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REVIVING THE QUADRATIC SERIES OF AU–YEUNG

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Abstract. In this paper we revive and bring to light the quadratic series of Au–Yeung

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{17}{4}\zeta(4),$$

where H_n denotes the nth harmonic number. We prove this series identity by using a technique based on the computation of a special logarithmic integral combined with Abel's summation formula.

1. Introduction and the main result

The identity

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{17}{4}\zeta(4)=\frac{17\pi^4}{360},$$

was discovered numerically by Enrico Au–Yeung an undergraduate student in the Faculty of Mathematics in Waterloo and proved rigorously by David Borwein and Jonathan Borwein in [3] who used Fourier series techniques combined with Parseval's formula for proving it. This quadratic series has become a classic in the theory of nonlinear harmonic series. We mention that a nonlinear harmonic series is a series which involves products of at least two harmonic numbers [4]. It appears as a problem in [7, Problem 2.6.1. p. 110], [5, Problem 3.70, p. 150] and a recent proof involving integrals of polylogarithm functions was given in [6].

In this paper we revive and bring to light the quadratic series of Au–Yeung by giving a proof of it which is based on the calculation of a quadratic logarithmic integral combined with Abel's summation formula. We mention that our results are not new and they exist in the mathematical literature.

We state below the theorem we are going to prove.

THEOREM 1 (A quadratic series of Au–Yeung) *The following equality holds*:

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{17}{4}\zeta(4)=\frac{17\pi^4}{360},$$

where H_n is the nth harmonic number defined, for $n\geqslant 1$, by $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$.

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Before we prove Theorem 1 we collect some results we need in our analysis.

Recall that, Abel's summation formula ([2, p. 55], [5, p. 258]) states that if $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ are two sequences of real numbers and $A_n = \sum\limits_{k=1}^{n} a_k$, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}). \tag{1}$$

We will also be using, in our calculations, the infinite version of the preceding formula

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}). \tag{2}$$

Next we prove the following two lemmas which are used in the proof of Theorem 1.

LEMMA 2 (Two logarithmic integrals and harmonic numbers) *Let $n \geqslant 1$ be an integer. The following equalities hold*:

(a) $\displaystyle\int_0^1 x^{n-1} \ln(1-x)\mathrm{d}x = -\frac{H_n}{n}$;

(b) $\displaystyle\int_0^1 x^{n-1} \ln^2(1-x)\mathrm{d}x = \frac{2}{n}\sum_{k=1}^{n}\frac{H_k}{k} = \frac{H_n^2}{n} + \frac{1}{n}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)$.

Proof. (a) We have,

$$\begin{aligned}
\int_0^1 x^{n-1} \ln(1-x)\mathrm{d}x &= \int_0^1 x^{n-1}\left(\int_0^x -\frac{1}{1-t}\mathrm{d}t\right)\mathrm{d}x \\
&= -\int_0^1 \frac{1}{1-t}\left(\int_t^1 x^{n-1}\mathrm{d}x\right)\mathrm{d}t \\
&= -\frac{1}{n}\int_0^1 \frac{1-t^n}{1-t}\mathrm{d}t \\
&= -\frac{1}{n}\int_0^1 \left(1+t+t^2+\cdots+t^{n-1}\right)\mathrm{d}t \\
&= -\frac{H_n}{n}.
\end{aligned}$$

(b) We have,

$$\begin{aligned}
\int_0^1 x^{n-1}\ln^2(1-x)\mathrm{d}x &= \int_0^1 x^{n-1}\left(\int_0^x -2\frac{\ln(1-t)}{1-t}\mathrm{d}t\right)\mathrm{d}x \\
&= -2\int_0^1 \frac{\ln(1-t)}{1-t}\left(\int_t^1 x^{n-1}\mathrm{d}x\right)\mathrm{d}t \\
&= \frac{-2}{n}\int_0^1 \ln(1-t)\frac{1-t^n}{1-t}\mathrm{d}t \\
&= -\frac{2}{n}\int_0^1 \ln(1-t)\left(1+t+t^2+\cdots+t^{n-1}\right)\mathrm{d}t \\
&= \frac{2}{n}\sum_{k=1}^{n}\frac{H_k}{k},
\end{aligned}$$

where the last equality follows based on part (a) of the lemma.

It remains to prove that

$$H_n^2+1+\frac{1}{2^2}+\cdots+\frac{1}{n^2} = 2\sum_{k=1}^{n}\frac{H_k}{k}.$$

It is worth mentioning that, the previous formula is known in the literature. For example, it appears in [1, Equation (3,62)] and it can be proved by mathematical induction. However, we give below another proof of it which is based on the summation by parts formula. To see this, we use formula (1), with $a_k = \frac{1}{k}$ and $b_k = H_k$, and we get that

$$\begin{aligned}
\sum_{k=1}^{n}\frac{H_k}{k} &= \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)H_{n+1} - \sum_{k=1}^{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)\frac{1}{k+1} \\
&= H_nH_{n+1} - \sum_{k=1}^{n}\frac{H_{k+1}-\frac{1}{k+1}}{k+1} \\
&= H_nH_{n+1} - \sum_{k=1}^{n}\frac{H_{k+1}}{k+1} + \sum_{k=1}^{n}\frac{1}{(k+1)^2} \\
&= H_nH_{n+1} - \sum_{m=2}^{n+1}\frac{H_m}{m} + \sum_{m=2}^{n+1}\frac{1}{m^2} \\
&= H_nH_{n+1} - \sum_{m=1}^{n}\frac{H_m}{m} - \frac{H_{n+1}}{n+1} + \sum_{m=1}^{n}\frac{1}{m^2} + \frac{1}{(n+1)^2} \\
&= H_n^2+1+\frac{1}{2^2}+\cdots+\frac{1}{n^2} - \sum_{m=1}^{n}\frac{H_m}{m},
\end{aligned}$$

and the lemma is proved.

LEMMA 3 (A special harmonic sum) *The following equality holds*:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right) = \frac{7}{4}\zeta(4).$$

Proof. We apply Abel's summation formula (2), with $a_n = \frac{1}{n^2}$ and $b_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$, and we have

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}}{n^2} \\
&= \lim_{n\to\infty} \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \right) \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{(n+1)^2} \right) \\
&\quad - \sum_{n=1}^{\infty} \frac{\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}}{(n+1)^2} \\
&= \zeta^2(2) - \sum_{n=1}^{\infty} \frac{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n+1)^2}}{(n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^4} \\
&= \zeta^2(2) - \sum_{m=2}^{\infty} \frac{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{m^2}}{m^2} + \sum_{m=2}^{\infty} \frac{1}{m^4} \\
&= \zeta^2(2) - S + \zeta(4) \\
&= \frac{7}{2}\zeta(4) - S.
\end{aligned}$$

We used that $\zeta^2(2) = \frac{5}{2}\zeta(4)$ since $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$ [9, p. 605].

The summation identity in Lemma 3 is known in the literature. It is a special case of the following series formula involving the generalized harmonic number (see [10, Formula (1.1)]):

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^p} = \frac{1}{2}\left(\zeta^2(p) + \zeta(2p)\right).$$

Now we are ready to prove Theorem 1.

Proof. We have, based on part (b) of Lemma 2, that

$$\int_0^1 \frac{x^{n-1}}{n} \ln^2(1-x)\mathrm{d}x = \frac{H_n^2}{n^2} + \frac{1}{n^2}\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right),$$

and it follows that

$$\sum_{n=1}^{\infty} \int_0^1 \frac{x^{n-1}}{n} \ln^2(1-x)\mathrm{d}x = \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2}\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right). \tag{3}$$

Using Tonelli's theorem [8, p. 309] we are allowed to bring the sum under the integral and we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\int_0^1 \frac{x^{n-1}}{n}\ln^2(1-x)\mathrm{d}x &= \int_0^1 \sum_{n=1}^{\infty}\frac{x^{n-1}}{n}\ln^2(1-x)\mathrm{d}x \\
&= -\int_0^1 \frac{\ln^3(1-x)}{x}\mathrm{d}x \\
&= -\int_0^1 \frac{\ln^3 y}{1-y}\mathrm{d}y \\
&= -\int_0^1 \ln^3 y \sum_{k=0}^{\infty} y^k \mathrm{d}y \\
&= -\sum_{k=0}^{\infty}\int_0^1 y^k \ln^3 y\,\mathrm{d}y \\
&= 6\sum_{k=0}^{\infty}\frac{1}{(k+1)^4} \\
&= 6\zeta(4),
\end{aligned} \tag{4}$$

since $\int_0^1 y^k \ln^3 y\,\mathrm{d}y = -\frac{6}{(k+1)^4}$. Combining (3), (4) and Lemma 3 we get that

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2 = \frac{17}{4}\zeta(4),$$

and the theorem is proved.

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