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UPPER ESTIMATE FOR GENERAL COMPLEX BASKAKOV–SZÁSZ OPERATOR

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Abstract. In the present article, we study general complex Baskakov-Szász operators and establish an upper estimate for these operators attached to analytic functions of exponential growth on compact disks.

1. Introduction

It is known that Baskakov operators are based on the negative binomial distribution. There are several integral modifications of the well known Baskakov operators available in the literature. The most common are Baskakov-Kantorovich and Baskakov Durrmeyer type operators. Apart from these there are other hybrid operators having different basis functions in summation and integration. In order to approximate integrable functions on the interval $[0,\infty)$ Gupta and Srivastava [12] proposed the Baskakov-Szász operators in case of real variables and studied some approximation properties. Very recently Agrawal et al [2] proposed the generalization of such operators based on a parameter $a>0$ and studied some approximation properties in case of real variables. In case of a complex variable, we can write the generalized operators as

$$L_n^a(f,z) = n\sum_{k=0}^{\infty} W_{n,k}^a(z)\int_0^{\infty} s_{n,k}(t)f(t)dt, \tag{1}$$

where

$$W_{n,k}^a(z) = e^{-\frac{az}{1+z}}\frac{P_k(n,a)}{k!}\frac{z^k}{(1+z)^{n+k}}, \qquad s_{n,k}(t) = e^{-nt}(nt)^k/k!$$

and $P_k(n,a) = \sum_{i=0}^{k}\binom{k}{i}(n)_i a^{k-i}$, with the rising factorial given by $(n)_i = n(n+1)\dots(n+i-1)$, $(n)_0 = 1$. As $\sum_{k=0}^{\infty} W_{n,k}^a(x) = 1$ and $\int_0^{\infty} s_{n,k}(t)dt = 1/n$, these operators reproduce constant functions. Also in special case $a=0$, these operators include the well known Baskakov-Szász operators (see e.g. [12]).

On the complex operators the commendable work was done by the pioneer S. G. Gal who presented results on overconvergence of several complex operators in his book [4] and references therein. Different forms of complex integral operators have been discussed in the recent years, we refer some of the papers in this direction due

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to Gal and Gupta [6], [7] and [8], Agarwal and Gupta [1] and Gupta [10] etc. Very recently Gupta in [11] compiled some of the results on complex type operators. Also the recent book by Gal [5] contains interesting generalizations and extensions of the results of [4].

For the special case $a=0$ the first author [9] established some results. But the operators $L_n^a(f,z)$ provides rational functions and are different from those considered for special case $a=0$. The present paper deals with the study of the complex generalized Baskakov-Szász operators (1). Here, we estimate the upper bound for these operators.

2. Basic results

In the sequel, we need the following lemmas.

LEMMA 1. *If we denote $T_{n,m}^a(z)=L_n^a(t^m,z)$, then there holds the following recurrence relation:*

$$T_{n,m+1}^a(z)=\frac{z(1+z)}{n}T'_{n,m}(z)+\left[\frac{m+1}{n}+z+\frac{az}{n(1+z)}\right]T_{n,m}(z).$$

Further by simple computation, we have

$$\begin{aligned}
L_n^a(1,z)&=1;\\
L_n^a(t,z)&=z+\frac{az}{n(1+z)}+\frac{1}{n};\\
L_n^a(t^2,z)&=z^2+\frac{z^2}{n}+\frac{4z}{n}+\frac{a^2z^2}{n^2(1+z)^2}+\frac{2az^2}{n(1+z)}+\frac{4az}{n^2(1+z)}+\frac{2}{n^2}.
\end{aligned}$$

The proof of the above lemma is similar as done in Lemma 3 of [2] for the real case, we omit the details.

We denote $\mathbb{D}_R=\{z\in\mathbb{C}:|z|<R\}$. By H_R, we mean the class of functions satisfying:

$f:[R,+\infty)\cup\overline{\mathbb{D}}_R\to\mathbb{C}$ is continuous in $(R,+\infty)\cup\overline{\mathbb{D}}_R$, analytic in $\mathbb{D}_R$ i.e. $f(z)=\sum_{k=0}^{\infty}c_kz^k$, for all $z\in\mathbb{D}_R$.

LEMMA 2. *Suppose that $f:[R,+\infty)\bigcup\overline{\mathbb{D}_R}$ is analytic in $\mathbb{D}_R$ and there exists $B,C>0$ such that $|f(x)|\leqslant Ce^{Bx}$, for all $x\in[R,+\infty)$. Denoting $f(z)=\sum_{k=0}^{\infty}c_kz^k$, $z\in\mathbb{D}_R$, we have $L_n^a(f)(z)=\sum_{k=0}^{\infty}c_kL_n^a(e_k)(z)$, for all $|z|\leqslant r$ with $Re(z)\geqslant 0$ and $n\in\mathbb{N}$ with $n>B/(1-h)$, where $h=\sqrt{r^2/(1+r^2)}$ and $r<R$.*

Proof. For any $m\in\mathbb{N}$, let us define

$$f_m(z)=\sum_{j=0}^{m}c_jz^j \text{ if } |z|\leqslant r \text{ and } f_m(x)=f(x) \text{ if } x\in(r,+\infty).$$

Since $|f_m(z)|\leqslant\sum_{j=0}^{\infty}|c_j|\cdot r^j:=C_r$, for all $|z|\leqslant r$ and $m\in\mathbb{N}$, f is continuous on $[r,R]$. Obviously from the hypothesis on f it follows that $|f_m(x)|\leqslant C_{r,R}e^{Bx}$, for all

$x \in [0,+\infty)$ and any $m \in \mathbb{N}$. This implies that for each fixed $m,n \in \mathbb{N}$, $n > B$ and $|z| \leqslant r$ with $Re(z) \geqslant 0$, we have

$$\begin{aligned}|L_n^a(f_m)(z)| &\leqslant C_{r,R}|(1+z)^{-n}|\left(\sum_{j=0}^{\infty}|e^{-\frac{az}{1+z}}|\frac{P_j(n,a)}{j!}\left(\frac{|z|}{|1+z|}\right)^j n\int_0^{\infty} e^{-nt}\cdot\frac{n^j}{j!}t^j e^{Bt}dt\right)\\ &= C_{r,R}|(1+z)^{-n}|\sum_{j=0}^{\infty}|e^{-\frac{az}{1+z}}|\frac{P_j(n,a)}{j!}\left(\frac{|z|}{|1+z|}\right)^j\cdot\frac{n^{j+1}}{(n-B)^{j+1}}\\ &\leqslant C_{r,R}|(1+z)^{-n}|\sum_{j=0}^{\infty}\frac{P_j(n,a)}{j!}h^j\cdot\frac{n^{j+1}}{(n-B)^{j+1}},\end{aligned}$$

where $h = \sqrt{\frac{r^2}{1+r^2}} < 1$, taking into account that for $z = x+iy$ with $x \geqslant 0$ we have

$$\begin{aligned}\left(\frac{|z|}{|1+z|}\right)^2 &= \frac{x^2+y^2}{1+2x+(x^2+y^2)}\\ &\leqslant \frac{x^2+y^2}{1+(x^2+y^2)} \leqslant \frac{r^2}{1+r^2}.\end{aligned}$$

We apply the ratio test to the last series, denoting $a_j = \frac{P_j(n,a)}{j!}h^j\cdot\frac{n^{j+1}}{(n-B)^{j+1}}$, we get $\frac{a_{j+1}}{a_j} = \frac{P_{j+1}(n,a)}{P_j(n,a)(j+1)}\cdot\frac{hn}{n-B}$, where $\frac{hn}{n-B} < 1$ is equivalent to $n > \frac{B}{1-h}$. Therefore, if $n > \frac{B}{1-h}$ then there exists j_0, such that $\frac{P_{j+1}(n,a)}{P_j(n,a)(j+1)}\cdot\frac{hn}{n-B} < 1$ for all $j \geqslant j_0$, therefore $L_n^a(f_m)(z)$ is well-defined for $n > \frac{B}{1-h}$.

Denoting

$$f_{m,k}(z) = c_k e_k(z) \text{ if } |z| \leqslant r \text{ and } f_{m,k}(x) = \frac{f(x)}{m+1} \text{ if } x \in (r,\infty),$$

clearly each $f_{m,k}$ is of exponential growth on $[0,\infty)$ and that $f_m(z) = \sum_{k=0}^m f_{m,k}(z)$. By the linearity property, we have

$$L_n^a(f_m)(z) = \sum_{k=0}^m c_k L_n^a(e_k)(z),$$

it is sufficient to prove that $\lim_{m\to\infty} L_n^a(f_m)(z) = L_n^a(f)(z)$ for any fixed $n \in \mathbb{N}$ with $n > B/(1-h)$ and $|z| \leqslant r$ with $Re(z) \geqslant 0$. But this is immediate from $\lim_{m\to\infty}\|f_m - f\|_r = 0$, from $\|f_m - f\|_{B[0,+\infty)} \leqslant \|f_m - f\|_r$ and from the inequality

$$\begin{aligned}|L_n^a(f_m)(z) - L_n^a(f)(z)| &\leqslant |(1+z)^{-n}|\cdot e^{n|z|}\cdot\|f_m - f\|_{B[0,\infty)}\\ &\leqslant M_{r,n}\|f_m - f\|_r.\end{aligned}$$

Here $\|\cdot\|_{B[0,+\infty)}$ denotes the uniform norm on $C[0,+\infty)$-the space of all complex-valued bounded functions on $[0,+\infty)$. $\square$

3. Upper estimate

Our main result is the following theorem for upper bound.

THEOREM 1. *Let* $f \in H_R$, $(r+1)+\sqrt{(r+1)^2+r(r+2)} < R < +\infty$ *and suppose that there exist* $M > 0$ *and* $A \in (\frac{1}{R}, 1)$, *with the property that* $|c_k| \leqslant M\frac{A^k}{\Gamma(k+a)}$, *for all* $a > 0$, $k = 0, 1, \ldots$, *(which implies* $|f(z)| \leqslant Me^{A|z|}$ *for all* $z \in \mathbb{D}_R$*) and* $|f(x)| \leqslant Ce^{Bx}$, *for all* $x \in [R, +\infty)$.

Let $1 \leqslant r < r+2 < (r+2)\cdot\frac{R+r}{R-r} < \frac{1}{A}$ *and* $h = \sqrt{r^2/(1+r^2)}$. *Then for all* $|z| \leqslant r$ *with* $Re(z) \geqslant 0$ *and and* $n \in N$ *with* $n > B/(1-h)$, *we have*

$$|L_n^a(f)(z) - f(z)| \leqslant \frac{C_{r,a,A}}{n},$$

where

$$C_{r,a,A} = M\sum_{k=2}^{\infty}(k+a)\left((r+2)\cdot\frac{R+r}{R-r}A\right)^k < \infty.$$

Proof. By using the recurrence relation of Lemma 1, we have

$$T_{n,k+1}^a(z) = \frac{z(1+z)}{n}(T_{n,k}^a(z))' + \left[\frac{nz+k+1}{n} + \frac{az}{n(1+z)}\right]T_{n,k}^a(z),$$

for all $z \in \mathbb{C}$, $k \in \{0,1,2,\ldots\}, n \in N$. From this we immediately get the recurrence formula

$$\begin{aligned} T_{n,k}^a(z) - z^k &= \frac{z(1+z)}{n}[T_{n,k-1}^a(z) - z^{k-1}]' + \left[\frac{nz+k}{n} + \frac{az}{n(1+z)}\right][T_{n,k-1}^a(z) - z^{k-1}] \\ &\quad + \left[\frac{(2k-1)+(k-1)z}{n} + \frac{az}{n(1+z)}\right]z^{k-1}, \end{aligned}$$

for all $z \in \mathbb{C}$, $k,n \in N$. Now for $1 \leqslant r < R$, if we denote the norm – $||\cdot||_r$ in $C(\overline{\mathbb{D}}_r)$, where $\overline{\mathbb{D}}_r = \{z \in \mathbb{C} : |z| \leqslant r\}$, then by a linear transformation, with $T_{n,k}^a(z) = \frac{P_{k,n}(z)}{(1+z)^k}$ where $P_{k,n}(z)$ is a polynomial of degree $\leqslant k$ the Bernstein's inequality in the closed unit disk for rational functions as given in [3] and also in Corollary 1.10.4 in [4], becomes $|(T_{n,k}^a(z))'(z)| \leqslant \frac{R+r}{R-r}\cdot\frac{k}{r}||T_{n,k}^a||_r$, for all $|z| \leqslant r$. Thus from the above recurrence relation with $h = \frac{|z|}{|1+z|} \leqslant \sqrt{\frac{r^2}{1+r^2}} < 1$, we get

$$\begin{aligned} ||T_{n,k}^a - e_k||_r &\leqslant \frac{r(1+r)}{n}\cdot||T_{n,k-1}^a - e_{k-1}||_r\frac{k-1}{r}\frac{R+r}{R-r} + \frac{nr+k+a}{n}||T_{n,k-1}^a - e_{k-1}||_r \\ &\quad + \frac{(k+a)(r+2)}{n}r^{k-1}, \end{aligned}$$

which, by using the notation $\eta = r+2$, implies

$$\begin{aligned} ||T_{n,k}^a - e_k||_r &\leqslant \left(r + \frac{(2+r)(k+a)}{n}\right)\cdot\frac{R+r}{R-r}\cdot||T_{n,k-1}^a - e_{k-1}||_r + \frac{(k+a)}{n}(2+r)r^{k-1} \\ &= \left(r + \frac{\eta(k+a)}{n}\right)\cdot\frac{R+r}{R-r}\cdot||T_{n,k-1}^a - e_{k-1}||_r + \frac{(k+a)}{n}\eta\cdot r^{k-1}. \end{aligned}$$

In what follows we prove by mathematical induction with respect to k that for $n \geqslant \eta$, this recurrence implies

$$||T^a_{n,k} - e_k||_r \leqslant \frac{\eta \cdot \Gamma(k+a+1)}{n} \cdot r^{k-1} \cdot \left(\frac{R+r}{R-r}\right)^{k-1} \quad \text{for all } k \geqslant 1. \qquad (2)$$

Indeed for $k = 1$ it is trivial, as the left-hand side is $\frac{az}{n(1+z)} + \frac{1}{n}$. Suppose that it is valid for k, the above recurrence relation implies that

$$||T^a_{n,k+1} - e_{k+1}||_r \leqslant \left(r + \frac{\eta \cdot (k+a+1)}{n}\right) \cdot \frac{\eta \cdot \Gamma(k+a+1)}{n} r^{k-1} \cdot \left(\frac{R+r}{R-r}\right)^k + \frac{\eta \cdot (k+a+1)}{n} r^k \cdot \left(\frac{R+r}{R-r}\right)^k.$$

It remains to prove that

$$\left(r + \frac{\eta \cdot (k+a+1)}{n}\right) \cdot \frac{\eta \cdot \Gamma(k+a+1)}{n} r^{k-1} + \frac{\eta \cdot (k+a+1)}{n} r^k \leqslant \frac{\eta \cdot \Gamma(k+a+2)}{n} r^k,$$

or after simplifications, equivalently to

$$\left(r + \frac{\eta \cdot (k+a+1)}{n}\right) \cdot \Gamma(k+a+1) + r(k+a+1) \leqslant \Gamma(k+a+2) \cdot r,$$

for all $k \in \mathbb{N}$ and $r \geqslant 1$.

Since by $n \geqslant \eta$, we get

$$\left(r + \frac{\eta \cdot (k+a+1)}{n}\right) \cdot \Gamma(k+a+1) + r(k+a+1) \leqslant (r+k+a+1) \cdot \Gamma(k+a+1) + r(k+a+1),$$

it is good enough if we prove that

$$(r+k+a+1) \cdot \Gamma(k+a+1) + r(k+a+1) \leqslant \Gamma(k+a+2) \cdot r.$$

But this last inequality is obviously valid for all $k \geqslant 1$ (and fixed $r \geqslant 1$).

From the hypothesis on f, by Lemma 2 we can write

$$L^a_n(f)(z) = \sum_{k=0}^{\infty} c_k L^a_n(e_k)(z) = \sum_{k=0}^{\infty} c_k T^a_{n,k}(z), \text{ for all } z \in \mathbb{D}_R, \ Re(z) \geqslant 0, \ n > B/(1-h),$$

which from the hypothesis on c_k immediately implies for all $|z| \leqslant r$ with $Re(z) \geqslant 0$ and $n \in \mathbb{N}$ with $n > B/(1-h)$,

$$\begin{aligned} |L^a_n(f)(z) - f(z)| &\leqslant \sum_{k=2}^{\infty} |c_k|.|T^a_{n,k}(z) - e_k(z)| \\ &\leqslant \sum_{k=2}^{\infty} M \frac{A^k}{\Gamma(k+a)} (r+2) \frac{\Gamma(k+a+1)}{n} r^{k-1} \cdot \left(\frac{R+r}{R-r}\right)^{k-1} \\ &\leqslant \frac{M}{n} \sum_{k=2}^{\infty} (k+a) \left((r+2) A \frac{R+r}{R-r}\right)^k = \frac{C_{r,a,A}}{n}, \end{aligned}$$

where

$$C_{r,a,A} = M\sum_{k=2}^{\infty}(k+a)\left((r+2)A\frac{R+r}{R-r}\right)^k < \infty$$

for all $1 \leqslant r \leqslant (r+2)\frac{R+r}{R-r} < \frac{1}{A}$, taking into account that the series $\sum_{k=1}^{\infty} u^k$ is uniformly convergent in any compact disk included in the open unit disk. □

REMARK 1. It is observed that the generalized Baskakov-Szász operators discussed here provides rational functions and these operators behave differently than the usual Baskakov-Szász operators discussed in [9]. At this moment we are not able to obtain asymptotic formula and we will discuss that elsewhere.

REMARK 2. Actually the conditions $\frac{1}{A} < R$ and $(r+2)\frac{R+r}{R-r} < \frac{1}{A}$ imply the inequality

$$(r+2)\frac{R+r}{R-r} < R,$$

which is true if

$$R > (r+1) + \sqrt{(r+1)^2 + r(r+2)},$$

and the last inequality should replace the condition $2 < R < +\infty$ (considered for special case $a = 0$ in [9]) in the formulation of Theorem 1.

REMARK 3. In particular, if f is an entire function, then under the growth conditions in Theorem 1, the property $L_n^a(f)(z) = \sum_{k=0}^{\infty} c_k L_n^a(e_k)(z)$ follows by simple direct calculation (without to need Lemma 1) and the upper estimate in Theorem 1 holds in any semi-disk $|z| \leqslant r$, $Re(z) \geqslant 0$. On the other hand, if f is supposed to be analytic only on $\mathbb{D}_R$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, $|z| < R$, without to require to be defined (and of exponential growth) on $[0, +\infty)$ too, then one can consider the approximation operator denoted by $L_n^{*a}(f)(z) = \sum_{k=0}^{\infty} c_k \cdot L_n^a(e_k)(z)$, $|z| < R$, which evidently will satisfy the estimate in Theorem 1.

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