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SOLUTIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS THAT EXTEND

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Abstract. This note discusses the question: *When do nonlinear fractional differential equations of arbitrary order have solutions that extend to a maximal interval of existence?* We show that a growth condition on the right-hand side of the equation ensures that solutions will extend. The method uses a classical approach from analysis, namely the divergence of an infinite series. It is interesting to note that the growth condition is related to the order of the fractional differential equation involved. A YouTube video from the author that is designed to complement this research is available at `http://tinyurl.com/Extend-FDE`.

1. Introduction

This article discusses the question:
When do nonlinear fractional differential equations of arbitrary order have solutions that extend to a maximal interval of existence?

In particular, the discussion is centered around the following initial value problem (IVP) of arbitrary order $q > 0$

$$D^q\left(x - T_{\lceil q\rceil -1}[x]\right) = f(t,x); \tag{1.1}$$

$$x(0) = A_0,\ x'(0) = A_1, \ldots,\ x^{(\lceil q\rceil -1)}(0) = A_{\lceil q\rceil -1}; \tag{1.2}$$

where: $\lceil q\rceil$ is the integer such that $q-1 < \lceil q\rceil \leqslant q$; D^q represents the Riemann–Liouville fractional differentiation operator of arbitrary order $q > 0$ (a precise definition is found in (2.1) a little later); $f : I \times D \subset \mathbb{R}^2 \to \mathbb{R}$; $T_{\lceil q\rceil -1}[x]$ is the Maclaurin polynomial of order $\lceil q\rceil - 1$ of $x = x(t)$; $a > 0$ and the A_i are constants.

The left-hand side of (1.1) is known as the Caputo derivative of x of order $q > 0$, with the notation ${}^C D^q(x) := D^q\left(x - T_{\lceil q\rceil -1}[x]\right)$ also used in the literature. Observe that the classical derivatives of the function x (from order zero to order $\lceil q\rceil - 1$ each at $t = 0$) are present in (1.1) and (1.2). This form was suggested by Caputo [1] responding to a need for improved accuracy in modelling the initial conditions of phenomena.

Works such as [2, 6, 7, 8, 10, 11, 12, 13, 14, 15] and the monographs [4, 5, 9] have analyzed qualitative and quantitative aspects of solutions to (1.1), (1.2). The methods employed in the above literature include: the sequential technique of successive approximations; and the classical fixed-point approaches of Banach and Schauder.

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In contrast to the aforementioned works the approach herein involves determining when solutions to (1.1), (1.2) will extend to their maximal interval of existence. The results in this note advance our cornerstone knowledge of solutions to nonlinear IVPs involving fractional differential equations and have been motivated by the case $q=1$ in [16].

2. Preliminaries

To understand the notation used throughout this short section contains some preliminary definitions and associated notation.

A solution to the IVP (1.1), (1.2) on an interval I is defined to be a q-th (fractionally)-differentiable function $x : I \subseteq \mathbb{R} \to \mathbb{R}$ such that the points $(t, x(t))$ lie in $I \times D$ for all $t \in I \subseteq \mathbb{R}$ and $x(t)$ satisfies: (1.1) for all $t \in I$; and (1.2).

Define the Riemann–Liouville fractional derivative and integral of order $q > 0$ of a function y, respectively, by:

$$D^q y(t) := \frac{d^{\lceil q \rceil}}{dt^{\lceil q \rceil}} \frac{1}{\Gamma(\lceil q \rceil - q)} \int_0^t (t-s)^{\lceil q \rceil - 1 - q} y(s)\, ds; \tag{2.1}$$

$$I^q y(t) := \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\, ds;$$

with the Caputo derivative defined via

$$^C D^q y(t) := D^q \left(y - T_{\lceil q \rceil - 1}[y] \right)(t).$$

Consider the rectangle

$$R := \{(t,u) \in [0,a] \times \mathbb{R} : |u - A_0| \leqslant b\}, \quad a > 0,\ b > 0$$

and let $M > 0$ be a constant such that

$$|f(t,u)| \leqslant M, \quad \text{for all } (t,u) \in R.$$

We shall need the following local existence result from [2, Theorem 2.1] which is proved using Schauder's fixed-point theorem or using Weierstass' polynomial approximation theorem [15].

THEOREM 2.1. *If $f : R \to \mathbb{R}$ is continuous then the IVP* (1.1), (1.2) *has at least one solution x on $[0, \alpha]$ such that $(t, x(t)) \in R$ for all $t \in [0, \alpha]$ with*

$$\alpha := \min \left\{ a, \left[\frac{b\Gamma(q+1)}{M} \right]^{1/q} \right\}.$$

3. Main Results

In this section the main results are presented. Under certain growth conditions, it is shown that solutions to fractional IVPs have solutions that extend to intervals of the type $[0,a]$ or $[0,\infty)$.

THEOREM 3.1. *Let $f:[0,a]\times\mathbb{R}\to\mathbb{R}$ be continuous. If there is a continuous, nondecreasing function $\varphi:[0,\infty)\to(0,\infty)$ such that*

$$|f(t,u)|\leqslant[\varphi(|u|)]^q,\ \textit{for all } (t,u)\in[0,a]\times\mathbb{R};$$
$$\int_d^\infty\frac{ds}{\varphi(s)}=\infty,\ \textit{for some constant } d\geqslant 0;$$

then the fractional IVP (1.1), (1.2) has at least one solution x that extends to the interval $[0,a]$.

Proof. Consider the general fractional IVP

$$D^q\left(x-\mathscr{T}_{\lceil q\rceil-1}[x]\right)=f(t,x); \tag{3.1}$$
$$x(t_0)=x_0,\ x'(t_0)=x_1,\ldots,\ x^{(\lceil q\rceil-1)}(t_0)=x_{\lceil q\rceil-1}; \tag{3.2}$$

where $\mathscr{T}_{\lceil q\rceil-1}[x]$ is the Taylor polynomial of order $\lceil q\rceil-1$ of $x=x(t)$ at $t=t_0$. (The definition of D^q in (2.1) will then have the 0 in the integral sign replaced with t_0.) Also consider the general rectangle

$$\mathscr{R}:=\{(t,u)\in[t_0,a]\times\mathbb{R}:|u-x_0|\leqslant b\},\quad b>0.$$

Let $\mathscr{M}>0$ be a constant such that

$$|f(t,u)|\leqslant\mathscr{M}\quad\text{for all } (t,u)\in\mathscr{R}.$$

Now, since f is continuous on $\mathscr{R}$, by Theorem 2.1 we know that the problem (3.1), (3.2) has at least one solution x on $[t_0,t_0+\alpha]$, where

$$\alpha:=\min\left\{a,\left[\frac{b\Gamma(q+1)}{\mathscr{M}}\right]^{1/q}\right\}.$$

Now let's reconsider (3.1), (3.2) by associating: t_0 with 0; x_0 with A_0; each x_i with A_i; and let's choose $b=1$. We can thus form the rectangle

$$R_0:=\{(t,u):t\in[0,a],\ |u-A_0|\leqslant 1\}$$

and so $|u|\leqslant|A_0|+1$. Consequently, a bound M_0 on f over R_0 can be obtained via

$$\begin{aligned}|f(t,u)|&\leqslant[\varphi(|u|)]^q\\&\leqslant[\varphi(|A_0|+1)]^q\\&=:M_0.\end{aligned}$$

Thus, the fractional IVP (3.1), (3.2) has at least one solution x on $[t_0, t_0 + t_1] = [0, t_1]$ with

$$\begin{aligned} t_1 &= \min\left\{a, \left[\frac{\Gamma(q+1)}{[\varphi(|A_0|+1)]^q}\right]^{1/q}\right\} \\ &= \min\left\{a, \frac{[\Gamma(q+1)]^{1/q}}{\varphi(|A_0|+1)}\right\} \end{aligned}$$

such that $|x(t) - A_0| \leqslant 1$ for all $t \in [0, t_1]$, so $|x(t_1)| \leqslant |A_0| + 1$.

Now we reconsider (3.1), (3.2) by associating: t_0 with t_1; x_0 with $x(t_1)$; each x_i with $x^{(i)}(t_1)$; and let's choose $b = 1$. We can thus form the rectangle

$$R_1 := \{(t,u) : t \in [t_1, a],\ |u - x(t_1)| \leqslant 1\}$$

and so $|u| \leqslant |x(t_1)| + 1 \leqslant |A_0| + 2$. Consequently, a bound M_1 on f over R_1 can be obtained via

$$\begin{aligned} |f(t,u)| &\leqslant [\varphi(|u|)]^q \\ &\leqslant [\varphi(|x(t_1)|+1)]^q \\ &\leqslant [\varphi(|A_0|+2)]^q \\ &=: M_1. \end{aligned}$$

Thus, the fractional IVP (3.1), (3.2) has at least one solution x on $[t_1, t_1 + t_2]$ with

$$\begin{aligned} t_2 &= \min\left\{a, \left[\frac{\Gamma(q+1)}{[\varphi(|A_0|+2)]^q}\right]^{1/q}\right\} \\ &= \min\left\{a, \frac{[\Gamma(q+1)]^{1/q}}{\varphi(|A_0|+2)}\right\} \end{aligned}$$

such that $|x(t) - x(t_1)| \leqslant 1$ for all $t \in [t_1, t_1 + t_2]$, so $|x(t_1 + t_2)| \leqslant |A_0| + 2$.

We continue in the above fashion and reconsider (3.1), (3.2) by associating: t_0 with $t_1 + \cdots + t_{m-1}$; x_0 with $x(t_1 + t_2 + \cdots + t_{m-1})$; each x_i with $x^{(i)}(t_1 + t_2 + \cdots + t_{m-1})$; and choosing $b = 1$. We can thus form the rectangle

$$R_{m-1} := \{(t,u) : t \in [0, a],\ |u - x(t_1 + \cdots + t_{m-1})| \leqslant 1\}$$

and so $|u| \leqslant |x(t_1 + \cdots t_{m-1})| + 1 \leqslant |A_0| + m$. Consequently, a bound M_{m-1} on f over R_{m-1} can be obtained via

$$\begin{aligned} |f(t,u)| &\leqslant [\varphi(|u|)]^q \\ &\leqslant [\varphi(|x(t_1 + \cdots + t_{m-1})|+1)]^q \\ &\leqslant [\varphi(|A_0|+m)]^q \\ &=: M_{m-1}. \end{aligned}$$

Thus, the fractional IVP (3.1), (3.2) has at least one solution x on $[t_1+\cdots+t_{m-1}, t_1+\cdots+t_m]$ with

$$t_m = \min\left\{a, \left[\frac{\Gamma(q+1)}{[\varphi(|A_0|+m)]^q}\right]^{1/q}\right\}$$
$$= \min\left\{a, \frac{[\Gamma(q+1)]^{1/q}}{\varphi(|A_0|+m)}\right\}$$

such that $|x(t)-x(t_1+\cdots+t_{m-1})| \leqslant 1$ for all $t \in [t_1+\cdots+t_{m-1}, t_1+\cdots+t_m]$, so $|x(t_1+\cdots+t_m)| \leqslant |A_0|+m$.

Combining the above we have thus shown the existence of a solution to (1.1), (1.2) on the interval $[0, \mathbf{T}_m]$ with

$$\mathbf{T}_m := \min\{a, t_1+\cdots+t_m\}, \quad (t_0 = 0).$$

Consider

$$\lim_{m\to\infty}(t_1+\cdots+t_m) = \sum_{i=1}^{\infty}\frac{[\Gamma(q+1)]^{1/q}}{\varphi(|A_0|+i)}$$
$$= \infty$$

where we have applied the integral test for series. Thus

$$\lim_{m\to\infty}\mathbf{T}_m = a$$

and we see that solutions extend to the entire interval $[0,a]$. $\square$

A pair of corollaries are now presented for special cases of the φ from Theorem 3.1.

COROLLARY 3.2. *Let $f : [0,a]\times\mathbb{R}\to\mathbb{R}$ be continuous. If there is are non-negative constants C and D such that*

$$|f(t,u)| \leqslant [C|u|+D]^q, \text{ for all } (t,u)\in[0,a]\times\mathbb{R}$$

then the fractional IVP (1.1), (1.2) *has at least one solution x that extends to the interval $[0,a]$.*

Proof. The conditions of Theorem 3.1 hold with $\varphi(s) = Cs+D$ and the result follows. $\square$

COROLLARY 3.3. *Let $f : [0,a]\times\mathbb{R}\to\mathbb{R}$ be continuous. If there is a non-negative constans D_1 such that*

$$|f(t,u)| \leqslant D_1, \text{ for all } (t,u)\in[0,a]\times\mathbb{R}$$

then the fractional IVP (1.1), (1.2) *has at least one solution x that extends to the interval $[0,a]$.*

Proof. The conditions of Theorem 3.1 hold with $\varphi(s) = D_1^{1/q}$ and the result follows. $\square$

If f is autonomous (ie, independent of t) in (1.1) then we can extend solutions over the half-line by using a similar sequential approach as in the proof of Theorem 3.1. Consider

$$D^q\left(x - T_{\lceil q\rceil-1}[x]\right) = f(x); \tag{3.3}$$

$$x(0) = A_0,\ x'(0) = A_1, \ldots,\ x^{(\lceil q\rceil-1)}(0) = A_{\lceil q\rceil-1}. \tag{3.4}$$

THEOREM 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If there is a continuous, nondecreasing function $\varphi : [0,\infty) \to (0,\infty)$ such that*

$$|f(u)| \leqslant [\varphi(|u|)]^q,\ \textit{for all } u \in \mathbb{R};$$

$$\int_d^\infty \frac{ds}{\varphi(s)} = \infty,\ \textit{for some constant } d \geqslant 0;$$

then the autonomous fractional IVP (3.3), (3.4) has at least one solution x that extends to the half-line $[0,\infty)$.

Proof. Consider the general autonomous fractional IVP

$$D^q\left(x - \mathscr{T}_{\lceil q\rceil-1}[x]\right) = f(x); \tag{3.5}$$

$$x(t_0) = x_0,\ x'(t_0) = x_1, \ldots,\ x^{(\lceil q\rceil-1)}(t_0) = x_{\lceil q\rceil-1}; \tag{3.6}$$

where $\mathscr{T}_{\lceil q\rceil-1}[x]$ is the Taylor polynomial of order $\lceil q\rceil - 1$ of $x = x(t)$ at $t = t_0$. Also consider the general interval

$$\mathscr{R} := \{u \in \mathbb{R} : |u - x_0| \leqslant b\}, \quad b > 0.$$

Let $\mathscr{M} > 0$ be a constant such that

$$|f(u)| \leqslant \mathscr{M} \quad \text{for all } u \in R.$$

Now, since f is continuous and independent of t, by Theorem 2.1 we know that the problem (3.1), (3.2) has at least one solution x on $[t_0, t_0 + \alpha]$, where

$$\alpha := \left[\frac{b\Gamma(q+1)}{\mathscr{M}}\right]^{1/q}.$$

Rerunning the proof of Theorem 3.1 (the details for omitted for brevity) we obtain the existence of a solution to (3.3), (3.4) on the interval $[0, \mathbf{T}_m]$ with

$$\mathbf{T}_m = t_1 + \cdots + t_m, \quad (t_0 = 0).$$

Consider

$$\lim_{m\to\infty} \mathbf{T}_m = \sum_{i=1}^\infty \frac{[\Gamma(q+1)]^{1/q}}{\varphi(|A_0| + i)} = \infty$$

where we have applied the integral test for series. Thus we see that solutions extend to the half-line $[0,\infty)$. $\square$

The following two corollaries to Theorem 3.4 provide special cases of φ. Their proofs are very similar to those of the earlier corollaries and so are omitted for brevity.

COROLLARY 3.5. *Let $f:\mathbb{R}\to\mathbb{R}$ be continuous. If there is are non-negative constants C and D such that*

$$|f(t,u)| \leqslant [C|u|+D]^q, \text{ for all } u\in\mathbb{R}$$

then the autonomous fractional IVP (3.3), (3.4) *has at least one solution x that extends to the interval $[0,\infty)$.*

COROLLARY 3.6. *Let $f:\mathbb{R}\to\mathbb{R}$ be continuous. If there is a non-negative constant D_1 such that*

$$|f(u)| \leqslant D_1, \text{ for all } u\in\mathbb{R}$$

then the autonomous fractional IVP (3.3), (3.4) *has at least one solution x that extends to the helf-line $[0,\infty)$.*

EXAMPLE 3.7. The fractional IVP (3.3), (3.4) with $q=1/2$ and

$$f(u) := \sqrt{|u|+2}$$

has at least one solution on the half-line $[0,\infty)$.

Proof. We may choose $C=1$ and $D=2$ so that the conditions of Corollary 3.5 are satisfied. Hence we conclude that our problem has at least one solution that extends to the half-line $[0,\infty)$. $\square$

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