Journal of Classical Analysis
Volume 7, Number 1 (2015), 25–37 doi:10.7153/jca-07-03

# ON APPROXIMATION BY PHILLIPS TYPE MODIFIED BERNSTEIN OPERATOR IN A MOBILE INTERVAL

MINAKSHI DHAMIJA, RYOZI SAKAI AND NAOKANT DEO

*Abstract.* In the present paper we study a Phillips type modified Bernstein operator $M_n$, where the function is defined in the mobile interval $\left[0,1-\frac{1}{n+1}\right]$ and obtain its $m-th$ order moment. We establish some direct results in simultaneous approximation for this modified Bernstein operator.

## 1. Introduction

Bernstein [1] introduced the operator $B_n : C[0,1] \rightarrow C[0,1]$ given by

$$B_n(f,x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1}$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad x \in [0,1], \quad n \in N.$$

Deo et al. [5] proposed a generalized form of the Bernstein operator $(B_n f)$ in the interval $\left[0,1-\frac{1}{n+1}\right]$ given by

$$V_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{2}$$

where

$$p_{n,k}(x) = \left(1+\frac{1}{n}\right)^n \binom{n}{k} x^k \left(1-\frac{1}{n+1}-x\right)^{n-k}, \quad x \in \left[0,1-\frac{1}{n+1}\right].$$

If $n$ is sufficiently large then this operator (2) closely resemble to original form of Bernstein operator (1). In the year 2008, Deo et al. [5] defined a Durrmeyer form of operator (2) and studied simultaneous approximation by the linear combination of modified Bernstein-Durrmeyer operator and very recently, Jung et al. [16] studied pointwise approximation by Durrmeyer type operator of (2) in the mobile interval $x \in \left[0,1-\frac{1}{n+1}\right]$.

*Mathematics subject classification* (2010): 41A25, 41A36.
*Keywords and phrases*: Bernstein operator, Phillips operator, Simultaneous approximation, direct results, mobile interval.


Paper JCA-07-03

In [5], a Philips type modification for the same operator (2) has also been proposed and given by $M_n : C\left[0, 1-\frac{1}{n+1}\right] \to C\left[0, 1-\frac{1}{n+1}\right]$ such that

$$M_n(f,x) = \frac{(n+1)^2}{n}\sum_{k=1}^{n} p_{n,k}(x)\int_0^{\frac{n}{n+1}} p_{n,k-1}(t)f(t)\,dt + \left(1+\frac{1}{n}\right)^n\left(1-\frac{1}{n+1}-x\right)^n f(0). \tag{3}$$

The generalized Bernstein operators (2) and its Phillips type modification (3) also approximate the functions having singularities at point 1. Several mathematicians studied Phillips type modifications for various operators (see [5], [6], [12], [13], [14]). Deo [3, 4] obtained some direct results and Voronovskaya type asymptotic formula for the Beta operator and exponential-type operators in simultaneous approximation. Asymptotic behaviour of differentiated Bernstein operator and its variant have been discussed by Gonska et al. [8], [9] and [10] and Floater [7].

In the present paper, we establish some direct results which includes the asymptotic behaviour of differentiated modified Bernstein operator (3).

## 2. Basic results

In this section, we consider some basic results which are necessary to prove our main theorems.

LEMMA 1. *For $m \in N^0$ (the set of nonnegative integers); if we define:*

$$M_n\left((t-x)^m,x\right) = T_{n,m}(x) = \frac{(n+1)^2}{n}\sum_{k=1}^{n} p_{n,k}(x)\int_0^{\frac{n}{n+1}} p_{n,k-1}(t)(t-x)^m dt + \left(1+\frac{1}{n}\right)^n\left(1-\frac{1}{n+1}-x\right)^n (-x)^m, \tag{4}$$

*then*

$$T_{n,0}(x) = 1,\ T_{n,1}(x) = \frac{-2x}{n+2},$$
$$T_{n,2}(x) = \frac{2n^2}{(n+1)(n+2)(n+3)}x - \frac{2(n-3)}{(n+2)(n+3)}x^2, \tag{5}$$

*and for $m \geqslant 1$, there holds the recurrence relation*

$$(n+m+2)\,T_{n,m+1}(x) = x\left(\frac{n}{n+1}-x\right)\left[T'_{n,m}(x)+2mT_{n,m-1}(x)\right] + \left[m\left(\frac{n}{n+1}-2x\right)-2x\right]T_{n,m}(x). \tag{6}$$

*Consequently,*

$$T_{n,m}(x) = \begin{cases} O\left(n^{-m/2}\right), & m\ (even) \\ O\left(n^{-(m+1)/2}\right), & m\ (odd). \end{cases}$$

*Proof.* The values of $T_{n,0}(x)$ and $T_{n,1}(x)$ can easily follow from definition. We prove the recurrence relation as follows:

$$\begin{aligned} T'_{n,m}(x) &= \frac{(n+1)^2}{n}\sum_{k=1}^{n} p'_{n,k}(x)\int_0^{\frac{n}{n+1}} p_{n,k-1}(t)(t-x)^m dt \\ &\quad - m\frac{(n+1)^2}{n}\sum_{k=1}^{n} p_{n,k}(x)\int_0^{\frac{n}{n+1}} p_{n,k-1}(t)(t-x)^{m-1} dt \\ &\quad - n\left(1+\frac{1}{n}\right)^n\left(\frac{n}{n+1}-x\right)^{n-1}(-x)^m \\ &\quad - m\left(1+\frac{1}{n}\right)^n\left(\frac{n}{n+1}-x\right)^{n}(-x)^{m-1}. \end{aligned} \tag{7}$$

Using the identity

$$x\left(\frac{n}{n+1}-x\right)p'_{n,k}(x) = n\left(\frac{k}{n+1}-x\right)p_{n,k}(x),$$

then we obtain

$$\begin{aligned} &x\left(\frac{n}{n+1}-x\right)T'_{n,m}(x) \\ &= \frac{(n+1)^2}{n}\sum_{k=1}^{n} n\left(\frac{k}{n+1}-x\right)p_{n,k}(x)\int_0^{\frac{n}{n+1}} p_{n,k-1}(t)(t-x)^m dt \\ &\quad - mx\left(\frac{n}{n+1}-x\right)T_{n,m-1}(x) + n\left(\frac{n+1}{n}\right)^n\left(\frac{n}{n+1}-x\right)^n(-x)^{m+1}. \end{aligned}$$

Therefore,

$$\begin{aligned} &x\left(\frac{n}{n+1}-x\right)\left[T'_{n,m}(x)+mT_{n,m-1}(x)\right] \\ &= \frac{(n+1)^2}{n}\sum_{k=1}^{n}\left(\frac{kn}{n+1}-nx\right)p_{n,k}(x)\int_0^{\frac{n}{n+1}} p_{n,k-1}(t)(t-x)^m dt \\ &\quad + n\left(\frac{n+1}{n}\right)^n(-x)^{m+1}\left(\frac{n}{n+1}-x\right)^n \\ &= \frac{(n+1)^2}{n}\sum_{k=1}^{n} p_{n,k}(x)\int_0^{\frac{n}{n+1}}\left[(k-1)\frac{n}{n+1}-nt+n(t-x)+\frac{n}{n+1}\right] \\ &\quad \times p_{n,k-1}(t)(t-x)^m dt + n\left(\frac{n+1}{n}\right)^n(-x)^{m+1}\left(\frac{n}{n+1}-x\right)^n \\ &= \frac{(n+1)^2}{n}\sum_{k=1}^{n} p_{n,k}(x)\int_0^{\frac{n}{n+1}} t\left(\frac{n}{n+1}-t\right)p'_{n,k-1}(t)(t-x)^m dt \\ &\quad + nT_{n,m+1}(x) + \left(\frac{n}{n+1}\right)T_{n,m}(x) - \left(\frac{n+1}{n}\right)^{n-1}\left(\frac{n}{n+1}-x\right)^n(-x)^m \end{aligned}$$

$$= \frac{(n+1)^2}{n}\sum_{k=1}^{n} p_{n,k}(x)\int_0^{\frac{n}{n+1}}\left[\left(\frac{n}{n+1}-2x\right)(t-x)-(t-x)^2+x\left(\frac{n}{n+1}-x\right)\right]$$
$$\times p'_{n,k-1}(t)(t-x)^m dt + nT_{n,m+1}(x)+\left(\frac{n}{n+1}\right)T_{n,m}(x)$$
$$-\left(\frac{n+1}{n}\right)^{n-1}\left(\frac{n}{n+1}-x\right)^n(-x)^m$$
$$= -(m+1)\left(\frac{n}{n+1}-2x\right)\left[T_{n,m}(x)-\left(\frac{n+1}{n}\right)^n\left(\frac{n}{n+1}-x\right)^n(-x)^m\right]$$
$$+(m+2)\left[T_{n,m+1}(x)-\left(\frac{n+1}{n}\right)^n\left(\frac{n}{n+1}-x\right)^n(-x)^{m+1}\right]$$
$$-mx\left(\frac{n}{n+1}-x\right)\left[T_{n,m-1}(x)-\left(\frac{n+1}{n}\right)^n\left(\frac{n}{n+1}-x\right)^n(-x)^{m-1}\right]$$
$$+nT_{n,m+1}(x)+\left(\frac{n}{n+1}\right)T_{n,m}(x)-\left(\frac{n+1}{n}\right)^{n-1}\left(\frac{n}{n+1}-x\right)^n(-x)^m$$
$$= \left[\left(\frac{n}{n+1}\right)-(m+1)\left(\frac{n}{n+1}-2x\right)\right]T_{n,m}(x)$$
$$+(n+m+2)T_{n,m+1}(x)-mx\left(\frac{n}{n+1}-x\right)T_{n,m-1}(x).$$

Hence

$$(n+m+2)T_{n,m+1}(x) = x\left(\frac{n}{n+1}-x\right)\left[T'_{n,m}(x)+2mT_{n,m-1}(x)\right]$$
$$+\left[m\left(\frac{n}{n+1}-2x\right)-2x\right]T_{n,m}(x).$$

This completes the proof of the recurrence relation. If putting $m=1$ in this recurrence relation then we may easily obtain the value of $T_{n,2}(x)$. □

LEMMA 2. [5] *For $m \in N^0$; if the mth order moment is defined by*

$$\mu_{n,m}(x) = \sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n+1}-x\right)^m, \quad m = 0,1,2......,$$

*then we have,* $\mu_{n,0}(x)=1, \mu_{n,1}(x)=0$ *and*

$$n\mu_{n,m+1}(x) = x\left(\frac{n}{n+1}-x\right)\left[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right].$$

*Consequently, for every $x \in [0, 1-\frac{1}{n+1}]$, we have*

*(i)* $\mu_{n,m}(x)$ *is a polynomial in $x$ of degree $\leqslant m$,*

*(ii)* $\mu_{n,m}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$,

*where* $[\alpha]$ *denotes integral part of* $\alpha$, *i.e.,*

$$\mu_{n,m}(x) = \begin{cases} O\left(n^{-m/2}\right), & m \text{ (even)} \\ O\left(n^{-(m+1)/2}\right), & m \text{ (odd)}. \end{cases}$$

LEMMA 3. [5] *There exists the polynomials* $q_{i,j,r}(x)$ *independent of* $n$ *and* $k$ *such that*

$$x^r\left(\frac{n}{n+1}-x\right)^r \frac{d^r}{dx^r}p_{n,k}(x) = \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^{i+j}\left(\frac{k}{n+1}-x\right)^j q_{i,j,r}(x)p_{n,k}(x).$$

LEMMA 4. *For* $\nu \in N^0$, *if we define:*

$$M_n(t^\nu,x) = \frac{(n+1)^2}{n}\sum_{k=1}^{n} p_{n,k}(x)\int_0^{\frac{n}{n+1}} p_{n,k-1}(t)t^\nu dt + \nu^*\left(\frac{n+1}{n}\right)^n\left(\frac{n}{n+1}-x\right)^n,$$

*where* $\nu^*=1$ *if* $\nu=0$, $\nu^*=0$ *if* $\nu \geqslant 1$, *then*

$$M_n(1,x)=1,\ \ M_n(t;x)=\frac{n}{n+2}x,\ \ M_n(t^2;x)=\frac{n!(n+1)!}{(n-2)!(n+3)!}x^2+\frac{2(n!)^3}{((n-1)!)^2(n+3)!}x,$$

*and for* $\nu \geqslant 0$, *there holds the recurrence relation*

$$(n+\nu+2)M_n(t^{\nu+1},x) = x\left(\frac{n}{n+1}-x\right)M'_n(t^\nu,x)+\left\{nx+\nu\frac{n}{n+1}\right\}M_n(t^\nu,x).$$

*Thus*

$$M_n(t^\nu,x) = \frac{(n!)(n+1)!}{(n-\nu)!(n+\nu+1)!}x^\nu+\nu(\nu-1)\frac{(n!)^3}{(n-1)!(n-\nu+1)!(n+\nu+1)!}x^{\nu-1} + O_x(n^{-1}),$$

*where* $O_x(n^{-1})$ *is a polynomial in x with degree* $(\nu-2)$ *and order* $n^{-1}$.

*Proof.* The values of $M_n(1,x)$, $M_n(t,x)$ and $M_n(t^2,x)$ can be obtained from definition. Now we prove the recurrence relation.

$$\begin{aligned} & x\left(\frac{n}{n+1}-x\right)M_n'(t^\nu,x) \\ &= \frac{(n+1)^2}{n}\sum_{k=1}^{n}\frac{(n+1)^2}{n}\sum_{k=1}^{n} x\left(\frac{n}{n+1}-x\right)p'_{n,k}(x)\int_0^{\frac{n}{n+1}} p_{n,k-1}(t)t^\nu dt \\ &= \frac{(n+1)^2}{n}\sum_{k=1}^{n} n\left(\frac{k}{n+1}-x\right)p_{n,k}(x)\int_0^{\frac{n}{n+1}} p_{n,k-1}(t)t^\nu dt \end{aligned}$$

$$
\begin{aligned}
&= \left\{ \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} \left\{ n\left(\frac{k-1}{n+1} - t\right) + nt + \frac{n}{n+1} \right\} p_{n,k-1}(t) t^{\nu} dt \right. \\
&\quad \left. - nx \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k-1}(t) t^{\nu} dt \right\} \\
&= \frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} \left(\frac{n}{n+1} - t\right) p'_{n,k-1}(t) t^{\nu} dt \\
&\quad + n\frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k-1}(t) t^{\nu+1} dt \\
&\quad + \frac{n}{n+1} \frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k-1}(t) t^{\nu} dt \\
&\quad - nx\frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k-1}(t) t^{\nu} dt \\
&= \frac{n}{n+1} \frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p'_{n,k-1}(t) t^{\nu+1} dt \\
&\quad - \frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p'_{n,k-1}(t) t^{\nu+2} dt \\
&\quad + n\frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k-1}(t) t^{\nu+1} dt \\
&\quad + \frac{n}{n+1} \frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k-1}(t) t^{\nu} dt \\
&\quad - nx\frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k-1}(t) t^{\nu} dt \\
&= -(\nu+1)\frac{n}{n+1} \frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k-1}(t) t^{\nu} dt \\
&\quad + (\nu+2)\frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k-1}(t) t^{\nu+1} dt \\
&\quad + n\frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k-1}(t) t^{\nu+1} dt \\
&\quad + \frac{n}{n+1} \frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k-1}(t) t^{\nu} dt \\
&\quad - nx\frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k-1}(t) t^{\nu} dt \\
&= -(\nu+1)\frac{n}{n+1} M_n(t^{\nu},x) + (\nu+2) M_n(t^{\nu+1},x) + nM_n(t^{\nu+1},x) \\
&\quad + \frac{n}{n+1} M_n(t^{\nu},x) - nxM_n(t^{\nu},x)
\end{aligned}
$$

$$= (n+\nu+2)M_n(t^{\nu+1},x) - \left\{nx + \nu\frac{n}{n+1}\right\}M_n(t^\nu,x).$$

Hence,

$$(n+\nu+2)M_n(t^{\nu+1},x) = x\left(\frac{n}{n+1}-x\right)M'_n(t^\nu,x) + \left\{nx+\nu\frac{n}{n+1}\right\}M_n(t^\nu,x).$$

Now we suppose,

$$M_n(t^\nu,x) = \frac{(n!)(n+1)!}{(n-\nu)!(n+\nu+1)!}x^\nu + \nu(\nu-1)\frac{(n!)^3}{(n-1)!(n-\nu+1)!(n+\nu+1)!}x^{\nu-1} + O_x(n^{-1}).$$

Then from recurrence relation, we get

$$\begin{aligned}
&(n+\nu+2)M_n(t^{\nu+1},x)\\
&= x\left(\frac{n}{n+1}-x\right)\left\{\nu\frac{(n!)(n+1)!}{(n-\nu)!(n+\nu+1)!}x^{\nu-1}\right.\\
&\quad\left.+\nu(\nu-1)^2\frac{(n!)^3}{(n-1)!(n-\nu+1)!(n+\nu+1)!}x^{\nu-2} + O'_x(n^{-1})\right\}\\
&\quad+\left\{nx+\nu\frac{n}{n+1}\right\}\left\{\frac{(n!)(n+1)!}{(n-\nu)!(n+\nu+1)!}x^\nu\right.\\
&\quad\left.+\nu(\nu-1)\frac{(n!)^3}{(n-1)!(n-\nu+1)!(n+\nu+1)!}x^{\nu-1} + O_x(n^{-1})\right\}\\
&= \left\{-\nu\frac{(n!)(n+1)!}{(n-\nu)!(n+\nu+1)!} + \frac{n(n!)(n+1)!}{(n-\nu)!(n+\nu+1)!}\right\}x^{\nu+1}\\
&\quad+\{\frac{n}{n+1}\nu\frac{(n!)(n+1)!}{(n-\nu)!(n+\nu+1)!} - \nu(\nu-1)^2\frac{(n!)^3}{(n-1)!(n-\nu+1)!(n+\nu+1)!}\\
&\quad+\nu\frac{n}{n+1}\frac{(n!)(n+1)!}{(n-\nu)!(n+\nu+1)!}\\
&\quad+\nu(\nu-1)n\frac{(n!)^3}{(n-1)!(n-\nu+1)!(n+\nu+1)!}\}x^\nu \ + O_x(n^{-1})\}\\
&= \frac{n!(n+1)!}{(n-\nu-1)!(n+\nu+1)!}\{-\nu\frac{1}{n-\nu} + \frac{n}{n-\nu}\}x^{\nu+1}\\
&\quad+\nu\frac{(n!)^3}{(n-1)!(n-\nu)!(n+\nu+1)!}\{1-(\nu-1)^2\frac{1}{n-\nu+1}\\
&\quad+1+(\nu-1)n\frac{1}{n-\nu+1}\}x^\nu \ + O_x(n^{-1})\}\\
&= \frac{n!(n+1)!}{(n-\nu-1)!(n+\nu+1)!}x^{\nu+1} + \nu(\nu+1)\frac{(n!)^3}{(n-1)!(n-\nu)!(n+\nu+1)!}x^\nu + O_x(n^{-1})\}.
\end{aligned}$$

Thus by induction, we have required result.

If putting $\nu = 2$ in $M_n(t^{\nu+1},x)$ then we may obtain the value of $M_n(t^3,x)$. □

## 3. Main results

In this section, we shall prove the following main results:

THEOREM 1. *Let* $f \in C\left[0, 1-\frac{1}{n+1}\right]$ *and let* $f^{(r)}(x)$ *exist at a point* $x \in \left(0, 1-\frac{1}{n+1}\right)$ *then*

$$M_n^{(r)}(f,x) = f^{(r)}(x) + o(1) \qquad as \qquad n \to \infty.$$

*Further if* $f^{(r)}(x) \in C\left(0, 1-\frac{1}{n+1}\right)$, *then above limit holds uniformly on* $\left(0, 1-\frac{1}{n+1}\right)$.

*Proof.* First applying Taylor's expansion of $f$, we get

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. Now

$$M_n(f,x) = \int_0^{\frac{n}{n+1}} W_n(x,t) f(t) dt,$$

where

$$W_n(x,t) = \frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) p_{n,k-1}(t) + \left(1+\frac{1}{n}\right)^n \left(\frac{n}{n+1} - x\right)^n \delta(t),$$

and $\delta(t)$ being a dirac delta function. Then

$$\begin{aligned} M_n^{(r)}(f,x) &= \int_0^{\frac{n}{n+1}} W_n^{(r)}(x,t) f(t) dt \\ &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^{\frac{n}{n+1}} W_n^{(r)}(x,t)(t-x)^i dt \\ &\quad + \int_0^{\frac{n}{n+1}} W_n^{(r)}(x,t) \varepsilon(t,x)(t-x)^r dt = R_1 + R_2. \end{aligned}$$

From Lemma 4, it follows that $\int_0^{\frac{n}{n+1}} W_n(x,t) t^\nu dt$ is a polynomial in $x$ of degree exactly $\nu$ and the coefficient of $x^\nu$ is

$$\frac{(n!)(n+1)!}{(n-\nu)!(n+\nu+1)!}.$$

Thus using binomial expansion of $(t-x)^m$ and Lemma 1, we have

$$\begin{aligned} R_1 &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{\nu=0}^{i} \binom{i}{\nu} (-x)^{i-\nu} \frac{\partial^r}{\partial x^r} \int_0^{\frac{n}{n+1}} W_n(x,t) t^\nu dt \\ &= \frac{f^{(r)}(x)}{r!} \frac{\partial^r}{\partial x^r} \int_0^{\frac{n}{n+1}} W_n(x,t) t^r dt \end{aligned}$$

$$= \frac{f^{(r)}(x)}{r!}\frac{\partial^r}{\partial x^r}\left[\frac{(n!)\,(n+1)!}{(n-r)!\,(n+r+1)!}x^r + O_x(n^{-1})\right]$$
$$= f^{(r)}(x) + o(1), \text{ as } n \to \infty.$$

From Lemma 3, we get

$$\begin{aligned}
|R_2| &= \left|\int_0^{\frac{n}{n+1}} W_n{}^{(r)}(x,t)\,\varepsilon(t,x)\,(t-x)^r dt\right| \\
&\leqslant \frac{(n+1)^2}{n}\sum_{k=1}^{n}\left|D^r p_{n,k}(x)\right|\int_0^{\frac{n}{n+1}} p_{n,k-1}(t)\,|\varepsilon(t,x)|\,|(t-x)^r|\,dt \\
&\quad + \left(1+\frac{1}{n}\right)^n \frac{n!}{(n-r)!}\left(\frac{n}{n+1}-x\right)^{n-r}|\varepsilon(0,x)|\,|(-x)^r| \\
&= \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \frac{(n+1)^2}{n}\frac{n^{i+j}\left|q_{i,j,r}(x)\right|}{\left\{x\left(\frac{n}{n+1}-x\right)\right\}^r}\sum_{k=1}^{n}\left|\frac{k}{n+1}-x\right|^j p_{n,k}(x) \\
&\quad \times \int_0^{\frac{n}{n+1}} p_{n,k-1}(t)\,|\varepsilon(t,x)|(t-x)^r dt \\
&\quad + \left(1+\frac{1}{n}\right)^n \frac{n!}{(n-r)!}\left(\frac{n}{n+1}-x\right)^{n-r}|\varepsilon(0,x)|\,x^r = R_3 + R_4.
\end{aligned}$$

Since, $\varepsilon(t,x) \to 0$ as $t \to x$ for a given $\varepsilon > 0$ there exist a $\delta > 0$ such that

$$|\varepsilon(t,x)| < \varepsilon \text{ whenever } 0 < |t-x| < \delta.$$

Further, if $\gamma$ is any integer $\geqslant r$ then we can find a constant $C_2 > 0$ such that $|\varepsilon(t,x)(t-x)^r| \leqslant C_2|t-x|^\gamma$ for $|t-x| \geqslant \delta$. Thus for some $C_1 > 0$, we may write

$$\begin{aligned}
R_3 &\leqslant \frac{(n+1)^2}{n} C_1 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^{i+j}\sum_{k=1}^{n}\left|\frac{k}{n+1}-x\right|^j p_{n,k}(x) \\
&\quad \times \left\{\varepsilon \int\limits_{|t-x|<\delta} p_{n,k-1}(t)|t-x|^r dt + \int\limits_{|t-x|\geqslant\delta} p_{n,k-1}(t) C_2 |t-x|^\gamma dt\right\} \\
&= R_5 + R_6,
\end{aligned}$$

where

$$C_1 = \sup_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \frac{\left|q_{i,j,r}(x)\right|}{\left\{x\left(\frac{n}{n+1}-x\right)\right\}^r}.$$

Applying Schwarz inequality for integration and summation respectively, we get

$$\begin{aligned}
R_5 &\leqslant \varepsilon \frac{(n+1)^2}{n} C_1 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^{i+j} \sum_{k=1}^{n} p_{n,k}(x) \left| \frac{k}{n+1} - x \right|^j \left( \int_o^{\frac{n}{n+1}} p_{n,k-1}(t)dt \right)^{1/2} \\
&\quad \times \left( \int_o^{\frac{n}{n+1}} p_{n,k-1}(t)(t-x)^{2r} dt \right)^{1/2} \\
&\leqslant \varepsilon C_1 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^{i+j} \left( \sum_{k=1}^{n} p_{n,k}(x) \left( \frac{k}{n+1} - x \right)^{2j} \right)^{1/2} \\
&\quad \times \left( \frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_o^{\frac{n}{n+1}} p_{n,k-1}(t)(t-x)^{2r} dt \right)^{1/2}.
\end{aligned}$$

Using Lemma 1 and Lemma 2, we have

$$R_5 \leqslant \varepsilon C_1 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^{i+j} O(n^{-j/2}) O(n^{-r/2}) = \varepsilon O(1) = o(1).$$

Once again applying the Schwarz inequality and Lemma 1 and Lemma 2, we have

$$\begin{aligned}
R_6 &\leqslant \frac{(n+1)^2}{n} C_2 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^{i+j} \sum_{k=1}^{n} p_{n,k}(x) \left| \frac{k}{n+1} - x \right|^j \left( \int_{|t-x|\geqslant\delta} p_{n,k-1}(t)dt \right)^{1/2} \\
&\quad \times \left( \int_{|t-x|\geqslant\delta} p_{n,k-1}(t)(t-x)^{2\gamma} dt \right)^{1/2} \\
&\leqslant C_2 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^{i+j} \left( \sum_{k=1}^{n} p_{n,k}(x) \left( \frac{k}{n+1} - x \right)^{2j} \right)^{1/2} \\
&\quad \times \left( \frac{(n+1)^2}{n} \sum_{k=1}^{n} p_{n,k}(x) \int_o^{\frac{n}{n+1}} p_{n,k-1}(t)(t-x)^{2\gamma} dt \right)^{1/2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
R_6 &\leqslant \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^{i+j} O(n^{-j/2}) O(n^{-\gamma/2}) \\
&= n^i O\left(n^{j/2}\right) O\left(n^{-\gamma/2}\right) \\
&= O\left(n^{\frac{r-\gamma}{2}}\right) = o(1).
\end{aligned}$$

Thus, due to arbitrariness of $\varepsilon > 0$, it follows that $R_3 = o(1)$. Also $R_4 \to 0$ as $n \to \infty$. Hence, $R_2 = o(1)$. Collecting the estimates of $R_1$ and $R_2$, we get the required result. $\square$

THEOREM 2. *Let* $f \in C\left[0,1-\frac{1}{n+1}\right]$ *and let* $f^{(r+2)}(x)$ *exist at a point* $x \in \left(0,1-\frac{1}{n+1}\right)$ *then*

$$\lim_{n\to\infty} n\left[M_n^{(r)}(f,x)-f^{(r)}(x)\right]$$
$$=-x\left[\frac{r(3r+1)}{2}+x\right]f^{(r+2)}(x)+\left[r-2x(1+r)\right]f^{(r+1)}(x)-r(r+1)f^{(r)}(x)$$

*and above convergence is uniform if* $f^{(r+2)}$ *is continuous on* $\left(0,1-\frac{1}{n+1}\right)$.

*Proof.* By Taylor's expansion of $f$, we have

$$f(t)=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^{r+2},$$

where $\varepsilon(t,x)\to 0$ *as* $t\to x$.
Now

$$n\left[M_n{}^{(r)}(f,x)-f^{(r)}(x)\right]=n\left[\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\int_0^{\frac{n}{n+1}}W_n{}^r(x,t)(t-x)^i dt-f^{(r)}(x)\right]$$
$$+\left[n\int_0^{\frac{n}{n+1}}W_n{}^r(x,t)\,\varepsilon(t,x)(t-x)^{r+2}dt\right]=E_1+E_2.$$

Let us estimate $E_1$

$$E_1=n\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\int_0^{\frac{n}{n+1}}W_n{}^{(r)}(x,t)\,t^j dt-nf^{(r)}(x).$$
$$=\frac{f^{(r)}(x)}{r!}n\left[M_n{}^{(r)}(t^r,x)-r!\right]$$
$$+\frac{f^{(r+1)}(x)}{(r+1)!}n\left[(r+1)(-x)M_n{}^{(r)}(t^r,x)+M_n{}^{(r)}\left(t^{r+1},x\right)\right]$$
$$+\frac{f^{(r+2)}(x)}{(r+2)!}n\left[\frac{(r+2)(r+1)}{2}x^2M_n{}^{(r)}(t^r,x)\right.$$
$$\left.+(r+2)(-x)M_n{}^{(r)}\left(t^{r+1},x\right)+M_n{}^{(r)}\left(t^{r+2},x\right)\right]$$
$$=nf^{(r)}(x)\left[\frac{(n!)(n+1)!}{(n-r)!\,(n+r+1)!}-1\right]$$
$$+\frac{f^{(r+1)}(x)}{(r+1)!}n\left[(r+1)(-x)(r!)\left\{\frac{(n!)(n+1)!}{(n-r)!\,(n+r+1)!}\right\}\right.$$
$$\left.+\frac{(n!)(n+1)!}{(n-r-1)!\,(n+r+2)!}(r+1)!x+r(r+1)\frac{(n!)^3}{(n-1)!\,(n-r)!\,(n+r+2)!}r!\right]$$

$$
\begin{aligned}
&+\frac{f^{(r+2)}(x)}{(r+2)!}n\left[\frac{(r+2)(r+1)}{2}x^2(r!)\frac{(n!)(n+1)!}{(n-r)!(n+r+1)!}+(r+2)(-x)\right.\\
&\times\left\{\frac{(n!)(n+1)!}{(n-r-1)!(n+r+2)!}(r+1)!x+r(r+1)\frac{(n!)^3}{(n-1)!(n-r)!(n+r+2)!}(r!)\right\}\\
&+\left\{\frac{(n!)(n+1)!}{(n-r-2)!(n+r+3)!}\right\}\frac{(r+2)!x^2}{2}\\
&\left.+(r+2)(r+1)\left\{\frac{(n!)^3}{(n-1)!(n-r-1)!(n+r+3)!}(r+1)!x\right\}+O_x(n^{-1})\right].
\end{aligned}
$$

In order to complete the proof of theorem, it is sufficient to show that $E_2 \to 0$ as $n \to \infty$, which can be proved along the lines of the proof of Theorem 1 and by using Lemma 1, 2 and 3. □

*Acknowledgements.* The authors are extremely thankful to the referee for valuable suggestions that improved the presentation of this paper.

## REFERENCES

[1] S. N. BERNSTEIN, *Démonstration du théoréme de Weierstrass fondée sur le calcul de probabilités*, Commun. Sco. Math. Charkov, **13 (2)**, 1912, 1–2.

[2] N. DEO, *Some approximation for the linear combinations of modified Beta operators*, AJMAA, **2 (2)**, Article 4, 2005, 1–12.

[3] N. DEO, *Direct result on the Durrmeyer variant of Beta operators*, SEA. Bull. Math., **32** (2008), 283–290.

[4] N. DEO, *Direct result on exponential-type operators*, Appl. Math. Comput., **204** (2008), 109–115.

[5] N. DEO, M. A. NOOR AND M. A. SIDDIQUI, *On approximation by a class of new Bernstein type operators*, Appl. Math. Comput., **201** (2008), 3951–3958.

[6] Z. FINTA AND V. GUPTA, *Direct and inverse estimates for Phillips type operators*, J. Math. Anal. Appl., **303** (2005), 627–642.

[7] M. S. FLOATER, *On the convergence of derivatives of Bernstein approximation*, J. Approx. Theory, **134** (2005), 130–135.

[8] H. GONSKA AND I. RASA, *Asymptotic behaviour of differentiated Bernstein polynomials*, Mat. Vesnik **61** (2009), 53–60.

[9] H. GONSKA, M. HEILMANN AND I. RASA, *Asymptotic behaviour of differentiated Bernstein polynomials revisited*, General Mathematics (Sibiu) **18** (2010), 45–53.

[10] H. GONSKA, M. HEILMANN AND I. RASA, *Kantorovich operators of order k*, Numer. Funct. Anal. Optimiz. **32** (2011), 717–738.

[11] V. GUPTA, *Error estimation for mixed summation-integral type operators*, J. Math. Anal. Appl., **313 (2)** (2006), 69–81.

[12] V. GUPTA, P. N. AGRAWAL, AND H. KARSLI, *Simultaneous approximation for the Phillips-Bézier operators*, JIPAM, **7 (4)** (2006), Art. 141, 1–7.

[13] V. GUPTA AND O. DOGRU, *Direct results for certain family of Integral type operators*, JIPAM, **6 (1)**, Art. 21, 2005, 1–11.

[14] V. GUPTA AND N. ISPIR, *On simultaneous approximation for some modified Bernstein-type operators*, IJMMS, **71** (2004), 604–612.

[15] V. GUPTA AND J. SINHA, *Direct results on certain Szasz-Mirakyan operators*, Appl. Math. Comput. **195** (2008), 230–239.

[16] H. S. JUNG, N. DEO AND M. DHAMIJA, *Pointwise approximation by Bernstein type operators in mobile interval*, Appl. Math. Comput., **214 (1)** (2014), 683–694.

(Received May 7, 2015)

*Minakshi Dhamija*
*Department of Applied Mathematics*
*Delhi Technological University*
*(Formerly Delhi College of Engineering)*
*Bawana Road, Delhi 110042, India*
*e-mail:* minakshidhamija11@gmail.com

*Ryozi Sakai*
*Department of Mathematics*
*Meijo University*
*Nagoya 468-8502, Japan*
*e-mail:* ryozi@crest.ocn.ne.jp

*Naokant Deo*
*Department of Applied Mathematics*
*Delhi Technological University*
*(Formerly Delhi College of Engineering)*
*Bawana Road, Delhi 110042, India*
*e-mail:* dr_naokant_deo@yahoo.com