Journal of
Classical
Analysis
Volume 6, Number 1 (2015), 73–84 doi:10.7153/jca-06-06

ON MAXIMAL AREA INTEGRAL PROBLEM FOR ANALYTIC FUNCTIONS IN THE STARLIKE FAMILY

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Abstract. For an analytic function f defined on the unit disk $|z| < 1$, let $\Delta(r, f)$ denote the area of the image of the subdisk $|z| < r$ under f, where $0 < r \leqslant 1$. In 1990, Yamashita conjectured that $\Delta(r, z/f) \leqslant \pi r^2$ for convex functions f and it was finally settled in 2013 by Obradović and et. al.. In this paper, we consider a class of analytic functions in the unit disk satisfying the subordination relation $zf'(z)/f(z) \prec (1+(1-2\beta)\alpha z)/(1-\alpha z)$ for $0 \leqslant \beta < 1$ and $0 < \alpha \leqslant 1$. We prove Yamashita's conjecture problem for functions in this class, which provides a partial solution to an open problem posed by Ponnusamy and Wirths.

1. Introduction, preliminaries, and main result

The univalent function has been the central object in the study of geometric function theory. Some of its natural geometric families act a prominent role in the theory of univalent functions [4, 6, 14] and their geometric properties. For instance, the classes of starlike, convex and close-to-convex, to name just a few. These classes have been familiarized and studied by many authors. It is interesting to observe that we can obtain many of their analytic properties by an integrated method. Study of various subclasses of the class of starlike functions have been appreciated by several authors. The class of starlike functions of order $\beta\,(0 \leqslant \beta < 1)$ was generated by Robertson [16] and has been then studied by Schild [19] and Merkes [10]. Marx [9] and Strohacker [20] proved that if $f(z)$ maps the unit disk onto a convex domain, then $f(z)$ is starlike of order $1/2$. Gabriel [5] showed that the class of starlike functions of order $1/2$ played an important role in the solution of differential equations. In 1968, Padmanabhan [13] discussed a different subfamily for the order of starlikeness. In this paper, we introduce a more general family than the family studied by Padmanabhan.

Define by $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$, the disk of radius r centred at the origin. The unit disk is then defined by $\mathbb{D} := \mathbb{D}_1$. Let $\mathscr{A}$ denote the family of all functions $f(z)$ analytic in $\mathbb{D}$ and normalized so that $f(0) = 0 = f'(0) - 1$, i.e. $f \in \mathscr{A}$ has the power series representation $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Denote by $\mathscr{S}$, the class of univalent functions $f \in \mathscr{A}$. The Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by the series

$$1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad |z| < 1,$$

Mathematics subject classification (2010): Primary 30C45; Secondary 30C55.
Keywords and phrases: Analytic, univalent and starlike functions, subordination, Dirichlet finite, area integral, Parseval-Gutzmer formula, Gaussian hypergeometric functions.

where a,b and c are complex numbers with c is neither zero nor a negative integer. Clearly, the shifted function $z_2F_1(a,b;c;z)$ belongs to $\mathscr{A}$. The notation $(a)_n$ denotes the shifted factorial and it is defined by

$$(a)_0 = 1, \ (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \geqslant 1.$$

Here, Γ stands for the usual gamma function. If either (or both) of a and b is (are) zero or a negative integer(s), then the series terminates.

For two analytic functions f and g in $\mathbb{D}$, we say that f is *subordinate* to g if

$$f(z) = g(w(z)), \quad |z| < 1,$$

for some analytic function w in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$. We express this symbolically by $f \prec g$. In particular, if g is univalent in $\mathbb{D}$, $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$ then $f \prec g$. For instance, one can easily see that $1/(1+z) \prec (1+z)/(1-z)$, $z \in \mathbb{D}$.

We denote by $\mathscr{S}t(\beta)$, the well-known class of starlike functions of order β. Analytically, for $f \in \mathscr{S}$, the starlike functions are characterized by the condition $\mathrm{Re}\,(zf'(z)/f(z)) > \beta$, where $0 \leqslant \beta < 1$, i.e. f has the subordination property,

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}, \quad z \in \mathbb{D}, \ 0 \leqslant \beta < 1.$$

The class $\mathscr{S}t := \mathscr{S}t(0)$ is the class of starlike functions, i.e. $f \in \mathscr{S}$ is starlike with respect to the origin, i.e. $tw \in f(\mathbb{D})$ whenever $t \in [0,1]$ and $w \in f(\mathbb{D})$.

Suppose that $f(z)$ is a function analytic in the unit disk $\mathbb{D}$. For $0 < r \leqslant 1$, we denote by $\Delta(r,f)$, the area of the image of the disk $\mathbb{D}_r$ under $f(z)$. Thus,

$$\Delta(r,f) = \iint_{\mathbb{D}_r} |f'(z)|^2 \, dxdy \quad (z = x+iy).$$

Computing this area is known as the *area problem for the function of type* f. The classical Parseval-Gutzmer formula for a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in $\overline{\mathbb{D}}_r$ states that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

By means of this formula, since $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$, we find

$$\Delta(r,f) = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}.$$

We call f a Dirichlet-finite function if $\Delta(1,f)$, the area covered by the mapping $z \to f(z)$ for $|z| < 1$, is finite. Our interest in this paper was originated by the work of Yamashita [21] and Ponnusamy et. al. [11, 12, 15]. Yamashita [21] initially conjectured that

$$\max_{f\in\mathscr{C}} \Delta\left(r, \frac{z}{f}\right) = \pi r^2$$

for each r, $0 < r \leqslant 1$, and the maximum is attained only by the rotations of the function $l(z) = z/(1-z)$. Here $\mathscr{C}$ denotes the class of convex functions i.e $f \in \mathscr{S}$ such that $f(\mathbb{D})$ is convex. This conjecture was recently settled in [11]. In fact the conjecture has been solved for a wider class of functions (the class of starlike functions of order β, $0 \leqslant \beta < 1$), which includes the class $\mathscr{C}$; see also Corollary 3.3. In [15], the Yamashita conjecture problem for the class of ϕ-spirallike functions of order β $(0 \leqslant \beta < 1)$ and $\phi \in (-\pi/2, \pi/2)$ have also been settled (see [8] for the definition of ϕ-spirallike function). Recent work in this direction can also be found in [12]. There are several other classes of analytic univalent functions having interesting geometric properties for which solution of the Yamashita conjecture problem would be of interesting to readers working in this field.

Our objective in this paper is to give a partial solution to a problem posed in [15] by considering the following subfamily of the family of starlike functions introduced by Padmanabhan [13].

DEFINITION 1.1. A function $f \in \mathscr{A}$ is said to be in $\mathscr{S}(\alpha)$, $0 < \alpha \leqslant 1$, if

$$\left|\left(\frac{zf'(z)}{f(z)} - 1\right) \Big/ \left(\frac{zf'(z)}{f(z)} + 1\right)\right| < \alpha, \quad \text{equivalently,} \quad \frac{zf'(z)}{f(z)} \prec \frac{1+\alpha z}{1-\alpha z},$$

for all $z \in \mathbb{D}$.

It is evident to see that $\mathscr{S}(1) \equiv \mathscr{S}t$ and $\mathscr{S}(\alpha) \subset \mathscr{S}t(\beta)$ for all $\alpha, \beta \in (0,1)$ with $\beta \leqslant (1-\alpha)/(1+\alpha)$. Also, the function $g(z) = z/(1-z) \in \mathscr{S}t(1/2)$ guarantees that this inclusion is proper. One can also verify that $k_\beta(z) := z/(1-z)^{2(1-\beta)}$ belongs to $\mathscr{S}t(\beta)$, whereas, the function $k_\alpha(z) := z/(1-\alpha z)^2 \in \mathscr{S}(\alpha)$. In this paper, we prove the Yamashita conjecture for functions in a more general family than $\mathscr{S}(\alpha)$. In particular, the conjecture will also follow for functions in $\mathscr{S}(\alpha)$. The generalization is now defined below.

DEFINITION 1.2. A function $f \in \mathscr{A}$ is said to belong to the class $\mathscr{S}(\alpha, \beta)$, $0 < \alpha \leqslant 1$, $0 \leqslant \beta < 1$, if

$$\left|\left(\frac{zf'(z)}{f(z)} - 1\right) \Big/ \left(\frac{zf'(z)}{f(z)} + 1 - 2\beta\right)\right| < \alpha,$$

i.e.,

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)\alpha z}{1-\alpha z}, \tag{1.1}$$

for all $z \in \mathbb{D}$.

A general form of this definition is earlier introduced by Aouf (see [1, Definition 2]). Definition of $\mathscr{S}(\alpha, \beta)$ says that the domain values of $zf'(z)/f(z)$ lie in the disk of radius $2\alpha(1-\beta)/(1-\alpha^2)$ centred at $(1+\alpha^2(1-2\beta))/(1-\alpha^2)$. We see that if $\beta = 0$, then Definition 1.2 turns into Definition 1.1. The function

$$k_{\alpha,\beta}(z) := z(1-\alpha z)^{-2(1-\beta)}, \quad z \in \mathbb{D},\ 0 < \alpha \leqslant 1,\ 0 \leqslant \beta < 1, \tag{1.2}$$

belongs to the family $\mathcal{S}(\alpha,\beta)$ and in this context it plays the role of extremal function for $\mathcal{S}(\alpha,\beta)$. Also, one notes that

$$\mathcal{S}(\alpha,\beta) \subset \mathcal{S}(\alpha) \subset \mathcal{S}t(\beta) \subset \mathcal{S}t;\ \mathcal{S}(1,\beta) = \mathcal{S}t(\beta),\ \mathcal{S}(\alpha,0) = \mathcal{S}(\alpha) \text{ and } \mathcal{S}(1) = \mathcal{S}t.$$

Consequently,

$$k_{1,\beta}(z) = k_\beta(z),\ k_{\alpha,0}(z) = k_\alpha(z),\ k_{1,0}(z) = k_1(z) = k(z). \tag{1.3}$$

In this article, our main aim is to examine the maximum area problem for the functions of type (z/f) so-called the Yamashita conjecture problem, when $f \in \mathcal{S}(\alpha,\beta)$. By looking into the behavior of $z/k_{\alpha,\beta}(z)$ (see Section 3), we expect the following theorem whose proof is given in Section 3.

THEOREM 1.3. (Main Theorem) *Let* $0 < \alpha \leqslant 1$ *and* $0 \leqslant \beta < 1$. *If* $f \in \mathcal{S}(\alpha,\beta)$ *and* z/f *is a non-vanishing analytic function in* $\mathbb{D}$, *then we have the maximal area*

$$\begin{aligned}\max_{f\in\mathcal{S}(\alpha,\beta)} \Delta\left(\rho, \frac{z}{f}\right) &= 4\pi\alpha^2(1-\beta)^2\rho^2{}_2F_1(2\beta-1, 2\beta-1; 2; \alpha^2\rho^2), \quad |z| < \rho \\ &=: A_{\alpha,\beta}(\rho)\end{aligned}$$

for all $\rho, 0 < \rho \leqslant 1$. *The maximum is attained only by the rotations of the function* $k_{\alpha,\beta}(z)$ *defined by* (1.2).

This generalizes the main results which are discussed in [11] and [21].

In Section 2, we prepare some basic results and use them to prove our main theorem in Section 3.

2. Preparatory results

If $f \in \mathcal{S}$ then $z/f(z)$ is non-vanishing in $\mathbb{D}$. Hence, it can be described as Taylor's series of the form

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}. \tag{2.1}$$

We first derive a coefficient estimate in series form for a function f of the form (2.1) when $f \in \mathcal{S}(\alpha,\beta)$.

LEMMA 2.1. *Let* $0 < \alpha \leqslant 1$, $0 \leqslant \beta < 1$, *and* $f \in \mathcal{S}(\alpha,\beta)$. *If* $g(z)$ *is a non-vanishing analytic function in* $\mathbb{D}$ *of the form* (2.1), *then it necessarily satisfies the coefficient inequality*

$$\sum_{k=1}^{\infty} \left(k^2 - (k-2(1-\beta))^2\alpha^2\right)|b_k|^2 \leqslant 4(1-\beta)^2\alpha^2.$$

Proof. Let $g(z) = z/f(z)$ be of the form (2.1). Note that the logarithmic derivative gives

$$\frac{zg'(z)}{g(z)} = 1 - \frac{zf'(z)}{f(z)}.$$

Since $f(z) \in \mathscr{S}(\alpha,\beta)$, the subordination relation (1.1) says that there exists an analytic function $w : \mathbb{D} \to \overline{\mathbb{D}}$ with $w(0) = 1$ such that

$$\frac{zf'(z)}{f(z)} = \frac{1+\alpha(1-2\beta)zw(z)}{1-\alpha zw(z)}, \quad z \in \mathbb{D},$$

and hence

$$\frac{g'(z)}{g(z)} = \frac{-2\alpha(1-\beta)w(z)}{1-\alpha zw(z)}.$$

Writing this in series form, we get

$$\sum_{k=1}^{\infty} kb_k z^{k-1} = \alpha\left(-2(1-\beta) + \sum_{k=1}^{\infty}(k-2(1-\beta))b_k z^k\right) w(z).$$

After a minor re-arrangement, we obtain

$$\sum_{k=1}^{n} kb_k z^{k-1} + \sum_{k=n+1}^{\infty} kb_k z^{k-1} = \alpha\left(-2(1-\beta) + \sum_{k=1}^{n-1}(k-2(1-\beta))b_k z^k\right) w(z)$$
$$+\alpha\left(\sum_{k=n}^{\infty}(k-2(1-\beta))b_k z^k\right) w(z).$$

By Clunie's method [2] (see also [3, 17, 18]), we obtain

$$\sum_{k=1}^{n} k^2|b_k|^2\rho^{2k-2} \leqslant \alpha^2\left(4(1-\beta)^2 + \sum_{k=1}^{n-1}(k-2(1-\beta))^2|b_k|^2\rho^{2k}\right),$$

equivalently,

$$\sum_{k=1}^{n} k^2|b_k|^2\rho^{2k-2} - \alpha^2\sum_{k=1}^{n-1}(k-2(1-\beta))^2|b_k|^2\rho^{2k} \leqslant 4(1-\beta)^2\alpha^2. \tag{2.2}$$

If we take $\rho = 1$ and allow $n \to \infty$, then we obtain the desired inequality

$$\sum_{k=1}^{\infty}\left(k^2 - (k-2(1-\beta))^2\alpha^2\right)|b_k|^2 \leqslant 4(1-\beta)^2\alpha^2.$$

The proof of our lemma is now complete. $\square$

We remark that the special choices $\alpha = 1$ and $\beta = 0$ turned Lemma 2.1 into the well-known Area Theorem for $f \in \mathscr{S}$ (see for instance [6, Theorem 11, p. 193 of Vol-2]).

We now prepare a lemma using a new technique introduced in [11] and this lemma plays an important role to prove our main theorem in this paper.

LEMMA 2.2. *Let* $0 < \alpha \leqslant 1,\ 0 \leqslant \beta < 1$, *and* $f \in \mathscr{S}(\alpha,\beta)$. *For* $|z| < \rho \leqslant 1$ *suppose that*

$$\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k \text{ and } (1+\alpha z)^{2-2\beta} = 1 + \sum_{k=1}^{\infty} (-1)^k c_k z^k.$$

Then the relation

$$\sum_{k=1}^{N} k|b_k|^2 \rho^{2k} \leqslant \sum_{k=1}^{N} k|c_k|^2 \rho^{2k} \tag{2.3}$$

is valid for all $N \in \mathbb{N}$.

Proof. We divide our proof into three steps.

Step-I: Clunie's method.

Rewrite (2.2) in the following form:

$$\sum_{k=1}^{n-1} \left(k^2 - (k-2(1-\beta))^2\alpha^2\rho^2\right)|b_k|^2\rho^{2k-2} + n^2|b_n|^2\rho^{2n-2} \leqslant 4(1-\beta)^2\alpha^2.$$

Multiply by ρ^2 on both sides we obtain

$$\sum_{k=1}^{n-1} \left(k^2 - (k-2(1-\beta))^2\alpha^2\rho^2\right)|b_k|^2\rho^{2k} + n^2|b_n|^2\rho^{2n} \leqslant 4(1-\beta)^2\alpha^2\rho^2. \tag{2.4}$$

The function $b(z) = (1+\alpha z)^{2-2\beta}$ clearly shows that the equality in (2.4) attains with $b_k = (-1)^k c_k$.

Step-II: Cramer's Rule.

We consider the inequalities (2.4) for $n = 1,\ldots,N$, and multiply the n-th coefficient by a factor $\lambda_{n,N}$ for each n. These factors are chosen in such a way that the addition of the left sides of the modified inequalities is equivalent to the left side of (2.3). The calculation of the factors $\lambda_{n,N}$ leads to the following system of linear equations:

$$k = k^2\lambda_{k,N} + \sum_{n=k+1}^{N} \lambda_{n,N}\left(k^2 - (k-2(1-\beta))^2\alpha^2\rho^2\right), \quad k = 1,\cdots,N. \tag{2.5}$$

Since the matrix of the system (2.5) is an upper triangular matrix with positive integers as diagonal elements, the solution of the system is uniquely determined. Cramer's rule allows us to write the solution of the system (2.5) in the form

$$\lambda_{n,N} = \frac{((n-1)\,!)^2}{(N\,!)^2}\text{Det } A_{n,N},$$

where $A_{n,N}$ is the $(N-n+1)\times(N-n+1)$ matrix constructed as follows:

$$A_{n,N} = \begin{bmatrix} n & n^2-(n-2(1-\beta))^2\alpha^2\rho^2 & \cdots & n^2-(n-2(1-\beta))^2\alpha^2\rho^2 \\ n+1 & (n+1)^2 & \cdots & (n+1)^2-(n+1-2(1-\beta))^2\alpha^2\rho^2 \\ \vdots & \vdots & \vdots & \vdots \\ N & 0 & \cdots & N^2 \end{bmatrix}.$$

Rest of the proof now similarly follows as explained in the proof of [11, Lemma 5]. Indeed, for the sake of completeness we provide the complete proof.

Determinants of these matrices can be found by expanding according to Laplace's rule with respect to the last row, wherein the first coefficient is N, the last one is N^2, and the rest of the entries are zeros. This expansion and a mathematical induction applies in the following formula. If $k \leqslant N-1$, then

$$\lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N}\left(1-\left(1-\frac{2(1-\beta)}{k}\right)^2 \alpha^2\rho^2\right)\prod_{m=k+1}^{N-1}\left(1-\left(1-\frac{2(1-\beta)}{m}\right)^2 \alpha^2\rho^2\right).$$

We see that for fixed $k \in \mathbb{N}, N \geqslant k$, the sequence $\{\lambda_{k,N}\}$ is a strictly non-increasing sequence i.e. $\lambda_{k,N} - \lambda_{k,N-1} < 0$ with

$$\lim_{N\to\infty} \lambda_{k,N} = \frac{1}{k} - \left(1-\left(1-\frac{2(1-\beta)}{k}\right)^2 \alpha^2\rho^2\right)\sum_{n=k+1}^{\infty}\frac{1}{n}\prod_{m=k+1}^{n-1}\left(\left(1-\frac{2(1-\beta)}{m}\right)^2 \alpha^2\rho^2\right). \tag{2.6}$$

Set $\lambda_k := \lim_{N\to\infty} \lambda_{k,N}$. To show that $\lambda_{k,N} > 0$ for all $N \in \mathbb{N}$, $k \in [1,N]$, it is enough to show that $\lambda_k \geqslant 0$ for $k \in \mathbb{N}$. This will be completed in Step III. But, before we proceed for the next step, we note that the proof of the said inequality is adequate for the proof of the assertion of the result, since, as we observed in Step-I, equality is obtained for $b_k = (-1)^k c_k$.

Step-III: Positivity of the Multipliers.

Let for an abbreviation

$$S_k = \sum_{n=k+1}^{\infty}\frac{1}{n}\prod_{m=k+1}^{n-1}\left(\left(1-\frac{2(1-\beta)}{m}\right)^2 \alpha^2\rho^2\right), \quad k \in \mathbb{N}.$$

We now show that

$$S_k \leqslant \frac{1}{k\left(1-\left(1-\frac{2(1-\beta)}{k}\right)^2 \alpha^2\rho^2\right)}.$$

From the equation (2.6), we get

$$\lambda_k = \frac{1}{k} - S_k + \left(1-\frac{2(1-\beta)}{k}\right)^2 \alpha^2\rho^2 S_k.$$

Again set for an abbreviation

$$T_k = \frac{1}{k} + \left(1-\frac{2(1-\beta)}{k}\right)^2 \alpha^2\rho^2 S_k.$$

It is enough to prove that

$$T_k \leqslant \frac{1}{k\left(1-\left(1-\frac{2(1-\beta)}{k}\right)^2 \alpha^2\rho^2\right)}. \tag{2.7}$$

To prove (2.7) we use the inequality

$$\frac{1}{n\left(1-\left(1-\frac{2(1-\beta)}{n}\right)^2\alpha^2\rho^2\right)} > \frac{1}{(n+1)\left(1-\left(1-\frac{2(1-\beta)}{n+1}\right)^2\alpha^2\rho^2\right)} \tag{2.8}$$

and the identity

$$\frac{1}{n\left(1-\left(1-\frac{2(1-\beta)}{n}\right)^2\alpha^2\rho^2\right)} = \frac{1}{n} + \frac{\left(1-\frac{2(1-\beta)}{n}\right)^2\alpha^2\rho^2}{n\left(1-\left(1-\frac{2(1-\beta)}{n}\right)^2\alpha^2\rho^2\right)} \tag{2.9}$$

which are admissible for each $n \in \mathbb{N}$. Repeated application of (2.8) and (2.9) for $n = k, k+1, \ldots, T$ results in the inequality

$$\begin{aligned}\frac{1}{k\left(1-\left(1-\frac{2(1-\beta)}{k}\right)^2\alpha^2\rho^2\right)} &> \sum_{n=k}^{T}\frac{1}{n}\prod_{m=k}^{n-1}\left(\left(1-\frac{2(1-\beta)}{m}\right)^2\alpha^2\rho^2\right)\\ &\quad + \frac{\prod_{m=k}^{T}\left(\left(1-\frac{2(1-\beta)}{m}\right)^2\alpha^2\rho^2\right)}{T\left(1-\left(1-\frac{2(1-\beta)}{T}\right)^2\alpha^2\rho^2\right)}\\ &=: S_{k,T} + R_{k,T}, \qquad \text{for } k \leqslant T.\end{aligned}$$

Since $R_{k,T} > 0$, allow the limit as $T \to \infty$, we get

$$\frac{1}{k\left(1-\left(1-\frac{2(1-\beta)}{k}\right)^2\alpha^2\rho^2\right)} \geqslant \lim_{T\to\infty} S_{k,T} = \sum_{n=k}^{\infty}\frac{1}{n}\prod_{m=k}^{n-1}\left(\left(1-\frac{2(1-\beta)}{m}\right)^2\alpha^2\rho^2\right) = T_k,$$

and we complete the inequality (2.7). Hence, the proof of Lemma 2.2 is complete. □

We now establish a preliminary result concerning necessary and sufficient conditions for a function to be in $\mathcal{S}(\alpha,\beta)$.

LEMMA 2.3. *Let* $0 < \alpha \leqslant 1$ *and* $0 \leqslant \beta < 1$. *Then* $f \in \mathcal{S}(\alpha,\beta)$ *if and only if* F *defined by* $F(z) = z(f(z)/z)^{\frac{1}{1-\beta}} \in \mathcal{S}(\alpha)$, $z \in \mathbb{D}$.

Proof. Let $F(z) = z(f(z)/z)^{\frac{1}{1-\beta}}$. Taking logarithm derivative on both sides and simplify, we get

$$z\frac{F'(z)}{F(z)} = 1 + \frac{1}{1-\beta}\left(z\frac{f'(z)}{f(z)} - 1\right).$$

By componendo and dividendo rule, we have

$$\left|\left(z\frac{F'(z)}{F(z)} - 1\right)\Big/\left(z\frac{F'(z)}{F(z)} + 1\right)\right| = \left|\left(z\frac{f'(z)}{f(z)} - 1\right)\Big/\left(z\frac{f'(z)}{f(z)} + 1 - 2\beta\right)\right|. \tag{2.10}$$

By Definition 1.2 we get $f(z) \in \mathcal{S}(\alpha,\beta)$ if and only if $F(z) \in \mathcal{S}(\alpha)$. □

3. Proof of the main Theorem

As an initial observation, from (1.2) we see that

$$\frac{z}{k_{\alpha,\beta}(z)} = (1-\alpha z)^{2(1-\beta)} = 1+\sum_{n=1}^{\infty} c_n z^n$$

where $c_n = \dfrac{(\zeta)_n}{(1)_n}\alpha^n$ and $\zeta = -2(1-\beta)$.

Hence, we apply the area formula for the function $z/k_{\alpha,\beta}(z)$ and obtain

$$\begin{aligned}
\pi^{-1}\Delta\left(\rho, \frac{z}{k_{\alpha,\beta}}\right) &= \sum_{n=1}^{\infty} n|c_n|^2\rho^{2n}, \quad |z|<\rho \\
&= \sum_{n=1}^{\infty} n\frac{(\zeta)_n(\zeta)_n}{(1)_n(1)_n}\alpha^{2n}\rho^{2n} \\
&= \zeta^2\alpha^2\rho^2\sum_{n=0}^{\infty}\frac{(\zeta+1)_n(\zeta+1)_n}{(2)_n(1)_n}\alpha^{2n}\rho^{2n} \\
&= 4\alpha^2(1-\beta)^2\rho^2{}_2F_1(2\beta-1,2\beta-1;2;\alpha^2\rho^2) \\
&= \pi^{-1}A_{\alpha,\beta}(\rho).
\end{aligned}$$

At this point let us write $A_{\alpha,\beta}(\rho)$, $0<\rho\leqslant 1$, in the following form:

$$A_{\alpha,\beta}(\rho) = 4\pi\alpha^2(1-\beta)^2\rho^2\sum_{n=0}^{\infty}\frac{(2\beta-1)_n^2}{(1)_n(2)_n}\alpha^{2n}\rho^{2n}.$$

Because the series on the right hand side has positive coefficients, $A_{\alpha,\beta}(\rho)$ is a non-decreasing and convex function of the real variable ρ. Thus, $A_{\alpha,\beta}(\rho)\leqslant A_{\alpha,\beta}(1)$, i.e.

$$A_{\alpha,\beta}(\rho)\leqslant 4\pi\alpha^2(1-\beta)^2\sum_{n=0}^{\infty}\frac{(2\beta-1)_n^2}{(1)_n(2)_n}\alpha^{2n}.$$

It is now time for us to prove the main theorem.

Proof. For $z\in\mathbb{D}$, we know by Lemma 2.3 that

$$f(z)\in\mathscr{S}(\alpha,\beta)\Longleftrightarrow F(z) = z\left(\frac{f(z)}{z}\right)^{\frac{1}{1-\beta}}\in\mathscr{S}(\alpha).$$

Further $F(z)\in\mathscr{S}(\alpha)$ gives

$$\frac{z}{F(z)} = \left(\frac{z}{f(z)}\right)^{\frac{1}{1-\beta}} \prec (1-\alpha z)^2 = \left(\frac{z}{k_{\alpha,\beta}(z)}\right)^{\frac{1}{1-\beta}},$$

i.e.

$$\frac{z}{f(z)}\prec(1-\alpha z)^{2(1-\beta)} = \frac{z}{k_{\alpha,\beta}(z)}.$$

If

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \text{ and } \frac{z}{k(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad |z| < \rho,$$

then the extension of Rogosinski's result observed by Goluzin [4, Theorem 6.3, p. 193] yields

$$\sum_{n=1}^{\infty} n|b_n|^2 \rho^{2n} \leqslant \sum_{n=1}^{\infty} n|c_n|^2 \rho^{2n}.$$

That is

$$\Delta\left(\rho, \frac{z}{f}\right) \leqslant \Delta\left(\rho, \frac{z}{k_{\alpha,\beta}}\right) = \pi\zeta^2\alpha^2\rho^2{}_2F_1(\zeta+1, \zeta+1; 2; \alpha^2\rho^2), \quad \zeta+1 = 2\beta-1$$

whenever the sequence $\{n\rho^{2n}\}$ is decreasing function of $\rho, 0 < \rho \leqslant \frac{1}{\sqrt{2}}$. Thus, the theorem is obviously true for $0 < \rho \leqslant \frac{1}{\sqrt{2}}$. On other hand, in order to present a proof to include the case $\rho > \frac{1}{\sqrt{2}}$, it suffices to prove

$$\sum_{n=1}^{N} n|b_n|^2 \rho^{2n} \leqslant \sum_{n=1}^{N} n|c_n|^2 \rho^{2n}, \quad N \in \mathbb{N}, \ \rho \in (0,1).$$

This follows from Lemma 2.2 and hence the proof of Theorem 1.3 is complete. $\square$

If we choose $\beta = 0$ in Theorem 1.3, then we get the following Yamashita conjecture problem solved for functions in the Padmanabhan class $\mathscr{S}(\alpha)$:

THEOREM 3.1. *Let for $0 < \alpha \leqslant 1$, $f \in \mathscr{S}(\alpha)$ and z/f be a non-vanishing analytic function in $\mathbb{D}$. Then we have*

$$\max_{f \in \mathscr{S}(\alpha)} \Delta\left(\rho, \frac{z}{f}\right) = 2\pi\alpha^2\rho^2(2+\alpha^2\rho^2)$$

for all $\rho, 0 < \rho \leqslant 1$. The maximum is attained only by the rotation of the function $k_\alpha(z)$ defined by (1.3).

If $\alpha = 1$ and $\beta = 1/2$, then as a consequence of Theorem 1.3 we get

COROLLARY 3.2. [11, Theorem 2] *We have*

$$\max_{f \in \mathscr{S}t(1/2)} \Delta\left(\rho, \frac{z}{f}\right) = \pi\rho^2 \quad \textit{for } \ 0 < \rho \leqslant 1,$$

where the maximum is attained only by the rotation of the Koebe function $k(z)$ defined by (1.3).

Moreover, if we choose $\alpha = 1$ in Theorem 1.3, we get

COROLLARY 3.3. [11, Theorem 3] *Let* $f \in \mathcal{S}t(\beta)$ *for some* $0 \leqslant \beta < 1$. *Then we have*

$$\max_{f \in \mathcal{S}t(\beta)} \Delta\left(\rho, \frac{z}{f}\right) = 4\pi(1-\beta)^2\rho^2{}_2F_1(2\beta-1, 2\beta-1; 2; \rho^2) \quad \textit{for } 0 < \rho \leqslant 1,$$

where the maximum is attained only by the rotation of the function $k_\beta(z)$ *defined by* (1.3).

4. Concluding Remark

For $-1 \leqslant B < A \leqslant 1$, the Janowski class $\mathcal{S}^*(A,B)$ is defined by the subordination relation

$$\mathcal{S}^*(A,B) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \, z \in \mathbb{D} \right\}.$$

The class $\mathcal{S}^*(A,B)$ is introduced in [7] and studied by number of researchers in this field. It is evident that $\mathcal{S}^*(A,B) \subset \mathcal{S}t$. In [15], it has been reported that Yamashita's conjecture is an open problem to prove for convex functions of order β and more generally, for functions in the class $\mathcal{S}^*(A,B)$ and also for the class of functions f for which $zf'(z) \in \mathcal{S}^*(A,B)$. In particular, the choices $A = (1-2\beta)\alpha$ and $B = -\alpha$ turn the class $\mathcal{S}^*(A,B)$ into the class $\mathcal{S}(\alpha, \beta)$. Therefore, a partial solution to the above open problem has been solved in this paper.

Acknowledgements. This research was started in the mid 2013 when the authors were visited ISI Chennai Centre. We would like to thank Professors R. Parvatham and S. Ponnusamy for useful discussion in this topic. The second author also acknowledges the support of the National Board for Higher Mathematics, Department of Atomic Energy, India (grant no. 2/39(20)/2010-R & D-II). The authors thank the referees for useful comments on the paper.

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(Received August 11, 2014)

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