

## SIDELNIKOV INEQUALITY AND LEGENDRE POLYNOMIALS

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*Abstract.* Extremal arrangements (in various meanings of “extremal”) of points on a sphere have long been a subject of much interest (see [3] for a survey of early works).

One class of such extremal arrangements, consisting of “semidesigns” introduced in [4], is related to so-called Sidelnikov inequality [8]. In this paper we present a very simple proof of a discrete case of Sidelnikov inequality and show that it becomes equality on semidesigns.

### 1. Introduction

Let  $\mathbb{R}^3$  denote Euclidean 3-space,  $\langle x, y \rangle$  the scalar product,  $\|x\|$  the norm  $\sqrt{\langle x, x \rangle}$  and  $S^2$  the unit sphere  $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ .

The following is a discrete case of Sidelnikov inequality in  $\mathbb{R}^3$ .

**THEOREM 1.** *Let  $x_1, \dots, x_m$  be any vectors in  $S^2$  and  $k$  a positive integer. Then*

$$\sum_{i=1}^m \sum_{j=1}^m \langle x_i, x_j \rangle^{2k} \geq \frac{m^2}{2k+1}. \quad (1)$$

Sidelnikov’s proof of (1) is rather sophisticated. Goethals, Seidel [2] and Venkov [1] gave some other proofs. Our proof given below is based on Legendre polynomials.

### 2. Some properties of Legendre polynomials

Legendre polynomials  $\{P_n\}_{n=0}^\infty$  can be defined by the conditions:  $\deg P_n(t) = n$ ,  $P_n(1) = 1$  and

$$\int_{-1}^1 P_n(t)P_m(t) dt = 0, \quad n \neq m. \quad (2)$$

Any three consecutive polynomials  $P_{n-1}$ ,  $P_n$ ,  $P_{n+1}$  are connected by the following relation (see e.g. [6])

$$tP_n(t) = \frac{n+1}{2n+1}P_{n+1}(t) + \frac{n}{2n+1}P_{n-1}(t). \quad (3)$$

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Also, we have  $P_0(t) = 1$ ,  $P_1(t) = t$ .

We need the expansion of  $t^{2k}$  in terms of Legendre polynomials. For  $k = 1$  we get  $t^2 = tP_1(t) = \frac{2}{3}P_2(t) + \frac{1}{3}P_0(t)$ . Multiply this equation by  $t$  and apply (3). We obtain

$$t^3 = \frac{2}{3} \left[ \frac{3}{5}P_3(t) + \frac{2}{5}P_1(t) \right] + \frac{1}{3}P_1(t) = \frac{2}{5}P_3(t) + \frac{3}{5}P_1(t).$$

Multiplying by  $t$  again, we get

$$t^4 = \frac{8}{35}P_4(t) + \frac{4}{7}P_2(t) + \frac{1}{5}P_0(t).$$

Continuing this process ad infinitum, we obtain an expression of the form

$$t^{2k} = \sum_{l=0}^k c_l P_{2k-2l}(t), \quad (4)$$

with  $c_l > 0$  for  $l = 0, \dots, k$ . Also, we can find explicitly the last coefficient  $c_k$ . To this end, we integrate (4) over  $[-1, 1]$ :

$$\int_{-1}^1 t^{2k} dt = c_k \int_{-1}^1 dt,$$

hence  $c_k = 1/(2k + 1)$ .

In our proof of inequality (1), a decisive role is played by the so-called nonnegative definiteness of Legendre polynomials.

Let  $n$  be a positive integer. For any  $n > 0$  there is the “addition formula”

$$P_n(\langle x, y \rangle) = \frac{4\pi}{2n+1} \sum_{s=1}^{2n+1} Y_{ns}(x)Y_{ns}(y), \quad x, y \in S^2. \quad (5)$$

Here  $\{Y_{ns}\}$  are spherical functions of order  $n$ .

For any  $X = \{x_1, \dots, x_m\} \subset S^2$  consider the sum

$$S_n(X) = \sum_{i=1}^m \sum_{j=1}^n P_n(\langle x_i, x_j \rangle).$$

By (5) we can rewrite this as

$$S_n(X) = \frac{4\pi}{2n+1} \sum_{s=1}^{2n+1} \left[ \sum_{i=1}^m Y_{ns}(x_i) \right]^2, \quad (6)$$

which yields  $S_n(X) \geq 0$  — the above nonnegative definiteness.

### 3. Proof of theorem 1

Let again system  $X = \{x_1, \dots, x_m\}$  be a subset of  $S^2$ . Taking  $t = \langle x_i, x_j \rangle$  in (4) and summing over  $i, j = 1, \dots, m$ , we get

$$S := \sum_{i,j=1}^m \langle x_i, x_j \rangle^{2k} = \sum_{l=0}^k c_l S_{2k-2l}(X) \quad (7)$$

with  $c_l > 0$  and  $S_{2k-2l}(X) \geq 0$ , hence  $S \geq c_k S_0(X) = \frac{m^2}{2k+1}$ .  $\square$

### 4. Conditions for the equality case in (1)

Inequality (1) turns to equality on spherical semidesigns. Spherical semidesigns have been introduced in [4]. One of equivalent definitions is the following

DEFINITION 1. (Spherical Semidesign) A system  $\Phi = \{\phi_1, \dots, \phi_m\}$  on  $S^2$  is called spherical semidesign of order  $2k$ , iff

$$S_n(\Phi) := \sum_{i,j=1}^m P_n(\langle \phi_i, \phi_j \rangle) = 0 \quad \text{for } n = 2, 4, 6, \dots, 2k. \quad (8)$$

Let us compare this with the notion of spherical design introduced in [5]. One of the possible definitions for a spherical design of order  $p$  is the following: a system  $\Phi$  is a spherical design of order  $p$  iff  $S_n(\Phi) = 0$  for all  $n = 1, 2, 3, \dots, p$  (see [7]).

THEOREM 2. Let  $X = \{x_1, \dots, x_m\}$  be any system on  $S^2$ . The equation

$$\sum_{i,j=1}^m \langle x_i, x_j \rangle^{2k} = \frac{m^2}{2k+1} \quad (9)$$

holds if and only if  $X$  is a spherical semidesign of order  $2k$ .

*Proof.* Let  $S$  denote the left-hand side of (9). The proof is based on representation (7). We rewrite it in the form

$$S = \sum_{l=0}^{k-1} c_l S_{2k-2l}(X) + \frac{m^2}{2k+1},$$

where  $c_l > 0$ ,  $l = 0, \dots, k-1$ . If  $S = m^2/(2k+1)$  then

$$\sum_{l=0}^{k-1} c_l S_{2k-2l}(X) = 0. \quad (10)$$

From  $S_{2k-2l}(X) \geq 0$ ,  $l = 0, \dots, k-1$ , it follows that  $S_{2k-2l}(X) = 0$  for all  $l = 0, \dots, k-1$ , that is,  $S_n(X) = 0$  for  $n = 2, 4, \dots, 2k$ . By the definition 1, the system  $X$  is a spherical semidesign of order  $2k$ .

Conversely, if  $X$  is a spherical semidesign of order  $2k$ , then (10) holds which yields  $S = m^2/(2k+1)$ .  $\square$

EXAMPLE 1. Consider an icosahedron inscribed in  $S^2$ . Let the points  $\{x_1, \dots, x_6, -x_1, \dots, -x_6\}$  be its vertices. Then  $\langle x_i, x_j \rangle = \pm 1/\sqrt{5}$  for  $i \neq j$ . Hence

$$\sum_{i,j=1}^6 \langle x_i, x_j \rangle^4 = 6 \cdot 1 + 30 \cdot \frac{1}{25} = \frac{36}{5}.$$

For  $k = 2$ ,  $m = 6$  we have  $m^2/(2k+1) = \frac{36}{5}$ . By theorem 2 the system  $X = \{x_1, \dots, x_6\}$  is a spherical semidesign of order 4. The set of all 12 vertices of icosahedron is a spherical design of order 5.

If we take cube, octahedron or dodecahedron inscribed in  $S^2$ , then the set of vertices for each of these polyhedra is a spherical design of order 3. At the same time, their “halves” are spherical semidesigns of order 2.

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