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# CERTAIN NON–LINEAR DIFFERENTIAL POLYNOMIALS HAVING COMMON POLES SHARING A NON ZERO POLYNOMIAL WITH FINITE WEIGHT

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*Abstract.* With the notion of weighted sharing we study the uniqueness property of meromorphic functions having common poles when certain non-inear differential polynomials share a non zero polynomial function. Our theorems in the paper will improve, extend and supplement a number of recent results in a more compact and convenient way.

## 1. Introduction, definitions and results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane.

Let $f$ and $g$ be two non-constant meromorphic functions and let $a$ be a finite complex number. We say that $f$ and $g$ share $a$ CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that $f$ and $g$ share $a$ IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that $f$ and $g$ share $\infty$ CM, if $1/f$ and $1/g$ share $0$ CM, and we say that $f$ and $g$ share $\infty$ IM, if $1/f$ and $1/g$ share $0$ IM.

We adopt the standard notations of value distribution theory (see [6]). We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\longrightarrow\infty$, outside of a possible exceptional set of finite linear measure.

A finite value $z_0$ is said to be a fixed point of $f(z)$ if $f(z_0)=z_0$. For a positive integer $m$ and a number $\mu$, let $m^*=\chi_\mu m$, where $\chi_\mu=0$ if $\mu=0$ and $\chi_\mu=1$ if $\mu\neq 0$. Throughout this paper, we need the following definition.

$$\Theta(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,a;f)}{T(r,f)},$$

where $a$ is a value in the extended complex plane.

We start with the following famous theorem of W.K. Hayman (see [5], Corollary of Theorem 9) obtained in 1959.

THEOREM A. *Let $f$ be a transcendental meromorphic function and $n(\geqslant 3)$ is an integer. Then $f^nf'=1$ has infinitely many solutions.*

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In 1997, Yang and Hua obtained the following uniqueness result corresponding to *Theorem A* :

THEOREM B. [17] *Let $f$ and $g$ be two non-constant meromorphic functions, $n \geqslant 11$ be a positive integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share $a$ CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c_1$, $c_2$ and $c$ are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant $t$ such that $t^{n+1} = 1$.*

Using the idea of sharing fixed points, in 2002, M.L. Fang and H.L. Qiu further extended *Theorem B* in the following manner.

THEOREM C. [4] *Let $f$ and $g$ be two non-constant meromorphic functions, and let $n \geqslant 11$ be a positive integer. If $f^n f' - z$ and $g^n g' - z$ share $0$ CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where $c_1$, $c_2$ and $c$ are three nonzero complex numbers satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a complex number $t$ such that $t^{n+1} = 1$.*

For the past few years researchers have become more interested in the value sharing of nonlinear differential polynomials which are the $k$-th derivative of some linear expression of $f$ and $g$.

In 2010, J.F. Xu, F. Lu and H.X. Yi proved the following results.

THEOREM D. [15] *Let $f$ and $g$ be two non-constant meromorphic functions, and let $n$, $k$ be two positive integers with $n > 3k + 10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share $z$ CM, $f$ and $g$ share $\infty$ IM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where $c_1$, $c_2$ and $c$ are three constants satisfying $4n^2 (c_1 c_2)^n c^2 = -1$ or $f \equiv tg$ for a constant $t$ such that $t^n = 1$.*

THEOREM E. [15] *Let $f$ and $g$ be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n}$, and let $n$, $k$ be two positive integers with $n \geqslant 3k + 12$. If $(f^n (f-1))^{(k)}$ and $(g^n (g-1))^{(k)}$ share $z$ CM, $f$ and $g$ share $\infty$ IM, then $f \equiv g$.*

In the mean time in 2008 Zhang and Lin [21, 22] obtained a more generalised result for entire function as follows.

THEOREM F. [21, 22] *Let $f$ and $g$ be two non-constant entire functions, and $n$, $m$, $k$ be three positive integers with $n > 2k + m^* + 4$. Suppose $(f^n(\mu f^m + \lambda))^{(k)}$, $(g^n(\mu g^m + \lambda))^{(k)}$ share $1$ CM, where $\lambda$, $\mu$ are constants such that $|\lambda| + |\mu| \neq 0$. If*

*(i) $\lambda\mu \neq 0$, and gcd $(n,m) = d$, then $f^d \equiv g^d$; especially when $d = 1$, $f \equiv g$. or while $m = 1$ and $\Theta(\infty, f) > 2/n$, then $f \equiv g$;*

*(ii) if $\lambda\mu = 0$, then either $f = tg$, where $t$ is a constant satisfying $t^{n+m^*} = 1$ or $f = c_1 e^{cz}$, $g = c_2 e^{-cz}$, where $c_1$, $c_2$ and $c$ are three constants such that $(-1)^k \lambda^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1$ or $(-1)^k \mu^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1$.*

In 2001 an idea of gradation of sharing of values was introduced in {[8], [9]} which measures how close a shared value is to being share CM or to being shared IM. This notion is known as weighted sharing and is defined as follows.

DEFINITION 1. [8, 9] Let $k$ be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f$, $g$ share the value $a$ with weight $k$.

The definition implies that if $f$, $g$ share a value $a$ with weight $k$ then $z_0$ is an $a$-point of $f$ with multiplicity $m\ (\leqslant k)$ if and only if it is an $a$-point of $g$ with multiplicity $m\ (\leqslant k)$ and $z_0$ is an $a$-point of $f$ with multiplicity $m\ (> k)$ if and only if it is an $a$-point of $g$ with multiplicity $n\ (> k)$, where $m$ is not necessarily equal to $n$.

We write $f$, $g$ share $(a,k)$ to mean that $f$, $g$ share the value $a$ with weight $k$. Clearly if $f$, $g$ share $(a,k)$, then $f$, $g$ share $(a,p)$ for any integer $p$, $0 \leqslant p < k$. Also we note that $f$, $g$ share a value $a$ IM or CM if and only if $f$, $g$ share $(a,0)$ or $(a,\infty)$ respectively. If $a$ is a small function we define that $f$ and $g$ share $(a,l)$ which means $f$ and $g$ share $a$ with weight $l$ if $f-a$ and $g-a$ share $(0,l)$.

With the notion of weighted sharing in 2011, X. Q. Lin [12] improved *Theorem F* as follows.

THEOREM G. [12] *Let $f$ and $g$ be two non-constant entire functions, and let $n$, $m$, and $k$ be three positive integers. Suppose $(f^n(\mu f^m+\lambda))^{(k)}$, $(g^n(\mu g^m+\lambda))^{(k)}$ share $(1,l)$, where $\lambda$, $\mu$ are constants such that $|\lambda|+|\mu|\neq 0$ and one of the following conditions holds.:*

*(i)* $l=2$ *and* $n>2k+m^*+4$;

*(ii)* $l=1$ *and* $n>\frac{5k+3m^*+9}{2}$;

*(iii)* $l=0$ *and* $n>5k+4m^*+7$.

*then conclusion of Theorem F holds.*

In 2012 Wang and Luo [13] investigated *Theorem F* for meromorphic functions and replaced value sharing by fixed point sharing.

THEOREM H. [13] *Let $f$ and $g$ be two transcendental meromorphic functions and $n$, $m$, $k$ be three positive integers with $n>3k+m^*+7$. Suppose $(f^n(\mu f^m+\lambda))^{(k)}$, $(g^n(\mu g^m+\lambda))^{(k)}$ share $(z,\infty)$, $f$, $g$ share $(\infty,0)$; where $\lambda(\neq 0)$, $\mu$ be constants. then one of the following results holds:*

*(i) if $\mu=0$, then either $f=tg$, where $t$ is a constant satisfying $t^n=1$, or $k=1$, $f=c_1e^{cz^2}$, $g=c_2e^{-cz^2}$, where $c_1$, $c_2$ and $c$ are three constants such that $4\lambda^2(c_1c_2)^n[nc]^2=-1$.*

*(ii) $\mu\neq 0$ and $m\geqslant 2$ and gcd $(n,m)=1$, then $f\equiv g$.*

*(iii) If $\mu\neq 0$ and $m=1$ then either $f\equiv g$ or*

$$f=-\frac{\lambda h(h^n-1)}{\mu(h^{n+1}-1)},\quad f=-\frac{\lambda(h^n-1)}{\mu(h^{n+1}-1)},$$

*where $h$ is a non-constant meromorphic function.*

Also J. Wang, W. Lu and Y. Chen [14] investigated the IM value sharing counterpart of *Theorem H* as follows.

THEOREM I. [14] *Let $f$ and $g$ be two non-constant meromorphic functions, and $n$, $k$, $m$ be three positive integers with $n>9k+6m^*+13$. Suppose $(f^n(\mu f^m+\lambda))^{(k)}$, $(g^n(\mu g^m+\lambda))^{(k)}$ share $(1,0)$, where $\lambda$, $\mu$ are constants such that $|\lambda|+|\mu|\neq 0$, and $f$, $g$ share $(\infty,0)$.*

*(i) If* $\lambda\mu \neq 0$, $m > 1$ *and* $(n, n+m) = 1$, *or while* $m = 1$ *and* $\Theta(\infty, f) > 2/n$, *then* $f \equiv g$;

*(ii) if* $\lambda\mu = 0$, *then either* $f = tg$, *where* $t$ *is a constant satisfying* $t^{n+m^*} = 1$ *or* $f = c_1 e^{cz^2}$, $g = c_2 e^{-cz^2}$, *where* $c_1$, $c_2$ *and* $c$ *are three constants such that* $(-1)^k \lambda^2 (c_1 c_2)^{n+m^*}$
$[(n+m^*)c]^{2k} = 1$ *or* $(-1)^k \mu^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1$.

The purpose of the paper is to unify all the above mentioned theorems into a single result under relaxed sharing hypothesis, which will improve, extend and generalize all the results discussed above in a large extent. We present the main result as follows.

THEOREM 1. *Let* $f$ *and* $g$ *be two transcendental meromorphic functions sharing* $(\infty, 0)$; $(f^n(\mu f^m + \lambda))^{(k)}$, $(g^n(\mu g^m + \lambda))^{(k)}$ *share* $(p(z), l)$, *where* $p(z)$ *be a nonzero polynomial with* $deg(p) = r$, $\lambda$, $\mu$ *are constants such that* $|\lambda| + |\mu| \neq 0$ *and* $n$, $m$, $k$ *be three positive integers. Suppose one of the following conditions hold:*

*(a)* $l \geqslant 3$ *and* $n > \max\{3k + m^* + 6, k + 2r\}$;

*(b)* $l = 2$ *and* $n > \max\{3k + m^* + 8, k + 2r\}$;

*(c)* $l = 1$ *and* $n > \max\{4k + \frac{3m^*}{2} + 9, k + 2r\}$;

*(d)* $l = 0$ *and* $n > \max\{9k + 4m^* + 14, k + 2r\}$.

*Then*

*(i) if* $\lambda\mu \neq 0$ *and (a)* $m = 1$, $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$; *or (b)* $m \geqslant 2$ *and for some constant* $t$, *satisfying* $t^d \equiv 1$,
*we have* $f \equiv tg$, *where* $d = (n+m, n)$.

*(ii) if* $\lambda\mu = 0$, *then either* $f = tg$, *where* $t$ *is a constant satisfying* $t^{n+m^*} = 1$ *or*

*if* $p(z)$ *is not a constant, then* $f = c_1 e^{cQ(z)}$, $g = c_2 e^{-cQ(z)}$, *where* $Q(z) = \int_0^z p(z)dz$, $c_1$, $c_2$ *and* $c$ *are constants such that* $a_{m^*}^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^2 = -1$;

*if* $p(z)$ *is a nonzero constant* $b$, *then* $f = c_3 e^{cz}$, $g = c_4 e^{-cz}$, *where* $c_3$, $c_4$ *and* $c$ *are constants such that* $(-1)^k a_{m^*}^2 (c_3 c_4)^{n+m^*} [(n+m^*)c]^{2k} = b^2$, *where* $a_{m^*} = \mu$, *when* $m^* = m$ *and* $a_{m^*} = \lambda$, *when* $m^* = 0$.

THEOREM 2. *Let* $f$ *and* $g$ *be two transcendental entire functions sharing* $(\infty, 0)$; $(f^n(\mu f^m + \lambda))^{(k)}$, $(g^n(\mu g^m + \lambda))^{(k)}$ *share* $(p(z), l)$, *where* $p(z)$ *be a nonzero polynomial with* $deg(p) = r$, $\lambda$, $\mu$ *are constants such that* $|\lambda| + |\mu| \neq 0$ *and* $n$, $m$, $k$ *be three positive integers. Suppose one of the following conditions holds:*

*(a)* $l \geqslant 2$ *and* $n > \max\{2k + m^* + 4, k + 2r\}$;

*(b)* $l = 1$ *and* $n > \max\{\frac{5k + 3m^* + 9}{2}, k + 2r\}$;

*(c)* $l = 0$ *and* $n > \max\{4k + 3m^* + 6, k + 2r\}$.

*Then*

*(i) if* $\lambda\mu \neq 0$ *and (a)* $m = 1$; *or (b)* $m \geqslant 2$ *and for some constant* $t$, *satisfying* $t^d \equiv 1$, *we have* $f \equiv tg$, *where* $d = (n+m, n)$.

*(ii)if* $\lambda\mu = 0$, *then either* $f = tg$, *where* $t$ *is a constant satisfying* $t^{n+m^*} = 1$ *or*

*if $p(z)$ is not a constant, then $f = c_1 e^{cQ(z)}$, $g = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z)dz$, $c_1$, $c_2$ and $c$ are constants such that $a_{m^*}^2 (c_1c_2)^{n+m^*}[(n+m^*)c]^2 = -1$;*

*if $p(z)$ is a nonzero constant $b$, then $f = c_3 e^{cz}$, $g = c_4 e^{-cz}$, where $c_3$, $c_4$ and $c$ are constants such that $(-1)^k a_{m^*}^2 (c_3c_4)^{n+m^*}[(n+m^*)c]^{2k} = b^2$, where $a_{m^*} = \mu$, when $m^* = m$ and $a_{m^*} = \lambda$, when $m^* = 0$.*

REMARK 1. In both the theorems when $p(z)$ is a constant $f$ and $g$ can be taken as non-constant instead of transcendental.

We now explain following definitions and notations which are used in the paper.

DEFINITION 2. [7] Let $p$ be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r,a;f \mid\geqslant p)$ ($\overline{N}(r,a;f \mid\geqslant p)$)denotes the counting function (reduced counting function) of those $a$-points of $f$ whose multiplicities are not less than $p$.

(ii) $N(r,a;f \mid\leqslant p)$ ($\overline{N}(r,a;f \mid\leqslant p)$)denotes the counting function (reduced counting function) of those $a$-points of $f$ whose multiplicities are not greater than $p$.

DEFINITION 3. {11, cf.[18]} For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer $p$ we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geqslant 2) + \dots \overline{N}(r,a;f \mid\geqslant p)$. Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

DEFINITION 4. Let $a,b \in \mathbb{C} \cup \{\infty\}$. Let $p$ be a positive integer. We denote by $\overline{N}(r,a;f \mid \geqslant p \mid g = b)$ ($\overline{N}(r,a;f \mid \geqslant p \mid g \neq b)$) the reduced counting function of those $a$-points of $f$ with multiplicities $\geqslant p$, which are the $b$-points (not the $b$-points) of $g$.

DEFINITION 5. {cf.[1], 2} Let $f$ and $g$ be two non-constant meromorphic functions such that $f$ and $g$ share the value 1 IM. Let $z_0$ be a 1-point of $f$ with multiplicity $p$, a 1-point of $g$ with multiplicity $q$. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of $f$ and $g$ where $p > q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of $f$ and $g$ where $p = q = 1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of $f$ and $g$ where $p = q \geqslant 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.

DEFINITION 6. {cf.[1], 2} Let $k$ be a positive integer. Let $f$ and $g$ be two non-constant meromorphic functions such that $f$ and $g$ share the value 1 IM. Let $z_0$ be a 1-point of $f$ with multiplicity $p$, a 1-point of $g$ with multiplicity $q$. We denote by $\overline{N}_{f>k}(r,1;g)$ the reduced counting function of those 1-points of $f$ and $g$ such that $p > q = k$. $\overline{N}_{g>k}(r,1;f)$ is defined analogously.

DEFINITION 7. [8, 9] Let $f$, $g$ share a value $a$ IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those $a$-points of $f$ whose multiplicities differ from the multiplicities of the corresponding $a$-points of $g$.

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

DEFINITION 8. Let $a,b_1,b_2,\dots,b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;f \mid g \neq b_1,b_2,\dots,b_q)$ the counting function of those $a$-points of $f$, counted according to multiplicity, which are not the $b_i$-points of $g$ for $i = 1,2,\dots,q$.

## 2. Lemmas

Let $F$ and $G$ be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by $H$ the function as follows:

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right), \tag{2.1}$$

and

$$V=\left(\frac{F'}{F-1}-\frac{F'}{F}\right)-\left(\frac{G'}{G-1}-\frac{G'}{G}\right). \tag{2.2}$$

LEMMA 1. [13] *Let $f$ be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ..., $a_0(z)$ be meromorphic functions such that $T(r,a_i(z))=S(r,f)$ for $i=0,1,2,\ldots,n$. Then*

$$T(r,a_nf^n+a_{n-1}f^{n-1}+\ldots+a_1f+a_0)=nT(r,f)+S(r,f).$$

LEMMA 2. [20] *Let $f$ be a non-constant meromorphic function, and $p$, $k$ be positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right)\leqslant T\left(r,f^{(k)}\right)-T(r,f)+N_{p+k}(r,0;f)+S(r,f), \tag{2.3}$$

$$N_p\left(r,0;f^{(k)}\right)\leqslant k\overline{N}(r,\infty;f)+N_{p+k}(r,0;f)+S(r,f). \tag{2.4}$$

LEMMA 3. [10] *If $N(r,0;f^{(k)}\mid f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of $f$, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}\mid f\neq 0)\leqslant k\overline{N}(r,\infty;f)+N(r,0;f\mid<k)+k\overline{N}(r,0;f\mid\geqslant k)+S(r,f).$$

LEMMA 4. *Suppose that $f$ and $g$ be two non-constant meromorphic functions. Let $F=[f^n(\mu f^m+\lambda)]^{(k)}$, $G=[g^n(\mu g^m+\lambda)]^{(k)}$, where $n$, $k$, $m$ are positive integers. If $f$, $g$ share $\infty$ IM and $V\equiv 0$, then $F\equiv G$.*

*Proof.* Suppose $V\equiv 0$. Then by integration we obtain

$$1-\frac{1}{F}\equiv A(1-\frac{1}{G}).$$

If $z_0$ is a pole of $f$ then it is a pole of $g$. Hence from the definition of $F$ and $G$ we have $\frac{1}{F(z_0)}=0$ and $\frac{1}{G(z_0)}=0$. So $A=1$ and hence $F\equiv G$. $\square$

LEMMA 5. [11] *Let $f_1$ and $f_2$ be two non-constant meromorphic functions satisfying $\overline{N}(r,0;f_i)+\overline{N}(r,\infty;f_i)=S(r;f_1,f_2)$ for $i=1,2$. If $f_1^sf_2^t-1$ is not identically zero for arbitrary integers $s$ and $t(|s|+|t|>0)$, then for any positive $\varepsilon$, we have*

$$N_0(r,1;f_1,f_2)\leqslant\varepsilon T(r)+S(r;f_1,f_2),$$

*where* $N_0(r,1;f_1,f_2)$ *denotes the reduced counting function related to the common* $1$*-points of* $f_1$ *and* $f_2$ *and* $T(r)=T(r,f_1)+T(r,f_2)$, $S(r;f_1,f_2)=o(T(r))$ *as* $r\longrightarrow\infty$ *possibly outside a set of finite linear measure.*

LEMMA 6. [6] *Suppose that* $f$ *is a non-constant meromorphic function,* $k\geqslant 2$ *is an integer. If*

$$N(r,\infty,f)+N(r,0;f)+N(r,0;f^{(k)})=S(r,\frac{f'}{f}),$$

*then* $f=e^{az+b}$*, where* $a\neq 0$*,* $b$ *are constants.*

LEMMA 7. *Let* $f$ *and* $g$ *be two non-constant meromorphic functions and* $k$*,* $m$*,* $n>3k+m^*$ *be three positive integers. If* $[f^n(\mu f^m+\lambda)]^{(k)}\equiv[g^n(\mu g^m+\lambda)]^{(k)}$*, then* $f^n(\mu f^m+\lambda)\equiv g^n(\mu g^m+\lambda)$.

*Proof.* We have $[f^n(\mu f^m+\lambda)]^{(k)}\equiv[g^n(\mu g^m+\lambda)]^{(k)}$.
When $k\geqslant 2$, integrating we get

$$[f^n(\mu f^m+\lambda)]^{(k-1)}\equiv[g^n(\mu g^m+\lambda)]^{(k-1)}+c_{k-1}.$$

If possible suppose $c_{k-1}\neq 0$.
Now in view of *Lemma* 2 for $p=1$ and using the second fundamental theorem we get

$$\begin{aligned}
&(n+m^*)T(r,f)\\
\leqslant\ & T(r,[f^n(\mu f^m+\lambda)]^{(k-1)})-\overline{N}(r,0;[f^n(\mu f^m+\lambda)]^{(k-1)})+N_k(r,0;f^n(\mu f^m+\lambda))\\
&+S(r,f)\\
\leqslant\ & \overline{N}(r,0;[f^n(\mu f^m+\lambda)]^{(k-1)})+\overline{N}(r,\infty;f)+\overline{N}(r,c_{k-1};[f^n(\mu f^m+\lambda)]^{(k-1)})\\
&-\overline{N}(r,0;[f^n(\mu f^m+\lambda)]^{(k-1)})+N_k(r,0;f^n(\mu f^m+\lambda))+S(r,f)\\
\leqslant\ & \overline{N}(r,\infty;f)+\overline{N}(r,0;[g^n(\mu g^m+\lambda)]^{(k-1)})+k\overline{N}(r,0;f)+N(r,0;\mu f^m+\lambda)+S(r,f)\\
\leqslant\ & \{k+1+m^*\}\,T(r,f)+(k-1)\overline{N}(r,\infty;g)+N_k(r,0;g^n(\mu g^m+\lambda))+S(r,f)\\
\leqslant\ & \{k+1+m^*\}\,T(r,f)+(k-1)\,\overline{N}(r,\infty;g)+k\,\overline{N}(r,0;g)+N(r,0;\mu g^m+\lambda)\\
&+S(r,f)\\
\leqslant\ & \{k+1+m^*\}\,T(r,f)+\{2k-1+m^*\}\,T(r,g)+S(r,f)+S(r,g)\\
\leqslant\ & \{3k+2m^*\}\,T(r)+S(r).
\end{aligned}$$

Similarly we get

$$(n+m^*)\,T(r,g)\leqslant\{3k+2m^*\}\,T(r)+S(r).$$

Combining these we get

$$(n-m^*-3k)\,T(r)\leqslant S(r),$$

which is a contradiction since $n>3k+m^*$.

Therefore $c_{k-1}=0$ and so $[f^n(\mu f^m+\lambda)]^{(k-1)} \equiv [g^n(\mu g^m+\lambda)]^{(k-1)}$. Repeating $k-1$ times, we obtain

$$f^n(\mu f^m+\lambda) \equiv g^n(\mu g^m+\lambda)+c_0.$$

If $k=1$, clearly integrating once we obtain the above. If possible suppose $c_0 \neq 0$.

Now using the second fundamental theorem we get

$$\begin{aligned}
&(n+m^*)T(r,f)\\
&\leqslant \overline{N}(r,0;f^n(\mu f^m+\lambda))+\overline{N}(r,\infty;f^n(\mu f^m+\lambda))+\overline{N}(r,c_0;f^n(\mu f^m+\lambda))\\
&\leqslant \overline{N}(r,0;f)+m^*T(r,f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g^n(\mu g^m+\lambda))\\
&\leqslant (m^*+2)\,T(r,f)+\overline{N}(r,0;g)+m^*\,T(r,g)+S(r,f)\\
&\leqslant (m^*+2)\,T(r,f)+(m^*+1)\,T(r,g)+S(r,f)+S(r,g)\\
&\leqslant (3+2m^*)\,T(r)+S(r).
\end{aligned}$$

Similarly we get

$$(n+m^*)\,T(r,g) \leqslant (3+2m^*)\,T(r)+S(r).$$

Combining these we get

$$(n-m^*-3)\,T(r) \leqslant S(r),$$

which is a contradiction since $n>3+m^*$.
Therefore $c_0=0$ and so

$$f^n(\mu f^m+\lambda) \equiv g^n(\mu f^m+\lambda).$$

This completes the Lemma. □

LEMMA 8. *Suppose that $f$ and $g$ be two non-constant meromorphic functions. $F$, $G$ be defined as in Lemma 4 and $H \not\equiv 0$. If $f$, $g$ share $(\infty,0)$ and $F$, $G$ share $(1,k_1)$, then*

$$\begin{aligned}
&\big(n+m^*-k-1\big)\overline{N}(r,\infty;f) \leqslant (k+m^*+1)\{T(r,f)+T(r,g)\}+\overline{N}_*(r,1;F,G)\\
&+S(r,f)+S(r,g).
\end{aligned}$$

*Similar result holds for $g$ also.*

*Proof.* Suppose $\infty$ is an e.v.P. of $f$ and $g$ then the lemma follows immediately.

Next suppose $\infty$ is not an e.v.P of $f$ and $g$. Since $H \not\equiv 0$ from *Lemma* 4 we have $V \not\equiv 0$. We suppose that $z_0$ is a pole of $f$ with multiplicity $q$ and a pole of $g$ with multiplicity $r$. Clearly $z_0$ is a pole of $F$ with multiplicity $(n+m)q+k$ and a pole of $G$ with multiplicity $(n+m)r+k$. Noting that $f$, $g$ share $(\infty,0)$ from the definition of $V$ it is clear that $z_0$ is a zero of $V$ with multiplicity at least $n+m+k-1$. Now using the Milloux theorem [6], p. 55, and *Lemma* 1, we obtain from the definition of $V$ that

$$m(r,V)=S(r,f)+S(r,g).$$

Thus using Lemma 1 and (2.4) we get

$$\begin{aligned}
&\big(n+m^*+k-1\big)\overline{N}(r,\infty;f)\\
\leqslant\ & N(r,0;V)\\
\leqslant\ & T(r,V)+O(1)\\
\leqslant\ & N(r,\infty;V)+m(r,V)+O(1)\\
\leqslant\ & \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)\\
&+S(r,f)+S(r,g)\\
\leqslant\ & N_{k+1}(r,0;f^n(\mu f^m+\lambda))+N_{k+1}(r,0;g^n(\mu g^m+\lambda))+k\overline{N}(r,\infty;f)\\
&+k\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
\leqslant\ & N_{k+1}(r,0;f^n)+N_{k+1}(r,0;(\mu f^m+\lambda))+N_{k+1}(r,0;g^n)\\
&+N_{k+1}(r,0;(\mu g^m+\lambda))+2k\overline{N}(r,\infty;f)+\overline{N}_*(r,1;F,G)\\
&+S(r,f)+S(r,g)\\
\leqslant\ & (k+1)\overline{N}(r,0;f)+N(r,0;(\mu f^m+\lambda))+(k+1)\overline{N}(r,0;g)\\
&+N(r,0;(\mu g^m+\lambda))+2k\overline{N}(r,\infty;f)+\overline{N}_*(r,1;F,G)\\
&+S(r,f)+S(r,g).
\end{aligned}$$

This gives

$$\begin{aligned}
\big(n+m^*-k-1\big)\overline{N}(r,\infty;f)\leqslant (k+m^*+1)\{T(r,f)+T(r,g)\}+\overline{N}_*(r,1;F,G)\\
+S(r,f)+S(r,g).
\end{aligned}$$

This completes the proof of the lemma. □

LEMMA 9. *Let $f$, $g$ be two transcendental meromorphic functions and $F=\frac{[f^n(\mu f^m+\lambda)]^{(k)}}{p(z)}$, $G=\frac{[g^n(\mu g^m+\lambda)]^{(k)}}{p(z)}$, where $p(z)$ is a non zero polynomial with $deg(p)=r$, $n(\geqslant 1)$, $k(\geqslant 1)$, $m(\geqslant 2)$ are positive integers such that $n>3k+m^*+3$. If $f$, $g$ share $(\infty,0)$ and $H\equiv 0$ then either $[f^n(\mu f^m+\lambda)]^{(k)}[g^n(\mu g^m+\lambda)]^{(k)}\equiv p^2$ or $f^n(\mu f^m+\lambda)\equiv g^n(\mu g^m+\lambda)$.*

*Proof.* Since $H\equiv 0$, on integration we get

$$\frac{1}{F-1}\equiv\frac{bG+a-b}{G-1}, \tag{2.5}$$

where $a$, $b$ are constants and $a\neq 0$. From (2.5) it is clear that $F$ and $G$ share $(1,\infty)$. We now consider the following cases.

**Case 1.** Let $b\neq 0$ and $a\neq b$.

If $b=-1$, then from (2.5) we have

$$F\equiv\frac{-a}{G-a-1}.$$

Therefore

$$\overline{N}(r,a+1;G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)+S(r,f).$$

So in view of *Lemma* 2 and the second fundamental theorem we get

$$\begin{aligned}
&(n+m^*)\, T(r,g)\\
&\leqslant T(r,G)+N_{k+1}(r,0;g^n(\mu g^m+\lambda))-\overline{N}(r,0;G)+S(r,g)\\
&\leqslant \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,a+1;G)+N_{k+1}(r,0;g^n(\mu g^m+\lambda))-\overline{N}(r,0;G)+S(r,g)\\
&\leqslant \overline{N}(r,\infty;g)+N_{k+1}(r,0;g^n(\mu g^m+\lambda))+\overline{N}(r,\infty;f)+S(r,f)+S(r,g)\\
&\leqslant \overline{N}(r,\infty;g)+N_{k+1}(r,0;g^n)+N_{k+1}(r,0;(\mu g^m+\lambda))+\overline{N}(r,\infty;f)+S(r,f)+S(r,g)\\
&\leqslant 2\overline{N}(r,\infty;g)+(k+1)\overline{N}(r,0;g)+T(r,(\mu g^m+\lambda))+S(r,f)+S(r,g)\\
&\leqslant (k+m^*+3)\, T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set $I$ with infinite measure such that $T(r,f)\leqslant T(r,g)$ for $r\in I$. So for $r\in I$, $S(r,f)$ can be replaced by $S(r,g)$. So for $r\in I$, we get a contradiction from above since $n>3k+m^*+3$.

If $b\neq -1$, from (2.5) we obtain that

$$F-(1+\frac{1}{b})\equiv \frac{-a}{b^2[G+\frac{a-b}{b}]}.$$

So

$$\overline{N}(r,\frac{(b-a)}{b};G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)+S(r,f).$$

Using *Lemma* 2 and the same argument as used in the case when $b=-1$ we can get a contradiction.

**Case 2.** Let $b\neq 0$ and $a=b$.

If $b=-1$, then from (2.5) we have

$$FG\equiv p^2,$$

that is

$$[f^n(\mu f^m+\lambda)]^{(k)}[g^n(\mu g^m+\lambda)]^{(k)}\equiv p^2.$$

If $b\neq -1$, from (2.5) we have

$$\frac{1}{F}\equiv\frac{bG}{(1+b)G-1}.$$

Therefore

$$\overline{N}(r,\frac{1}{1+b};G)=\overline{N}(r,0;F).$$

So in view of *Lemma* 2 and the second fundamental theorem we get

$$\begin{aligned}
&(n+m^*)\, T(r,g)\\
&\leqslant T(r,G)+N_{k+1}(r,0;g^n(\mu g^m+\lambda))-\overline{N}(r,0;G)+S(r,g)\\
&\leqslant \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,\frac{1}{1+b};G)+N_{k+1}(r.0;g^n(\mu g^m+\lambda))\\
&-\overline{N}(r,0;G)+S(r,g)
\end{aligned}$$

$$\begin{aligned}
&\leqslant \overline{N}(r,\infty;g)+(k+1)\overline{N}(r,0;g)+T(r,(\mu g^m+\lambda))+\overline{N}(r,0;F)+S(r,g)\\
&\leqslant \overline{N}(r,\infty;g)+(k+1)\overline{N}(r,0;g)+T(r,(\mu g^m+\lambda))+(k+1)\overline{N}(r,0;f)\\
&\quad +T(r,(\mu f^m+\lambda))+k\overline{N}(r,\infty;f)+S(r,f)+S(r,g)\\
&\leqslant (k+m^*+2)\,T(r,g)+(2k+m^*+1)\,T(r,f)+S(r,f)+S(r,g).
\end{aligned}$$

So for $r\in I$ we have

$$(n+m^*)\,T(r,g)\leqslant (3k+2m^*+3)\,T(r,g)+S(r,g),$$

which is a contradiction since $n>3k+m^*+3$.

**Case 3.** Let $b=0$. From (2.5) we obtain

$$F\equiv \frac{G+a-1}{a}. \tag{2.6}$$

If $a\neq 1$ then from (2.6) we obtain

$$\overline{N}(r,1-a;G)=\overline{N}(r,0;F).$$

We can similarly deduce a contradiction as in *Case 2*. Therefore $a=1$ and from (2.6) we obtain

$$F\equiv G.$$

Then by the *Lemma* 7 we have

$$f^nP(f)\equiv g^nP(g).$$

□

LEMMA 10. *Let $f$, $g$ be two transcendental meromorphic functions and $p(z)$ be a non-constant polynomial, where $n$ and $k\geqslant 2$ be two positive integers. If $f=e^{\alpha}$ and $g=e^{\beta}$, where $\alpha$, $\beta$ are non-constant entire functions such that $[f^n]^{(k)}-p(z)$ and $[g^n]^{(k)}-p(z)$ share 0 CM, then $[f^n]^{(k)}[g^n]^{(k)}\not\equiv p^2$.*

*Proof.* Suppose

$$[f^n]^{(k)}[g^n]^{(k)}\equiv p^2. \tag{2.7}$$

From (2.7) we have

$$N(r,0;[f^n]^{(k)})=S(r,f) \quad\text{and}\quad N(r,0;[g^n]^{(k)})=S(r,g).$$

Let

$$F_1=\frac{[f^n]^{(k)}}{p} \quad \textit{and} \quad G_1=\frac{[g^n]^{(k)}}{p}. \tag{2.8}$$

We note that $T(r,F_1)\leqslant n(k+1)T(r,f)+S(r,f)$ and so $T(r,F_1)=O(T(r,f))$. By *Lemma* 2, one can obtain $T(r,f)=O(T(r,F_1))$. Hence $S(r,F_1)=S(r,f)$. Similarly we get $S(r,G_1)=S(r,g)$. From (2.7) we get

$$F_1G_1\equiv 1. \tag{2.9}$$

It is clearl that $T(r,F_1)=T(r,G_1)+O(1)$. So $S(r,F_1)=S(r,G_1)$. If $F_1 \equiv cG_1$, where $c$ is a nonzero constant, then $F_1$ is a constant and so $f$ is a polynomial, which contradicts our assumption. Hence $F_1 \not\equiv cG_1$ and so in the view of (2.9) we see that $F_1$ and $G_1$ share $-1$ IM.

Now by *Lemma* 2 we have

$$N(r,0;F_1) \leqslant nN(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f) \leqslant S(r,F_1).$$

Similarly we have

$$N(r,0;G_1) \leqslant nN(r,0;g)+k\overline{N}(r,\infty;g)+S(r,g) \leqslant S(r,G_1).$$

Also we see that

$$N(r,\infty;F_1)=S(r,F_1), \quad N(r,\infty;G_1)=S(r,G_1).$$

Let

$$f_1 = \frac{F_1}{G_1}.$$

and

$$f_2 = \frac{F_1-1}{G_1-1}.$$

Clearly $f_1$ is non-constant. If $f_2$ is a nonzero constant then $F_1$ and $G_1$ share $\infty$ CM and so from (2.9) we conclude that $F_1$ and $G_1$ have no poles.

Next we suppose that $f_2$ is non-constant. We see that

$$F_1 = \frac{f_1(1-f_2)}{f_1-f_2}, \quad G_1 = \frac{1-f_2}{f_1-f_2}.$$

Clearly

$$T(r,F_1) \leqslant 2[T(r,f_1)+T(r,f_2)]+O(1)$$

and

$$T(r,f_1)+T(r,f_2) \leqslant 4T(r,F_1)+O(1).$$

These give $S(r,F_1)=S(r;f_1,f_2)$. Also we note that

$$\overline{N}(r,0;f_i)+\overline{N}(r,\infty;f_i)=S(r;f_1,f_2)$$

for $i=1,2$.

We note that $\overline{N}(r,-1;F_1) \neq S(r,F_1)$, since otherwise by the second fundamental theorem $F_1$ will be a constant.

Also we see that

$$\overline{N}(r,-1;F_1) \leqslant N_0(r,1;f_1,f_2).$$

Thus we have

$$T(r,f_1)+T(r,f_2)\leqslant 4\,N_0(r,1;f_1,f_2)+S(r,F_1).$$

Then by *Lemma* 5 there exist two mutually prime integers $s$ and $t(|s|+|t|>0)$ such that

$$f_1^s f_2^t \equiv 1,$$

i.e.,

$$\left[\frac{F_1}{G_1}\right]^s\left[\frac{F_1-1}{G_1-1}\right]^t\equiv 1. \tag{2.10}$$

If either $s$ or $t$ is zero then we arrive at a contradiction and so $st\neq 0$.

We now consider following cases:

**Case (i):** Suppose $s>0$ and $t=-t_1$, where $t_1>0$. Then we have

$$\left[\frac{F_1}{G_1}\right]^s\equiv\left[\frac{F_1-1}{G_1-1}\right]^{t_1}. \tag{2.11}$$

Let $z_1$ be a pole of $F_1$ of multiplicity $p$. Then from (2.11) we see that $z_1$ must be a zero of $G_1$ of multiplicity $p$. Now from (2.11) we get $2s=t_1$, which is impossible. Hence $F_1$ has no pole. Similarly we can prove that $G_1$ also has no poles.

**Case (ii):** Suppose either $s>0$ and $t>0$ or $s<0$ and $t<0$. Then from (2.11) one can easily prove that $F_1$ and $G_1$ have no poles.

Consequently from (2.9) we see that $F_1$ and $G_1$ have no zeros.

Since $F_1$ and $G_1$ have no zeros and poles, we have

$$F_1\equiv e^{\gamma_1}G_1,$$

i.e.,

$$[f^n]^{(k)}\equiv e^{\gamma_1}[g^n]^{(k)}, \tag{2.12}$$

where $\gamma_1$ is a non-constant entire function.

First suppose that $\alpha$ and $\beta$ both are both transcendental entire functions. Moreover from (2.7) we see that we see that

$$N(r,0;[f^n]^{(k)})\leqslant N(r,0;p^2)=O(logr)$$

and we see that

$$N(r,0;[g^n]^{(k)})\leqslant N(r,0;p^2)=O(logr).$$

From this we get

$$N(r,\infty;f^s)+N(r,0;f^s)+N(r,0;[f^s]^{(k)})=S(r,n\alpha^{'})=S(r,\frac{[f^n]^{'}}{f^n}) \tag{2.13}$$

and

$$N(r,\infty;g^s)+N(r,0;g^s)+N(r,0;[g^s]^{(k)})=S(r,n\beta^{'})=S(r,\frac{[g^n]^{'}}{g^n}). \tag{2.14}$$

Then from (2.13), (2.14) and *Lemma* 6 we must have

$$f=e^{az+b}, g=e^{cz+d}, \tag{2.15}$$

where $a\neq 0$, $b$, $c\neq 0$ and $d$ are constants. But these types of $f$ and $g$ do not agree with the relation (2.7).

Next suppose $\alpha$, $\beta$ both are polynomials. Since $f=e^{\alpha}$ and $g=e^{\beta}$, it follows that

$$[f^n]^{(k)}=A[(\alpha^{'})^k+P_{k-1}(\alpha^{'})]e^{n\alpha},\quad [g^n]^{(k)}=B[(\beta^{'})^k+P_{k-1}(\beta^{'})]e^{n\beta},$$

where $A$, $B$ are non-zero constants, $P_{k-1}(\alpha^{'})$, $P_{k-1}(\beta^{'})$ are differential polynomials in $\alpha^{'}$ and $\beta^{'}$ of degree at most $k-1$ respectively. From (2.7) we see that $\alpha+\beta=C$, i.e., $\alpha^{'}=\beta^{'}$. So $deg(\alpha)=deg(\beta)$.

If $deg(\alpha)=deg(\beta)=1$, then from (2.7) we again get a contradiction. So we suppose $deg(\alpha)=deg(\beta)=\geqslant 2$. From (2.12) we see that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 0 CM. So we have for some non zero constant $D$

$$[(\alpha^{'})^k+P_{k-1}(\alpha^{'})]\equiv D[(\beta^{'})^k+Q_{k-1}(\beta^{'})],$$

which is impossible as $k\geqslant 2$.

Actually $[(\alpha^{'})^k+P_{k-1}(\alpha^{'})]$ and $[(\beta^{'})^k+Q_{k-1}(\beta^{'})]$ contain the terms $(\alpha^{'})^k+K(\alpha^{'})^{k-2}\alpha^{''}$ and $(\beta^{'})^k+K(\beta^{'})^{k-2}\beta^{''}$ respectively, where $K$ is a suitably chosen positive integer. But these two terms are not identical. □

LEMMA 11. *([19], Lemma 6) If $H\equiv 0$, then $F$, $G$ share 1 CM. If further $F$, $G$ share $\infty$ IM then $F$, $G$ share $\infty$ CM.*

LEMMA 12. *Let $f$ and $g$ be two transcendental meromorphic functions, let $p(z)$ be a nonzero polynomial with $deg(p)=r$; $n$, $k$ and $m$ be three positive integers with $n>k+2r$. Suppose that $H\equiv 0$. If $[f^n(\mu f^m+\lambda)]^{(k)}[g^n(\mu g^m+\lambda)]^{(k)}\equiv p^2$, where $\lambda$, $\mu$ are constants such that $|\lambda|+|\mu|\neq 0$, $f$ and $g$ share $(\infty,0)$; if $p(z)$ is not a constant, then $f=c_1e^{cQ(z)}$, $g=c_2e^{-cQ(z)}$, where $Q(z)=\int_0^z p(z)dz$, $c_1$, $c_2$ and $c$ are constants such that $a_{m^*}^2(c_1c_2)^{n+m^*}[(n+m^*)c]^2=-1$, if $p(z)$ is a nonzero constant $b$, then $f=c_3e^{cz}$, $g=c_4e^{-cz}$, where $c_3$, $c_4$ and $c$ are constants such that $(-1)^k a_{m^*}^2(c_3c_4)^{n+m^*}[(n+m^*)c]^{2k}=b^2$, where $a_{m^*}=\mu$, when $m^*=m$ and $a_{m^*}=\lambda$, when $m^*=0$. Also when $p(z)$ is a nonzero constant $b$, then $f$ and $g$ can be taken as non-constant.*

*Proof.* Since $H\equiv 0$. It follows from *Lemma* 11 that $F$, $G$ share 1 CM. So $[f^n]^{(k)}-p(z)$ and $[g^n]^{(k)}-p(z)$ share 0 CM except the zeros of $p(z)$. Let

$$[f^n(\mu f^m+\lambda)]^{(k)}[g^n(\mu g^m+\lambda)]^{(k)}\equiv p^2. \tag{2.16}$$

First suppose that $\lambda\mu\neq 0$

Note that $f$ and $g$ share $(\infty,0)$, we have $f\neq\infty$, $g\neq\infty$ Suppose that $z_0$ is a zero of $f$ of order $p$, then $z_0$ will be a zero of $[(f^n(\mu f^m+\lambda)]^{(k)}$ of order $np-k$. Since $n>k+2r$, we can deduce that $z_0$ must be a zero of $p^2(z)$ with order at least

$2r+1$. This is impossible. Thus $f$ has no zero. Similarly $g$ has no zero. So $f = e^{\alpha(z)}$, $g = e^{\beta(z)}$, where $\alpha(z)$ and $\beta(z)$ are two non constant entire functions. Then we get

$$(\mu f^{n+m})^{(k)} = t_2(\alpha', \alpha'', ..., \alpha^{(k)})e^{(n+m)\alpha}, \tag{2.17}$$

$$(\lambda f^n)^{(k)} = t_1(\alpha', \alpha'', ..., \alpha^{(k)})e^{n\alpha}, \tag{2.18}$$

where $t_i(\alpha', \alpha'', ..., \alpha^{(k)})$ $(i = 1, 2)$ are differential polynomials in $\alpha'$, $\alpha''$, . . . , $\alpha^{(k)}$. Obviously

$$t_i(\alpha', \alpha'', ..., \alpha^{(k)}) \not\equiv 0$$

for $i = 1, 2$. From (2.16) and (2.17) we obtain

$$\begin{aligned} &\overline{N}(r, 0; t_2(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{m\alpha(z)} + t_1(\alpha', \alpha'', \ldots, \alpha^{(k)})) \\ \leqslant\ &N(r, 0; p^2(z)) = S(r, f). \end{aligned} \tag{2.19}$$

Since $\alpha$ is an entire function, we obtain $T(r, \alpha^{(j)}) = S(r, f)$ for $j = 1, 2$. Hence $T(r, t_i) = S(r, f)$ for $i = 1, 2$.

So from (2.19) we obtain

$$\begin{aligned} mT(r, f) &= T(r, t_2 e^{m\alpha}) + S(r, f) \\ &\leqslant \overline{N}(r, 0; t_2 e^{m\alpha}) + \overline{N}(r, 0; t_2 e^{m\alpha} + t_1) + S(r, f) \\ &= S(r, f), \end{aligned}$$

which is a contradiction.

Hence we have $\lambda\mu = 0$. Here also $f = e^{\alpha}$ and $g = e^{\beta}$, where $\alpha$ and $\beta$ are two non constant entire function. Then from (2.16) we have

$$a_{m^*}^2 [f^{n+m^*}]^{(k)} [g^{n+m^*}]^{(k)} \equiv p^2. \tag{2.20}$$

Let $s = n + m^*$.

**Case1:** Let $deg(p(z)) = r(\geqslant 1)$. First suppose $k \geqslant 2$. Then from *Lemma* 10 we get a contradiction.

Next suppose $k = 1$. Suppose that $\alpha$ and $\beta$ are transcendental. Then from (2.20) we get

$$AB\alpha' \beta' e^{s(\alpha+\beta)} \equiv p^2(z), \tag{2.21}$$

where $AB = (n + m^*)^2 a_{m^*}^2$.

Let $\alpha + \beta = \gamma$. From (2.21) we know that $\gamma$ is not a constant since in that case we get a contradiction. Now from (2.21) we get

$$AB\alpha' (\gamma' - \alpha')e^{n\gamma} \equiv p^2(z). \tag{2.22}$$

We have $T(r, \gamma') = m(r, \gamma') = m(r, \frac{(e^{n\gamma})'}{e^{n\gamma}}) = S(r, e^{n\gamma})$. Thus from (2.22) we get

$$T(r, e^{n\gamma}) \leqslant T(r, \frac{p^2}{\alpha'(\gamma' - \alpha')}) + O(1)$$

$$\leqslant T(r,\alpha^{'})+T(r,\gamma^{'}-\alpha^{'})+O(\log r)+O(1)$$
$$\leqslant 2\,T(r,\alpha^{'})+S(r,\alpha^{'})+S(r,e^{n\gamma}),$$

which implies that $T(r,e^{n\gamma})=O(T(r,\alpha^{'}))$ and so $S(r,e^{n\gamma})$ can be replaced by $S(r,\alpha^{'})$. Thus we get $T(r,\gamma^{'})=S(r,\alpha^{'})$ and so $\gamma^{'}$ is a small with respect to $\alpha^{'}$. In view of (2.22) and by the second fundamental theorem for small functions we get

$$\begin{aligned}T(r,\alpha^{'}) &\leqslant \overline{N}(r,\infty;\alpha^{'})+\overline{N}(r,0;\alpha^{'})+\overline{N}(r,0;\alpha^{'}-\gamma^{'})+S(r,\alpha^{'})\\ &\leqslant O(logr)+S(r,\alpha^{'}),\end{aligned}$$

which shows that $\alpha^{'}$ is a polynomial and so $\alpha$ is a polynomial. Similarly we can prove that $\beta$ is also a polynomial. This contradicts the fact that $\alpha$ and $\beta$ are transcendental.

Next suppose without loss of generality that $\alpha$ is a polynomial and $\beta$ is a transcendental entire function. Then $\gamma$ is transcendental. So in view of (2.22) we can obtain

$$\begin{aligned}nT(r,e^{\gamma}) &\leqslant T(r,\frac{p^2}{\alpha^{'}(\gamma^{'}-\alpha^{'})})+O(1)\\ &\leqslant T(r,\alpha^{'})+T(r,\gamma^{'}-\alpha^{'})+S(r,\gamma)\\ &\leqslant\ T(r,\gamma^{'})+S(r,e^{\gamma})=S(r,e^{\gamma}),\end{aligned}$$

which leads to a contradiction. Thus $\alpha$ and $\beta$ both are polynomials. Also from (2.21) we can conclude that $\gamma(z)=\alpha(z)+\beta(z)\equiv C$ for a constant $C$ and so $\alpha^{'}(z)+\beta^{'}(z)\equiv 0$. Again from (2.21) we get $a_{m^*}^2(n+m^*)^2e^{s\gamma}\alpha^{'}\beta^{'}\equiv p^2(z)$. By computation we get

$$\alpha^{'}=cp(z),\beta^{'}=-cp(z). \tag{2.23}$$

Hence

$$\alpha=cQ(z)+l_1,\beta=-cQ(z)+l_2, \tag{2.24}$$

where $Q(z)=\int_0^z p(z)dz$ and $l_1$, $l_2$ are constants. Finally we take $f$ and $g$ as

$$f(z)=c_1e^{cQ(z)},g(z)=c_2e^{-cQ(z)},$$

where $c_1$, $c_2$ and $c$ are constants such that $a_{m^*}^2[(n+m^*)c]^2(c_1c_2)^{n+i}=-1$.

**Case 2:** Let $p(z)$ be a nonzero constant $b$. Obviously we get $f=e^{\alpha}$ and $g=e^{\beta}$, where $\alpha$ and $\beta$ are two non-constant entire functions. Proceeding in the same as above we get in view of (2.20), $\alpha=cz+l_3$, $\beta=-cz+l_4$. We can rewrite $f$ and $g$ as

$$f=c_3e^{cz},g=c_4e^{-cz},$$

where $c_3$, $c_4$ and $c$ are nonzero constants such that $(-1)^ka_{m^*}^2(c_3c_4)^{n+m^*}[(n+m^*)c]^{2k}=b^2$.

This completes the proof of the lemma. □

LEMMA 13. *Let $f$ and $g$ be two non-constant meromorphic (entire) functions and $n(\geqslant 2)$, $m$ be two distinct integers satisfying $n+m \geqslant d+7$ ($n+m \geqslant d+3$). Then for two constants $\lambda$, $\mu$, with $|\lambda|+|\mu| \neq 0$,*

$$f^n\left(\mu f^m+\lambda\right) \equiv g^n\left(\mu g^m+\lambda\right) \tag{2.25}$$

*implies the following.*

*(i) if $\lambda\mu \neq 0$ and*

*(a) $m=1$, $\Theta(\infty, f)+\Theta(\infty, g)>4/n$; or*

*(b) $m \geqslant 2$ and for some constant $t$, satisfying $t^d \equiv 1$,*

*we have $f \equiv tg$, where $d=(n+m, n)$.*

*(ii) if $\lambda\mu=0$, then $f=tg$, where $t$ is a constant satisfying $t^{n+m^*}=1$.*

*Proof.* First suppose $\lambda\mu \neq 0$.

Let $m=1$. In this case noting that $d=1=(n+1, n)$, proceeding in the same way as done in *Lemma 6* of [10] we can show when $\Theta(\infty, f)+\Theta(\infty, g)>4/n$, we have $f \equiv g$.

Next suppose $m \geqslant 2$. Let $f \not\equiv tg$ for a constant $t$ satisfying $t^d=1$. We put $h=\frac{f}{g}$. Then $h^d \not\equiv 1$, i.e., $(h-v_0)(h-v_1)\ldots(h-v_{d-1}) \not\equiv 0$, where $v_k=exp\left(\frac{2k\pi i}{d}\right)$, $k=0,1,2,\ldots,d-1$. First suppose that $h$ is constant. Now (2.25) implies

$$\mu g^m\left(h^{n+m}-1\right) \equiv -\lambda\left(h^n-1\right).$$

Since $gcd(n+m, n)=d$, eliminating $d$ common factors namely $h-v_k$, $k=0,1,\ldots,d-1$ from both sides we are left with

$$a g^m\left(h-\alpha_1\right)\left(h-\alpha_2\right)\ldots\left(h-\alpha_{n+m-d}\right) \equiv\left(h-\beta_1\right)\left(h-\beta_2\right)\ldots\left(h-\beta_{n-d}\right),$$

where $\alpha_i$ and $\beta_j$ are those zeros of $h^{n+m}-1$ and $h^n-1$ which are not the zeros of $h^d-1$, $i=1,2,\ldots,n+m-d$ and $j=1,2,\ldots,n-d$. Also we note that none of the $\alpha_i$'s coincides with $\beta_j$'s. So if $h=\alpha_i$ or $\beta_j$, then we have either $(h-\beta_1)(h-\beta_2)\ldots(h-\beta_{n-d}) \equiv 0$ or $g \equiv 0$ and in both case we get contradictions. Consequentely we assume neither $h^{n+m} \equiv 1$ nor $h^n \equiv 1$. Hence we may write

$$g^m=-\frac{\lambda}{\mu} \frac{h^n-1}{h^{n+m}-1}. \tag{2.26}$$

It follows from (2.26) that $g$ is a constant, which is impossible. So $h$ is non-constant. We observe that since a non-constant meromorphic function can not have more than two Picard exceptional values $h$ can take at least $n+m-d-2$ values among $u_j=exp\left(\frac{2j\pi i}{n+m}\right)$, where $j=0,1,2,\ldots,n+m-1$. Since $f^m$ has no simple pole $h-u_j$ has no simple zero for at least $n+m-d-2$ values of $u_j$, for $j=0,1,2,\ldots,n+m-1$ and for these $n+m-d-2$ values of $j$ within $j=0,1,2,\ldots,n+m-1$, we have $\Theta(u_j;h) \geqslant \frac{1}{2}$. So by the maximum deficiency sum we have $\frac{n+m-d-2}{2} \leqslant 2$ i.e., $n+m \leqslant d+6$, which leads to a contradiction as $n+m>d+7$.

When $f$ and $g$ are entire functions, proceeding in the same way we can obtain (2.26) where $h$ is non-constant. Since $g$ has no pole and $h$ can omit at most 2 values, we must have $n+m \leqslant d+2$, which is a contradiction.

Next suppose $\lambda\mu \neq 0$. Then from the give condition either $\lambda$ or $\mu$ will be zero. So we get $f = tg$, where $t$ is a constant satisfying $t^{n+m^*} = 1$. This proves the lemma. □

LEMMA 14. [3] *Let $f$ and $g$ be two non-constant meromorphic functions sharing $(1,k_1)$, where $2 \leqslant k_1 \leqslant \infty$. Then*

$$\overline{N}(r,1;f|=2) + 2\,\overline{N}(r,1;f|=3) + \ldots + (k_1-1)\,\overline{N}(r,1;f|=k_1) + k_1\,\overline{N}_L(r,1;f) + (k_1+1)\,\overline{N}_L(r,1;g) + k_1\,\overline{N}_E^{(k_1+1}(r,1;g) \leqslant N(r,1;g) - \overline{N}(r,1;g).$$

LEMMA 15. [2] *Let $f$, $g$ share $(1,1)$. Then*

$$\overline{N}_{f>2}(r,1;g) \leqslant \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_0(r,0;f^{'}) + S(r,f),$$

*where $N_0(r,0;f^{'})$ is the counting function of those zeros of $f^{'}$ which are not the zeros of $f(f-1)$.*

LEMMA 16. [2] *Let $f$ and $g$ be two non-constant meromorphic functions sharing $(1,0)$. Then*

$$\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) - \overline{N}_{f>1}(r,1;g) - \overline{N}_{g>1}(r,1;f) \leqslant N(r,1;g) - \overline{N}(r,1;g).$$

LEMMA 17. [2] *Let $f$, $g$ share $(1,0)$. Then*

$$\overline{N}_L(r,1;f) \leqslant \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f)$$

LEMMA 18. [2] *Let $f$, $g$ share $(1,0)$. Then*

$$(i) \quad \overline{N}_{f>1}(r,1;g) \leqslant \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f^{'}) + S(r,f)$$
$$(ii) \quad \overline{N}_{g>1}(r,1;f) \leqslant \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_0(r,0;g^{'}) + S(r,g).$$

## 3. Proofs of the theorems

**Proof of Theorem 1.** Let $F = [f^nP(f)]^{(k)}/p(z)$ and $G = [g^nP(g)]^{(k)}/p(z)$, where $P(w) = \mu w^m + \lambda$. It follows that $F$ and $G$ share $(1,l)$ except the zeros of $p(z)$ and $f$, $g$ share $(\infty,0)$.

**Case 1.** Let $H \not\equiv 0$.

**Subcase 1.1.** $l \geqslant 1$

From (2.1) it can be easily calculated that the possible poles of $H$ occur at (i) multiple zeros of $F$ and $G$, (ii) those 1 points of $F$ and $G$ whose multiplicities are different, (iii) poles of $F$ and $G$ with different multiplicities, (iv) zeros of $F^{'}$ $(G^{'})$ which are not the zeros of $F(F-1)(G(G-1))$.

Since $H$ has only simple poles we get

$$N(r,\infty;H) \tag{3.1}$$

$$\leqslant \overline{N}_*(r,\infty;f,g)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\geqslant 2)+\overline{N}(r,0;G|\geqslant 2)$$
$$+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'}),$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined.

Let $z_0$ be a simple zero of $F-1$ but $a(z_0)\neq 0,\infty$. Then $z_0$ is a simple zero of $G-1$ and a zero of $H$. So

$$N(r,1;F|=1)\leqslant N(r,0;H)\leqslant N(r,\infty;H)+S(r,f)+S(r,g). \tag{3.2}$$

While $l\geqslant 3$, using (3.1) and (3.2) we get

$$\begin{aligned}
&\overline{N}(r,1;F) \\
&\leqslant N(r,1;F|=1)+\overline{N}(r,1;F|\geqslant 2)\\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,0;F|\geqslant 2)+\overline{N}(r,0;G|\geqslant 2)+\overline{N}_*(r,1;F,G)\\
&\quad+\overline{N}(r,1;F|\geqslant 2)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g).
\end{aligned}\tag{3.3}$$

Now in view of *Lemmas* 14 and 3 we get

$$\begin{aligned}
&\overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F\,|\geqslant 2)+\overline{N}_*(r,1;F,G)\\
&\leqslant \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F|=2)+\overline{N}(r,1;F|=3)+\ldots+\overline{N}(r,1;F|=l)\\
&\quad+\overline{N}_E^{(l+1}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_*(r,1;F,G)\\
&\leqslant \overline{N}_0(r,0;G^{'})-\overline{N}(r,1;F|=3)-\ldots-(l-2)\overline{N}(r,1;F|=l)-(l-1)\overline{N}_L(r,1;F)\\
&\quad-l\overline{N}_L(r,1;G)-(l-1)\overline{N}_E^{(l+1}(r,1;F)+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_*(r,1;F,G)\\
&\leqslant \overline{N}_0(r,0;G^{'})+N(r,1;G)-\overline{N}(r,1;G)-(l-2)\overline{N}_L(r,1;F)-(l-1)\overline{N}_L(r,1;G)\\
&\leqslant N(r,0;G^{'}\mid G\neq 0)-(l-2)\overline{N}_L(r,1;F)-(l-1)\overline{N}_L(r,1;G)\\
&\leqslant \overline{N}(r,0;G)+\overline{N}(r,\infty;g)-(l-2)\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G)\\
&\leqslant N(r,0;G)+\overline{N}(r,\infty;g)-\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G).
\end{aligned}\tag{3.4}$$

Hence using (3.3), (3.4), *Lemmas* 2 and 8 we get from the second fundamental theorem that

$$\begin{aligned}
&(n+m^*)T(r,f)\\
&\leqslant T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)\\
&\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)\\
&\quad-N_0(r,0;F^{'})\\
&\leqslant \overline{N}(r,\infty,f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F)+N_{k+2}(r,0;f^nP(f))+\overline{N}(r,0;F|\geqslant 2)\\
&\quad+\overline{N}(r,0;G|\geqslant 2)+\overline{N}(r,1;F|\geqslant 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;G^{'})-N_2(r,0;F)\\
&\quad+S(r,f)+S(r,g)\\
&\leqslant 3\,\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^nP(f))+N_2(r,0;G)-\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G)
\end{aligned}\tag{3.5}$$

$$
\begin{aligned}
&+S(r,f)+S(r,g)\\
&\leqslant 3\,\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^nP(f))+k\,\overline{N}(r,\infty;g)+N_{k+2}(r,0;g^nP(g))\\
&\quad-\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leqslant (3+k)\,\overline{N}(r,\infty;f)+(k+2)\,\overline{N}(r,0;f)+T(r,P(f))+(k+2)\,\overline{N}(r,0;g)\\
&\quad+T(r,P(g))-\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leqslant (k+m^*+2)\,\{T(r,f)+T(r,g)\}+(3+k)\overline{N}(r,\infty;f)-\overline{N}_*(r,1;F,G)\\
&\quad+S(r,f)+S(r,g)\\
&\leqslant (k+m^*+2)\,\{T(r,f)+T(r,g)\}+\frac{(3+k)(k+m^*+1)}{n+m^*-k-1}\,\{T(r,f)+T(r,g)\}\\
&\leqslant \left[k+m^*+2+\frac{(3+k)(k+m^*+1)}{n+m^*-k-1}\right]\,\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),
\end{aligned}
$$

In a similar way we can obtain

$$
\begin{aligned}
&(n+m^*)T(r,g) \qquad (3.6)\\
&\leqslant \left[k+m^*+2+\frac{(3+k)(k+m^*+1)}{n+m^*-k-1}\right]\,\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).
\end{aligned}
$$

Adding (3.5) and (3.6) we get

$$
\left[n-m^*-2k-4-\frac{(6+2k)(k+m^*+1)}{n+m^*-k-1}\right]\,\{T(r,f)+T(r,g)\}\leqslant S(r,f)+S(r,g).
$$

Since the quantity in the third bracket can be written as

$$
\left[\frac{(n+m^*-k-1)^2-(2m^*+k+3)(n+m^*-k-1)-2(k+3)(k+m^*+1)}{n+m^*-k-1}\right], \qquad (3.7)
$$

by a simple computation one can easily verify that when

$$
\begin{aligned}
&n+m^*-k-1>2m^*+2k+5>\\
&\frac{2m^*+k+3+\sqrt{(2m^*+k+3)^2+8(k+3)(k+m^*+1)}}{2},
\end{aligned}
$$

i.e., when $n>3k+m^*+6$ we get a contradiction from (3.7).

While $l\geqslant 2$, like (3.3), (3.4) and not using *Lemma* 8 in (3.5) we can deduce a contradiction when $n>3k+m^*+7$. So we omit the detail.

While $l=1$, using *Lemmas* 3, 14, 15, (3.1) and (3.2) we get

$$
\begin{aligned}
&\overline{N}(r,1;F) \qquad (3.8)\\
&\leqslant N(r,1;F|=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geqslant 2)+\overline{N}(r,0;G|\geqslant 2)+\overline{N}_*(r,1;F,G)\\
&\quad+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')\\
&\quad+S(r,f)+S(r,g)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geqslant 2)+\overline{N}(r,0;G|\geqslant 2)+2\overline{N}_L(r,1;F)\\
&\quad +2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})\\
&\quad +S(r,f)+S(r,g)\\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geqslant 2)+\overline{N}(r,0;G|\geqslant 2)+\overline{N}_{F>2}(r,1;G)\\
&\quad +N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})\\
&\quad +S(r,f)+S(r,g)\\
&\leqslant \frac{3}{2}\,\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geqslant 2)+\frac{1}{2}\,\overline{N}(r,0;F)+\overline{N}(r,0;G|\geqslant 2)\\
&\quad +N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;G^{'})+\overline{N}_0(r,0;F^{'})\\
&\quad +S(r,f)+S(r,g)\\
&\leqslant \frac{3}{2}\,\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geqslant 2)+\frac{1}{2}\,\overline{N}(r,0;F)+\overline{N}(r,0;G|\geqslant 2)\\
&\quad +N(r,0;G^{'}|G\neq 0)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g)\\
&\leqslant \frac{3}{2}\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geqslant 2)+\frac{1}{2}\,\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad +\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g).
\end{aligned}
$$

Hence using (3.8), *Lemmas* 1 and 2 we get from second fundamental theorem that

$$
\begin{aligned}
&(n+m^*)T(r,f) \qquad\qquad (3.9)\\
&\leqslant T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)\\
&\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)\\
&\quad -N_0(r,0;F^{'})\\
&\leqslant \frac{5}{2}\,\overline{N}(r,\infty,f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\frac{1}{2}\overline{N}(r,0;F)+N_{k+2}(r,0;f^nP(f))\\
&\quad +N_2(r,0;G)-N_2(r,0;F)+S(r,f)+S(r,g)\\
&\leqslant \frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+\frac{1}{2}\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad +S(r,f)+S(r,g)\\
&\leqslant \frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+k\,\overline{N}(r,\infty;g)+N_{k+2}(r,0;g^nP(g))\\
&\quad +\frac{1}{2}\{k\overline{N}(r,\infty;f)+\overline{N}_{k+1}(r,0;f^nP(f))\}+S(r,f)+S(r,g)\\
&\leqslant \frac{5+k}{2}\,\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,\infty;g)+\frac{3k+5}{2}\,\overline{N}(r,0;f)+\frac{3}{2}\,T(r,P(f))\\
&\quad +(k+2)\,\overline{N}(r,0;g)+T(r,P(g))+S(r,f)+S(r,g)\\
&\leqslant (2k+5+\frac{3m^*}{2})\,T(r,f)+(2k+4+m^*)\,T(r,g)+S(r,f)+S(r,g)\\
&\leqslant (4k+9+\frac{5m^*}{2})\,T(r)+S(r).
\end{aligned}
$$

In a similar way we can obtain

$$(n+m^*)\,T(r,g) \leqslant \left(4k+9+\frac{5m^*}{2}\right)\,T(r)+S(r). \tag{3.10}$$

Combining (3.9) and (3.10) we see that

$$(n+m^*)\,T(r) \leqslant \left(4k+9+\frac{5m^*}{2}\right)\,T(r)+S(r),$$

i.e

$$\left(n-4k+9-\frac{3m^*}{2}\right)\,T(r) \leqslant S(r). \tag{3.11}$$

Since $n > 4k+9+\frac{3m^*}{2}$, (3.11) leads to a contradiction.

**Subcase 1.2.** $l=0$. Here (3.2) changes to

$$N_E^{1)}\left(r,1;F^{(k)}\mid=1\right) \leqslant N(r,0;H) \leqslant N(r,\infty;H)+S(r,F)+S(r,G) \tag{3.12}$$

using *Lemmas* 3, 16, 17, 18, (3.1) and (3.12) we get

$$\begin{aligned}
&\overline{N}(r,1;F) \qquad (3.13)\\
&\leqslant N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F\mid\geqslant 2)+\overline{N}(r,0;G\mid\geqslant 2)+\overline{N}_*(r,1;F,G)\\
&\quad+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})\\
&\quad+S(r,f)+S(r,g)\\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F\mid\geqslant 2)+\overline{N}(r,0;G\mid\geqslant 2)+2\overline{N}_L(r,1;F)\\
&\quad+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F\mid\geqslant 2)+\overline{N}(r,0;G\mid\geqslant 2)+\overline{N}_{F>1}(r,1;G)\\
&\quad+\overline{N}_{G>1}(r,1;F)+\overline{N}_L(r,1;F)+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;F^{'})\\
&\quad+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\leqslant 3\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;G^{'})+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g)\\
&\leqslant 3\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad+N(r,0;G^{'}\mid G\neq 0)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g)\\
&\leqslant 3\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad+\overline{N}(r,0;G)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g).
\end{aligned}$$

Hence using (3.13), *Lemmas* 1 and 2 we get from second fundamental theorem that

$$\begin{aligned} &(n+m^{*})T(r,f) \qquad (3.14)\\ \leqslant\ & T(r,F)+N_{k+2}(r,0;f^{n}P(f))-N_{2}(r,0;F)+S(r,f)\\ \leqslant\ & \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^{n}P(f))-N_{2}(r,0;F)\\ & -N_{0}(r,0;F^{'})+S(r,f)\\ \leqslant\ & 4\overline{N}(r,\infty,f)+3\overline{N}(r,\infty;g)+N_{2}(r,0;F)+\overline{N}(r,0;F)+N_{k+2}(r,0;f^{n}P(f))\\ & +N_{2}(r,0;G)+\overline{N}(r,0;G)-N_{2}(r,0;F)+S(r,f)+S(r,g)\\ \leqslant\ & 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^{n}P(f))+2\overline{N}(r,0;F)+N_{2}(r,0;G)\\ & +\overline{N}(r,0;G)+S(r,f)+S(r,g)\\ \leqslant\ & 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^{n}P(f))+2k\overline{N}(r,\infty;f)\\ & +2N_{k+1}(r,0;f^{n}P(f))+k\,\overline{N}(r,\infty;g)+N_{k+2}(r,0;g^{n}P(g))+k\overline{N}(r,\infty;g)\\ & +\overline{N}_{k+1}(r,0;g^{n}P(g))+S(r,f)+S(r,g)\\ \leqslant\ & (2k+4)\,\overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)+(3k+4)\overline{N}(r,0;f)+3T(r,P(f))\\ & +(2k+3)\,\overline{N}(r,0;g)+2T(r,P(g))+S(r,f)+S(r,g)\\ \leqslant\ & (5k+8+3m^{*})\,T(r,f)+(4k+6+2m^{*})\,T(r,g)+S(r,f)+S(r,g)\\ \leqslant\ & (9k+14+5m^{*})\,T(r)+S(r), \end{aligned}$$

where $T(r)=\max\{T(r,f),T(r,g)\}$. In a similar way we can obtain

$$(n+m^{*})\,T(r,g)\leqslant(9k+14+5m^{*})\;T(r)+S(r). \tag{3.15}$$

Combining (3.14) and (3.15) we see that

$$(n+m^{*})\,T(r)\leqslant(9k+14+5m^{*})\;T(r)+S(r),$$

i.e

$$(n-9k-14-4m^{*})\;T(r)\leqslant S(r). \tag{3.16}$$

Since $n>9k+14+4m^{*}$, (3.16) leads to a contradiction.

**Case 2.** Let $H\equiv0$. Then by *Lemma* 9 we obtain either

$$[f^{n}(\mu f^{m}+\lambda)]^{(k)}[g^{n}(\mu g^{m}+\lambda)]^{(k)}\equiv p^{2}$$

or

$$f^{n}(\mu f^{m}+\lambda)\equiv g^{n}(\mu g^{m}+\lambda).$$

So the theorem follows from *Lemma* 12 and 13. □

**Proof of Theorem 1.** Proceeding in the same way the proof of *Theorem* 2 can be carried out in the line of proof of *Theorem* 1. □

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