-1}p'(te^{i\theta})e^{i\theta}dt.\end{aligned}$$

This implies

$$|\{p(Re^{i\theta})\}^s-\{p(e^{i\theta})\}^s|\leqslant s\int_1^R |p(te^{i\theta})|^{s-1}|p'(te^{i\theta})|dt \tag{3.7}$$

Inequality (3.6) in conjunction with inequalities (3.6) and (1.1), yields for $n>2$,

$$\begin{aligned}&|\{p(Re^{i\theta})\}^s-\{p(e^{i\theta})\}^s|\\&\leqslant s\int_1^R (t^nM(p,1))^{s-1}[t^{n-1}M(p',1)-(t^{n-1}-t^{n-3})|p'(0)|]dt\\&=s\int_1^R t^{ns-1}\{M(p,1)\}^{s-1}M(p',1)-(t^{ns-1}-t^{ns-3})\{M(p,1)\}^{s-1}|p'(0)|]dt\\&=s\Bigg[\frac{R^{ns}-1}{ns}\{M(p,1)\}^{s-1}M(p',1)\\&\qquad-\left(\frac{R^{ns}-1}{ns}-\frac{R^{ns-2}-1}{ns-2}\right)\{M(p,1)\}^{s-1}|p'(0)|\Bigg]\end{aligned}$$

Which on combining with Lemma 2, yields for $n>2$

$$\begin{aligned}&|\{p(Re^{i\theta})\}^s-\{p(e^{i\theta})\}^s|\\&\leqslant\frac{R^{ns}-1}{k^{n-\mu+1}}\left[\frac{n|c_n|k^{2\mu}+|c_{n-\mu}|k^{\mu-1}}{n|c_n|(k^{2\mu}+k^{\mu-1})+\mu|c_{n-\mu}|(k^{\mu-1}+1)}\right]\{M(p,1)\}^s\\&\qquad-s\left(\frac{R^{ns}-1}{ns}-\frac{R^{ns-2}-1}{ns-2}\right)\{M(p,1)\}^{s-1}|p'(0)|\end{aligned}$$

from which we get the desired result. □

The proof of inequality (1.10) follows on the same lines as that of inequality (1.9), but instead of using inequality (2.3) of Lemma 3 we use inequality (2.4) of Lemma 3.

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