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GENERALISED ITERATION OF ENTIRE FUNCTIONS WITH FINITE ITERATED ORDER

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Abstract. In this paper, considering the generalised iteration of two entire functions we investigate the growth of iterated entire functions of finite iterated order to generalise some earlier results.

1. Introduction and definitions

For two transcendental entire functions $f(z)$ and $g(z)$ Clunie [4] showed that $\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,f)}=\infty$ and $\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}=\infty$. Singh [12] proved some comparative growth properties of $\log T(r,fg)$ and $T(r,f)$; also raised the question of investigating the comparative growth of $\log T(r,fg)$ and $T(r,g)$. During the past decades several authors [3, 4, 7, 8, 9, 10, 11, 12, 15] made close investigations on growth properties of composition of two entire functions with finite order to achieve various remarkable results. After this in 2009, Jin Tu et.al [14] investigate the growth of two composite entire functions of finite iterated order. In the present paper using the idea of generalised iteration introduced by Banerjee and Mondal [1], generalise the results of Jin Tu et.al [14] for generalised iterated entire functions with finite iterated order.

We do not explain the standard notations and definitions of the theory of meromorphic functions as those are available in [5].

Following Sato [13], we write $\log^{[0]}x=x$, $\exp^{[0]}x=x$ and for positive integer m, let $\log^{[m]}x=\log(\log^{[m-1]}x)$, $\exp^{[m]}x=\exp(\exp^{[m-1]}x)$.

In [2], Bernal introduced the notions of finite iterated order and finiteness degree of the order as follows.

DEFINITION 1.1. [2, 6] The iterated i order $\rho_i(f)$ of an entire function f is defined by

$$\rho_i(f)=\limsup_{r\to\infty}\frac{\log^{[i+1]}M(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log^{[i]}T(r,f)}{\log r}\quad (i\in\mathbb{N}).$$

Similarly, the iterated i lower order $\mu_i(f)$ of an entire function f is defined by

$$\mu_i(f)=\liminf_{r\to\infty}\frac{\log^{[i+1]}M(r,f)}{\log r}=\liminf_{r\to\infty}\frac{\log^{[i]}T(r,f)}{\log r}\quad (i\in\mathbb{N}).$$

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DEFINITION 1.2. [2, 6] The finiteness degree of the order of an entire function f is defined by

$$i(f) = \begin{cases} 0 & \text{if } f(z) \text{ is a polynomial;} \\ \min\{k \in \{1,2,\dots\}, \rho_k(f) < \infty\} & \text{if } f(z) \text{ is transcendental;} \\ \infty & \text{when } \rho_k(f) = \infty \text{ for all } k. \end{cases} \tag{1.1}$$

In 2012, Banerjee and Mondal [1] introduced a new type of iteration called generalised iteration.

DEFINITION 1.3. [1] Let $f(z)$ and $g(z)$ be entire functions and $\alpha \in (0,1]$ be any real number. Then the generalised iteration of $f(z)$ with respect to $g(z)$ is defined as follows:

$$\begin{aligned} f_{1,g}(z) &= (1-\alpha)z + \alpha f(z) \\ f_{2,g}(z) &= (1-\alpha)g_{1,f}(z) + \alpha f(g_{1,f}(z)) \\ f_{3,g}(z) &= (1-\alpha)g_{2,f}(z) + \alpha f(g_{2,f}(z)) \\ &\vdots \\ f_{n,g}(z) &= (1-\alpha)g_{n-1,f}(z) + \alpha f(g_{n-1,f}(z)) \end{aligned}$$

and so are

$$\begin{aligned} g_{1,f}(z) &= (1-\alpha)z + \alpha g(z) \\ g_{2,f}(z) &= (1-\alpha)f_{1,g}(z) + \alpha g(f_{1,g}(z)) \\ g_{3,f}(z) &= (1-\alpha)f_{2,g}(z) + \alpha g(f_{2,g}(z)) \\ &\vdots \\ g_{n,f}(z) &= (1-\alpha)f_{n-1,g}(z) + \alpha g(f_{n-1,g}(z)). \end{aligned}$$

Clearly all $f_{n,g}(z)$ and $g_{n,f}(z)$ are entire functions.

Throughout the paper we consider $f(z)$ and $g(z)$ are entire functions having finite iterated order if $\rho_p(f) < \infty$, $\rho_q(g) < \infty$and positive iterated lower order if $\mu_p(f) > 0$, $\mu_q(g) > 0$.

2. Known lemmas

Following lemmas will be needed in the sequel.

LEMMA 2.1. [10] *Let $f(z)$ and $g(z)$ be entire functions. If $M(r,g) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$ for any $\varepsilon > 0$, then*

$$T(r, f(g)) < (1+\varepsilon)T(M(r,g), f).$$

In particular if $g(0) = 0$, then $T(r, f(g)) \leqslant T(M(r,g), f)$ for all $r > 0$.

LEMMA 2.2. [4] *Let* $f(z)$ *and* $g(z)$ *be entire functions with* $g(0)=0$. *Let* β *satisfy* $0<\beta<1$ *and let* $c(\beta)=\frac{(1-\beta)^2}{4\beta}$. *Then for* $r>0$,

$$\begin{aligned} M(M(r,g),f) &\geqslant M(r,f(g)) \\ &\geqslant M(c(\beta)M(\beta r,g),f). \end{aligned}$$

Furthermore if $\beta=\frac{1}{2}$, *for sufficiently large* r

$$M(r,f(g))\geqslant M\left(\frac{1}{8}M\left(\frac{r}{2},g\right),f\right).$$

LEMMA 2.3. [5] *Let* $f(z)$ *and* $g(z)$ *be transcendental entire functions. Then*

$$\frac{T(r,f)}{T(r,g(f))}\to 0 \text{ as } r\to\infty.$$

3. Finite iterated order and finiteness degree of the order

THEOREM 3.1. *Let* $f(z)$ *and* $g(z)$ *be entire functions of finite iterated order and positive iterated lower order with* $i(f)=p,\ i(g)=q$.

(i) If n *is odd, then* $i(f_{n,g})=\frac{n+1}{2}p+\frac{n-1}{2}q$ *and* $\rho_{\frac{n+1}{2}p+\frac{n-1}{2}q}(f_{n,g})=\rho_p(f)$ *and*

(ii) if n *is even, then* $i(f_{n,g})=\frac{n}{2}(p+q)$ *and* $\rho_{\frac{n}{2}(p+q)}(f_{n,g})=\rho_q(g)$.

Proof. By Definition 1.1, we have for given $\varepsilon>0$ and for sufficiently large r

$$T(r,f)\leqslant \exp^{[p-1]}(r^{\rho_p(f)+\varepsilon}),\qquad M(r,g)\leqslant \exp^{[q]}(r^{\rho_q(g)+\varepsilon}).$$

For sufficiently large r, we have

$$\begin{aligned} T(r,f_{n,g}) &\leqslant T(r,g_{n-1,f})+T(r,f(g_{n-1,f}))+O(1) \\ &= (1+o(1))T(r,f(g_{n-1,f})), \quad \text{using Lemma 2.3} \\ &\leqslant 2T(M(r,g_{n-1,f}),f), \quad \text{using Lemma 2.1} \\ &\leqslant \exp^{[p-1]}\{M(r,g_{n-1,f})\}^{\rho_p(f)+2\varepsilon} \\ &= \exp^{[p]}\{(\rho_p(f)+2\varepsilon)\log M(r,g_{n-1,f})\} \qquad (3.1) \\ &\leqslant \exp^{[p]}[(\rho_p(f)+2\varepsilon)\{\log M(r,f_{n-2,g})+\log M(r,g(f_{n-2,g}))+O(1)\}] \\ &\leqslant \exp^{[p]}[(\rho_p(f)+2\varepsilon)\{\log M(M(r,f_{n-2,g}),g)+\log M(M(r,f_{n-2,g}),g) \\ &\quad +O(1)\}], \quad \text{using Lemma 2.2 and since } g \text{ is clearly transcendental} \\ &\leqslant \exp^{[p]}\{3(\rho_p(f)+2\varepsilon)\log M(M(r,f_{n-2,g}),g)\} \\ &\leqslant \exp^{[p]}[3(\rho_p(f)+2\varepsilon)\log\{\exp^{[q]}\{M(r,f_{n-2,g})\}^{\rho_q(g)+\varepsilon}\}] \\ &\leqslant \exp^{[p+q]}\{(\rho_q(g)+2\varepsilon)\log M(r,f_{n-2,g})\} \end{aligned}$$

$$\leqslant \exp^{[p+q]}[(\rho_p(g)+2\varepsilon)\{\log M(r,g_{n-3,f})+\log M(r,f(g_{n-3,f}))+O(1)\}]$$
$$\leqslant \exp^{[p+q]}[(\rho_p(g)+2\varepsilon)\{\log M(M(r,g_{n-3,f}),f)+\log M(M(r,g_{n-3,f})$$
$$+O(1)\}], \quad \text{using Lemma 2.2 and since } f \text{ is clearly transcendental}$$
$$\leqslant \exp^{[p+q]}\{3(\rho_q(g)+2\varepsilon)\log M(M(r,g_{n-3,f}),f)\}$$
$$\leqslant \exp^{[p+q]}[3(\rho_q(g)+2\varepsilon)\log\{\exp^{[p]}\{(M(r,g_{n-3,f})\}^{\rho_p(f)+\varepsilon}\}]$$
$$\leqslant \exp^{[2p+q]}\{(\rho_p(f)+2\varepsilon)\log M(r,g_{n-3,f})\}$$
$$\leqslant \exp^{[2p+q]}[(\rho_p(f)+2\varepsilon)\{\log M(r,f_{n-4,g})+\log M(r,g(f_{n-4,g}))+O(1)\}]$$
$$\leqslant \exp^{[2p+q]}[(\rho_p(f)+2\varepsilon)\{\log M(M(r,f_{n-4,g}),g)+\log M(M(r,f_{n-4,g}),g)$$
$$+O(1)\}] \quad \text{using Lemma 2.2 and since } g \text{ is clearly transcendental}$$
$$\leqslant \exp^{[2p+q]}\{3(\rho_p(f)+2\varepsilon)\log M(M(r,f_{n-4,g}),g)\}$$
$$\leqslant \exp^{[2p+q]}[3(\rho_p(f)+2\varepsilon)\log\{\exp^{[q]}\{M(r,f_{n-4,g})\}^{\rho_q(g)+\varepsilon}\}]$$
$$\leqslant \exp^{[2p+2q]}\{(\rho_q(g)+2\varepsilon)\log M(r,f_{n-4,g})\}.$$

Here two cases may arise.

Case (i). Suppose n is odd. Then

$$T(r,f_{n,g}) \leqslant \exp^{[2p+2q]}\{(\rho_q(g)+2\varepsilon)\log M(r,f_{n-4,g})\}$$
$$\vdots$$
$$\leqslant \exp^{[\frac{n-1}{2}p+\frac{n-1}{2}q]}\{(\rho_q(g)+2\varepsilon)\log M(r,f_{1,g})\}$$
$$\leqslant \exp^{[\frac{n-1}{2}(p+q)]}[(\rho_q(g)+2\varepsilon)\{\log M(r,z)+\log M(r,f)+O(1)\}]$$
$$\leqslant \exp^{[\frac{n-1}{2}(p+q)]}\{(\rho_q(g)+2\varepsilon)(1+o(1))\log M(r,f)\} \tag{3.2}$$
$$\leqslant \exp^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}\{\log(r^{\rho_p(f)+2\varepsilon})\}. \tag{3.3}$$

Therefore,

$$\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}T(r,f_{n,g})}{\log r} \leqslant \rho_p(f)+2\varepsilon, \quad r>r_0. \tag{3.4}$$

On the otherhand, since $i(f)=p,$ we have

$$\limsup_{r\to\infty}\frac{\log^{[p+1]}M(r,f)}{\log r}=\rho_p(f).$$

Since $\rho_p(f)>0$, there exists a sequence $\{r_m\}$ tending to infinity such that for given ε $[0<\varepsilon<\rho_p(f)]$ and for sufficiently large r_m, we have

$$M(r_m,f) \geqslant \exp^{[p]}(r_m^{\rho_p(f)-\varepsilon}). \tag{3.5}$$

We denote $\{r_m\}$, a sequence, tending to infinity, not necessarily the same at each occurence. Since $\mu_p(f)>0$, $\mu_q(g)>0$ and by the same reasoning as K. Niino and C.

C. Yang [11], for sufficiently large r_m, we have

$$
\begin{aligned}
T(r_m, f_{n,g}) &\geqslant T(r_m, f(g_{n-1,f})) - T(r_m, g_{n-1,f}) + O(1) \\
&= (1+o(1))T(r_m, f(g_{n-1,f})), \quad \text{using Lemma 2.3} \\
&\geqslant \frac{1}{3}(1+o(1))\log M\left(\frac{1}{8}M\left(\frac{r_m}{4}, g_{n-1,f}\right) + o(1), f\right) \\
&\geqslant \frac{1}{3}(1+o(1))\log M\left(\frac{1}{9}M\left(\frac{r_m}{4}, g_{n-1,f}\right), f\right) \\
&\geqslant \exp^{[p]}\left[\log\left\{M\left(\frac{r_m}{4}, g_{n-1,f}\right)\right\}^{\mu_p(f)-2\varepsilon}\right] \\
&\geqslant \exp^{[p]}\left\{(\mu_p(f)-2\varepsilon)T\left(\frac{r_m}{4}, g_{n-1,f}\right)\right\} \\
&\geqslant \exp^{[p]}\left[(\mu_p(f)-2\varepsilon)\left\{T\left(\frac{r_m}{4}, g(f_{n-2,g})\right) - T\left(\frac{r_m}{4}, f_{n-2,g}\right) + O(1)\right\}\right] \\
&= \exp^{[p]}\left\{(\mu_p(f)-2\varepsilon)(1+o(1))T\left(\frac{r_m}{4}, g(f_{n-2,g})\right)\right\}, \quad \text{using Lemma 2.3} \\
&\geqslant \exp^{[p]}\left\{\frac{1}{3}(\mu_p(f)-2\varepsilon)(1+o(1))\log M\left(\frac{1}{9}M\left(\frac{r_m}{4^2}, f_{n-2,g}\right), g\right)\right\} \\
&\geqslant \exp^{[p]}\left[\exp^{[q]}\left\{\log\left\{M\left(\frac{r_m}{4^2}, f_{n-2,g}\right)\right\}^{\mu_q(g)-2\varepsilon}\right\}\right] \\
&= \exp^{[p+q]}\left\{(\mu_q(g)-2\varepsilon)\log M\left(\frac{r_m}{4^2}, f_{n-2,g}\right)\right\} \qquad (3.6) \\
&\;\vdots \\
&\geqslant \exp^{[\frac{n-1}{2}(p+q)]}\left\{(\mu_q(g)-2\varepsilon)\log M\left(\frac{r_m}{4^{n-1}}, f_{1,g}\right)\right\} \\
&\geqslant \exp^{[\frac{n-1}{2}(p+q)]}\left\{(\mu_q(g)-2\varepsilon)(1+o(1))\log M\left(\frac{r_m}{4^{n-1}}, f\right)\right\} \qquad (3.7) \\
&= \exp^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}\{\log(r_m)^{\rho_p(f)-2\varepsilon}\}, \quad \text{using (3.5).} \qquad (3.8)
\end{aligned}
$$

Therefore,

$$
\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r_m, f_{n,g})}{\log r_m} \geqslant \rho_p(f) - 2\varepsilon, \quad \text{for } r = r_m \to \infty. \qquad (3.9)
$$

From (3.4) and (3.9), we get

$$
\limsup_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log r} = \rho_p(f).
$$

Therefore, $i(f_{n,g}) = \frac{n+1}{2}p + \frac{n-1}{2}q$ and

$$
\rho_{\frac{n+1}{2}p+\frac{n-1}{2}q}(f_{n,g}) = \rho_p(f). \qquad (3.10)
$$

Case (ii). Suppose n is even. Then

$$\begin{aligned}
T(r, f_{n,g}) &\leqslant \exp^{[2p+2q]}\{(\rho_q(g)+2\varepsilon)\log M(r, f_{n-4,g})\} \\
&\ \vdots \\
&\leqslant \exp^{[\frac{n}{2}p+\frac{n-2}{2}q]}\{(\rho_p(f)+2\varepsilon)\log M(r, g_{1.f})\} \\
&\leqslant \exp^{[\frac{n}{2}p+\frac{n-2}{2}q]}[(\rho_p(f)+2\varepsilon)\{\log M(r,z)+\log M(r,g)+O(1)\}] \\
&\leqslant \exp^{[\frac{n}{2}p+\frac{n-2}{2}q]}\{(\rho_p(f)+2\varepsilon)(1+o(1))\log M(r,g)\} \qquad (3.11) \\
&\leqslant \exp^{[\frac{n}{2}(p+q)]}\{\log(r^{\rho_q(g)+2\varepsilon})\}. \qquad (3.12)
\end{aligned}$$

Therefore,

$$\frac{\log^{[\frac{n}{2}(p+q)]} T(r, f_{n,g})}{\log r} \leqslant \rho_q(g)+2\varepsilon, \quad r > r_0. \qquad (3.13)$$

By similar argument as in case (i) and from (3.6), we have

$$\begin{aligned}
T(r_m, f_{n,g}) &\geqslant \exp^{[p+q]}\left\{(\mu_q(g)-2\varepsilon)\log M\left(\frac{r_m}{4^2}, f_{n-2,g}\right)\right\} \\
&\ \vdots \\
&\geqslant \exp^{[\frac{n}{2}p+\frac{n-2}{2}q]}\left\{(\mu_p(f)-2\varepsilon)\log M\left(\frac{r_m}{4^{n-1}}, g_{1,f}\right)\right\} \\
&\geqslant \exp^{[\frac{n}{2}p+\frac{n-2}{2}q]}\left\{(\mu_p(f)-2\varepsilon)(1+o(1))\log M\left(\frac{r_m}{4^{n-1}}, g\right)\right\} \qquad (3.14) \\
&\geqslant \exp^{[\frac{n}{2}p+\frac{n-2}{2}q]}\left[(\mu_p(f)-2\varepsilon)(1+o(1))\log\left\{\exp^{[q]}\left(\frac{r_m}{4^{n-1}}\right)^{\rho_q(g)-\varepsilon}\right\}\right] \\
&= \exp^{[\frac{n}{2}(p+q)]}\{\log({r_m}^{\rho_q(g)-2\varepsilon})\}. \qquad (3.15)
\end{aligned}$$

Therefore,

$$\frac{\log^{[\frac{n}{2}(p+q)]} T(r_m, f_{n,g})}{\log r_m} \geqslant \rho_q(g)-2\varepsilon, \text{ for } r = r_m \to \infty. \qquad (3.16)$$

From (3.13) and (3.16), we get

$$\limsup_{r\to\infty} \frac{\log^{[\frac{n}{2}(p+q)]} T(r, f_{n,g})}{\log r} = \rho_q(g).$$

Therefore,

$$i(f_{n,g}) = \frac{n}{2}(p+q)$$

and

$$\rho_{\frac{n}{2}(p+q)}(f_{n,g}) = \rho_q(g). \quad \square$$

COROLLARY 3.1. *Let $f(z)$ and $g(z)$ be entire functions of finite iterated order and positive iterated lower order with $p \leqslant i(f) \leqslant l$ and $i(g) = q$.*

(i) If n is odd, then

$$\frac{n+1}{2}p + \frac{n-1}{2}q \leqslant i(f_{n,g}) \leqslant \frac{n+1}{2}l + \frac{n-1}{2}q$$

and

$$\rho_{\frac{n+1}{2}p+\frac{n-1}{2}q}(f_{n,g}) \geqslant \rho_p(f), \qquad \rho_{\frac{n+1}{2}l+\frac{n-1}{2}q}(f_{n,g}) \leqslant \rho_p(f);$$

and

(ii) if n is even, then

$$\frac{n}{2}(p+q) \leqslant i(f_{n,g}) \leqslant \frac{n}{2}(l+q)$$

and

$$\rho_{\frac{n}{2}(p+q)}(f_{n,g}) \geqslant \rho_q(g), \rho_{\frac{n}{2}(l+q)}(f_{n,g}) \leqslant \rho_q(g).$$

Proof. Case (i). Suppose n is odd.
Let $i(f) = m$. Then $m = \min\{j : \rho_j(f) < \infty\}$.
So, $\rho_{m+k}(f) < \infty$, for $k = 0, 1, 2, \ldots$ and $\rho_{m-k}(f) = \infty$, for $k = 1, 2, \ldots$.
Now, since $i(f) = m$ and $i(g) = q$, from case (i) of Theorem 3.1, we have

$$i(f_{n,g}) = \frac{n+1}{2}m + \frac{n-1}{2}q. \tag{3.17}$$

Now $p \leqslant m \leqslant l$ gives

$$\frac{n+1}{2}p + \frac{n-1}{2}q \leqslant \frac{n+1}{2}m + \frac{n-1}{2}q \leqslant \frac{n+1}{2}l + \frac{n-1}{2}q$$

i.e.,

$$\frac{n+1}{2}p + \frac{n-1}{2}q \leqslant i(f_{n,g}) \leqslant \frac{n+1}{2}l + \frac{n-1}{2}q. \tag{3.18}$$

Now from (3.17), (3.18) and (3.10), we get

$$\rho_{\frac{n+1}{2}p+\frac{n-1}{2}q}(f_{n,g}) \geqslant \rho_{\frac{n+1}{2}m+\frac{n-1}{2}q}(f_{n,g}) = \rho_p(f),$$
$$\rho_{\frac{n+1}{2}l+\frac{n-1}{2}q}(f_{n,g}) \leqslant \rho_{\frac{n+1}{2}m+\frac{n-1}{2}q}(f_{n,g}) = \rho_p(f).$$

Case (ii). Suppose n is even.
Then the proof is omitted since it is as in case (i). □

COROLLARY 3.2. *Let $f(z)$ and $g(z)$ be entire functions of finite iterated order and positive iterated lower order.*

(i) If n is odd and $i(f_{n,g}) = \frac{n+1}{2}p + \frac{n-1}{2}q$ then

$$i(f) = p \text{ and } \rho_{\frac{n+1}{2}p+\frac{n-1}{2}q}(f_{n,g}) = \rho_p(f);$$

and

(ii) if n is even and $i(f_{n,g}) = \frac{n}{2}(p+q)$ then

$$i(g) = q \ \textit{and} \ \rho_{\frac{n}{2}(p+q)}(f_{n,g}) = \rho_q(g).$$

Proof. Case (i). Suppose n is odd.
Since $i(f_{n,g}) = \frac{n+1}{2}p + \frac{n-1}{2}q$, we have

$$\rho_{\frac{n+1}{2}p+\frac{n-1}{2}q-1}(f_{n,g}) = \infty \ \text{ and } \ \rho_{\frac{n+1}{2}p+\frac{n-1}{2}q}(f_{n,g}) < \infty.$$

Since $\rho_{\frac{n+1}{2}p+\frac{n-1}{2}q-1}(f_{n,g}) = \infty$, then for any arbitrary large λ

$$\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q-1]} T(r, f_{n,g})}{\log r} > \lambda,$$

for large values of r.

But from (3.2),for large r, we have

$$T(r, f_{n,g}) \leqslant \exp^{[\frac{n-1}{2}(p+q)]}\{(\rho_q(g) + 2\varepsilon)(1 + o(1)) \log M(r, f)\}.$$

Therefore, for all large r

$$\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q-1]} T(r, f_{n,g}) \leqslant \log^{[p]} M(r, f) + O(1)$$

i.e., $\frac{\log^{[p]} M(r,f)+O(1)}{\log r} \geqslant \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q-1]} T(r,f_{n,g})}{\log r} > \lambda.$
So,

$$\rho_{p-1}(f) = \infty. \tag{3.19}$$

Again $\rho_{\frac{n+1}{2}p+\frac{n-1}{2}q}(f_{n,g}) < \infty$. Let $\rho_{\frac{n+1}{2}p+\frac{n-1}{2}q}(f_{n,g}) = l < \infty$.

Then for given ε (> 0) there exists a sequence $\{r_m\}$ tending to infinity such that for large r_m, we get

$$\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r_m, f_{n,g})}{\log r_m} \leqslant l + \varepsilon.$$

Again from (3.7), we have

$$T(r_m, f_{n,g}) \geqslant \exp^{[\frac{n-1}{2}(p+q)]}\left\{(\mu_q(g) - 2\varepsilon)(1 + o(1)) \log M\left(\frac{r_m}{4^{n-1}}, f\right)\right\}.$$

Therefore,

$$\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r_m, f_{n,g}) \geqslant \log^{[p+1]} M\left(\frac{r_m}{4^{n-1}}, f\right) + O(1)$$

i.e.,

$$\frac{\log^{[p+1]} M(\frac{r_m}{4^{n-1}}, f) + O(1)}{\log r_m} \leqslant \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r_m, f_{n,g})}{\log r_m} \leqslant l + \varepsilon$$

i.e.,

$$\frac{\log^{[p+1]} M(r_m, f)}{\log r_m} \leqslant l+\varepsilon, \text{ for } r = r_m \to \infty$$

i.e.,

$$\rho_p(f) < \infty. \tag{3.20}$$

From (3.19) and (3.20), we get $i(f) = p$.
Again from (3.10), $\rho_{\frac{n+1}{2}p+\frac{n-1}{2}q}(f_{n,g}) = \rho_p(f)$.

Case (ii). Suppose n is even.
Then the proof is omitted since it is as in case (i). □

COROLLARY 3.3. *Let $f(z)$ and $g(z)$ be entire functions of finite iterated order and positive iterated lower order with $i(f_{n,g}) = p$ $(n \geqslant 2)$ and $\frac{1}{2} < \alpha \leqslant 1$ then $\rho_p(f) = 0$.*

Proof. Since $i(f_{n,g}) = p$, so $\rho_p(f_{n,g}) = \beta$ (say) $< \infty$. Then for any given ε (>0) and for sufficiently large r, we have

$$M(r, f_{n,g}) \leqslant \exp^{[p]}(r^{\beta+\varepsilon}). \tag{3.21}$$

Clearly f and g are transcendental. So we have for all sufficiently large r and arbitrary large m

$$\begin{aligned}
M(r^m, f) &\leqslant (2\alpha-1)M\Big(\frac{1}{8}M\Big(\frac{r}{2}, g_{n-1,f}\Big), f\Big)\\
&= \alpha M\Big(\frac{1}{8}M\Big(\frac{r}{2}, g_{n-1,f}\Big), f\Big) - (1-\alpha)M\Big(\frac{1}{8}M\Big(\frac{r}{2}, g_{n-1,f}\Big), f\Big)\\
&\leqslant \alpha M\Big(\frac{1}{8}M\Big(\frac{r}{2}, g_{n-1,f}\Big), f\Big) - (1-\alpha)M(r, g_{n-1,f})\\
&\leqslant \alpha M(r, f(g_{n-1,f})) - (1-\alpha)M(r, g_{n-1,f}), \quad \text{using Lemma 2.2}\\
&\leqslant M(r, f_{n,g})\\
&\leqslant \exp^{[p]}(r^{\beta+\varepsilon}), \quad \text{by (3.21).}
\end{aligned}$$

Therefore, $M(r, f) \leqslant \exp^{[p]}\{r^{\frac{\beta}{m}+\varepsilon'}\}$, where $\varepsilon' = \frac{\varepsilon}{m}$.
So, $\rho_p(f) \leqslant \frac{\beta}{m}$ and since m is arbitrarily large, we get $\rho_p(f) = 0$. □

4. Growth of generalised iterated entire functions

THEOREM 4.1. *Let $f(z)$, $g(z)$ be entire functions of finite iterated order and positive iterated lower order with $i(f) = p$, $i(g) = q$ and $\rho_q(g) < \mu_p(f)$.*
(i) If n is odd, then

$$\lim_{r\to\infty} \frac{\log^{[\frac{n-1}{2}(p+q)+1]} T(r, f_{n,g})}{T(r, f)} = 0$$

and

$$\lim_{r\to\infty}\frac{\log^{[\frac{n-1}{2}(p+q)+2]}M(r,f_{n,g})}{\log M(r,f)}=0$$

and

(ii) if n is even, then

$$\lim_{r\to\infty}\frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q]}T(r,f_{n,g})}{T(r,f)}=0$$

and

$$\lim_{r\to\infty}\frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q+1]}M(r,f_{n,g})}{\log M(r,f)}=0.$$

Proof. For sufficiently large r, we have

$$\exp^{[p-1]}(r^{\mu_p(f)-\varepsilon})\leqslant T(r,f)\leqslant \log M(r,f)\leqslant \exp^{[p-1]}(r^{\rho_p(f)+\varepsilon}). \tag{4.1}$$

Case (i). Suppose n is odd. Then for sufficiently large r and for given ε $[0<\varepsilon<\mu_p(f)]$ by (3.3) and (4.1), we have

$$\frac{\log^{[\frac{n-1}{2}(p+q)+1]}T(r,f_{n,g})}{T(r,f)}\leqslant\frac{\exp^{[p-2]}(r^{\rho_p(f)+2\varepsilon})}{\exp^{[p-1]}(r^{\mu_p(f)-\varepsilon})}.$$

Therefore, $\lim_{r\to\infty}\frac{\log^{[\frac{n-1}{2}(p+q)+1]}T(r,f_{n,g})}{T(r,f)}=0$.

For sufficiently large r,

$$\begin{aligned}
M(r,f_{n,g}) &\leqslant M(r,g_{n-1,f})+M(r,f(g_{n-1,f}))+O(1)\\
&\leqslant M(M(r,g_{n-1,f}),f)+M(M(r,g_{n-1,f}),f)+O(1),\\
&\qquad\text{using Lemma 2.2and since f is clearly transcendental}\\
&\leqslant (2+o(1))M(M(r,g_{n-1,f}),f)\\
&\leqslant (2+o(1))\exp^{[p]}\{M(r,g_{n-1,f})\}^{\rho_p(f)+\varepsilon}\\
&\leqslant \exp[\exp^{[p]}\{(\rho_p(f)+2\varepsilon)\log M(r,g_{n-1,f})\}]\\
&\ \vdots\\
&\leqslant \exp[\exp^{[\frac{n+1}{2}p+\frac{n-1}{2}q)]}\{\log(r^{\rho_p(f)+2\varepsilon})\}],\quad \text{using (3.1) and (3.3)}\\
&\leqslant \exp^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}(r^{\rho_p(f)+2\varepsilon}).
\end{aligned}\tag{4.2}$$

By (4.1), (4.2) and sufficiently large r and for any given ε $[0<\varepsilon<\mu_p(f)]$, we have

$$\frac{\log^{[\frac{n-1}{2}(p+q)+2]}M(r,f_{n,g})}{\log M(r,f)}\leqslant\frac{\exp^{[p-2]}\{r^{\rho_p(f)+2\varepsilon}\}}{\exp^{[p-1]}\{r^{\mu_p(f)-\varepsilon}\}}.$$

Therefore, $\lim_{r\to\infty}\frac{\log^{[\frac{n-1}{2}(p+q)+2]}M(r,f_{n,g})}{\log M(r,f)}=0$.

Case (ii). Suppose n is even. Then for sufficiently large r and for any given ε $[0 < 3\varepsilon < \mu_p(f) - \rho_q(g)]$, by (3.12) and (4.1), we have

$$\frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q]} T(r, f_{n,g})}{T(r,f)} \leqslant \frac{\exp^{[p-1]}\{r^{\rho_q(g)+2\varepsilon}\}}{\exp^{[p-1]}\{r^{\mu_p(f)-\varepsilon}\}}.$$

Therefore, $\lim_{r\to\infty} \frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q]} T(r,f_{n,g})}{T(r,f)} = 0.$
By similar reasoning as in case (i), we get

$$\lim_{r\to\infty} \frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q+1]} M(r, f_{n,g})}{\log M(r,f)} = 0. \quad \square$$

NOTE 4.1. When n is odd, the restriction $\rho_q(g) < \mu_p(f)$ may be relaxed.

THEOREM 4.2. *Let $f(z)$, $g(z)$ be entire functions of finite iterated order and positive iterated lower order with $i(f) = p$, $i(g) = q$ and $\rho_q(g) < \rho_p(f)$.*
(i) If n is odd, then

$$\liminf_{r\to\infty} \frac{\log^{[\frac{n-1}{2}(p+q)+1]} T(r, f_{n,g})}{T(r,f)} = 0$$

and

$$\liminf_{r\to\infty} \frac{\log^{[\frac{n-1}{2}(p+q)+2]} M(r, f_{n,g})}{\log M(r,f)} = 0$$

and
(ii) if n is even, then

$$\liminf_{r\to\infty} \frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q]} T(r, f_{n,g})}{T(r,f)} = 0,$$

and

$$\liminf_{r\to\infty} \frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q+1]} M(r, f_{n,g})}{\log M(r,f)} = 0.$$

Proof. There exists a sequence $\{r_m\} \to \infty$ such that for given ε (> 0) and for sufficiently large r_m, we have

$$T(r_m, f) \geqslant \exp^{[p-1]}(r_m^{\rho_p(f)-\varepsilon}). \tag{4.3}$$

Let n be even. Then using (4.3) instead of (4.1) we proceed as in Theorem 4.1 to get results. $\square$

THEOREM 4.3. *Let $f(z)$, $g(z)$ be entire functions of finite iterated order and positive iterated lower order with $i(f)=p$, $i(g)=q$ and $\mu_q(g)<\mu_p(f)$.*

(i) If n is odd, then

$$\liminf_{r\to\infty}\frac{\log^{[\frac{n-1}{2}(p+q)+1]}T(r,f_{n,g})}{T(r,f)}=0$$

and

$$\liminf_{r\to\infty}\frac{\log^{[\frac{n-1}{2}(p+q)+2]}M(r,f_{n,g})}{\log M(r,f)}=0$$

and

(ii) if n even, then

$$\liminf_{r\to\infty}\frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q]}T(r,f_{n,g})}{T(r,f)}=0$$

and

$$\liminf_{r\to\infty}\frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q+1]}M(r,f_{n,g})}{\log M(r,f)}=0.$$

Proof. Case (i). Suppose n is odd.

Given ε $[0<\varepsilon<\mu_p(f)]$ and for sufficiently large r, from (4.1) and (3.3), we get

$$\frac{\log^{[\frac{n-1}{2}(p+q)+1]}T(r,f_{n,g})}{T(r,f)}\leqslant\frac{\exp^{[p-2]}(r^{\rho_p(f)+2\varepsilon})}{\exp^{[p-1]}(r^{\mu_p(f)-\varepsilon})}.$$

Therefore, $\liminf_{r\to\infty}\frac{\log^{[\frac{n-1}{2}(p+q)+1]}T(r,f_{n,g})}{T(r,f)}=0.$

Case (ii). Suppose n is even. Thenthere exists a sequence $\{r_m\}\to\infty$ such that for sufficiently large r_m and for given ε (>0), we have from (3.11)

$$\begin{aligned} T(r_m,f_{n,g}) &\leqslant \exp^{[\frac{n}{2}p+(\frac{n}{2}-1)q]}\{(\rho_p(f)+2\varepsilon)(1+o(1))\log M(r,g)\}\\ &\leqslant \exp^{[\frac{n}{2}(p+q)]}\{\log(r_m)^{\mu_q(g)+2\varepsilon}\}. \end{aligned}\tag{4.4}$$

From (4.1) and (4.4), for chosen ε $[0<3\varepsilon<\mu_p(f)-\mu_q(g)]$ and for sufficiently large r_m, we get

$$\frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q]}T(r_m,f_{n,g})}{T(r_m,f)}\leqslant\frac{\exp^{[p-1]}(r_m)^{\mu_q(g)+2\varepsilon}}{\exp^{[p-1]}(r_m)^{\mu_p(f)-\varepsilon}}.$$

Therefore, $\liminf_{r\to\infty}\frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q]}T(r,f_{n,g})}{T(r,f)}=0.$ □

THEOREM 4.4. *Let $f(z)$, $g(z)$ be entire functions having positive iterated lower order and finite iterated order of $f(z)$ such that $\rho_p(f) < \rho_q(g)$.*
(i) If n is odd, then

$$\limsup_{r\to\infty} \frac{\log^{[\frac{n-1}{2}(p+q)-1]} T(r, f_{n,g})}{T(r,f)} = \infty$$

and

$$\limsup_{r\to\infty} \frac{\log^{[\frac{n-1}{2}(p+q)]} M(r, f_{n,g})}{\log M(r,f)} = \infty$$

and
(ii) if n is even, then

$$\limsup_{r\to\infty} \frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q]} T(r, f_{n,g})}{T(r,f)} = \infty$$

and

$$\limsup_{r\to\infty} \frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q+1]} M(r, f_{n,g})}{\log M(r,f)} = \infty.$$

Proof. There exists a sequence $\{r_m\} \to \infty$ such that for any given $\varepsilon(>0)$ and for sufficiently large r_m, we have

$$T(r_m, f) \leqslant \exp^{[p-1]}(r_m^{\rho_p(f)+\varepsilon}). \tag{4.5}$$

Case (i). Suppose n is odd. Then from relation (3.8) and (4.5), for chosen ε $[0 < 2\varepsilon < \rho_p(f)]$ and for sufficiently large $r_m,$ we have

$$\frac{\log^{[\frac{n-1}{2}(p+q)-1]} T(r_m, f_{n,g})}{T(r_m,f)} \geqslant \frac{\exp^{[p]}(r_m^{\rho_p(f)-2\varepsilon})}{\exp^{[p-1]}(r_m^{\rho_p(f)+\varepsilon})}.$$

Hence,

$$\limsup_{r\to\infty} \frac{\log^{[\frac{n-1}{2}(p+q)-1]} T(r, f_{n,g})}{T(r,f)} = \infty.$$

Case (ii). Suppose n is even. Then from relation (3.15) and (4.5), for chosen ε $[0 < 3\varepsilon < \rho_q(g) - \rho_p(f)]$ and for sufficiently large r_m, we have

$$\frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q]} T(r_m, f_{n,g})}{T(r_m,f)} \geqslant \frac{\exp^{[p-1]}(r_m^{\rho_q(g)-2\varepsilon})}{\exp^{[p-1]}(r_m^{\rho_p(f)+\varepsilon})}$$

and hence the result. □

NOTE 4.2. When n is odd, the restriction $\rho_p(f) < \rho_q(g)$ may be relaxed.

THEOREM 4.5. *Let $f(z),\ g(z)$ be entire functions having positive iterated lower order such that $\mu_p(f) < \mu_q(g)$.*

(i) If n is odd, then

$$\limsup_{r\to\infty} \frac{\log^{[\frac{n-1}{2}(p+q)-1]} T(r, f_{n,g})}{T(r,f)} = \infty$$

and

$$\limsup_{r\to\infty} \frac{\log^{[\frac{n-1}{2}(p+q)]} M(r, f_{n,g})}{\log M(r,f)} = \infty$$

and

(ii) if n is even, then

$$\limsup_{r\to\infty} \frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q]} T(r, f_{n,g})}{T(r,f)} = \infty$$

and

$$\limsup_{r\to\infty} \frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q+1]} M(r, f_{n,g})}{\log M(r,f)} = \infty.$$

Proof. There exists a sequence $\{r_m\} \to \infty$ such that for chosen ε $[0 < 2\varepsilon < \mu_p(f)]$ and for sufficiently large r_m, we have

$$T(r_m, f) \leqslant \exp^{[p-1]}(r_m^{\mu_p(f)+\varepsilon}). \tag{4.6}$$

Case (i). Suppose n is odd. Then from (3.7), for sufficiently large r_m, we have

$$\begin{aligned} T(r_m, f_{n,g}) &\geqslant \exp^{[\frac{n-1}{2}(p+q)]}\left\{(\mu_q(g)-2\varepsilon)(1+o(1))\log M\left(\frac{r_m}{4^{n-1}}, f\right)\right\} \\ &\geqslant \exp^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}\{\log(r_m)^{\mu_p(f)-2\varepsilon}\}. \end{aligned} \tag{4.7}$$

By (4.6) and (4.7), we have for sufficiently large r_m

$$\frac{\log^{[\frac{n-1}{2}(p+q)-1]} T(r_m, f_{n,g})}{T(r_m,f)} \geqslant \frac{\exp^{[p]}(r_m^{\mu_p(f)-2\varepsilon})}{\exp^{[p-1]}(r_m^{\mu_p(f)+\varepsilon})}.$$

So,

$$\limsup_{r\to\infty} \frac{\log^{[\frac{n-1}{2}(p+q)-1]} T(r, f_{n,g})}{T(r,f)} = \infty.$$

Case (ii). Suppose n is even. Then from (3.14), for sufficiently large r_m and for given ε $[0 < 3\varepsilon < \mu_q(g) - \mu_p(f)]$, we have

$$\begin{aligned} T(r_m, f_{n,g}) &\geqslant \exp^{[\frac{n}{2}p+(\frac{n}{2}-1)q]}\left\{(\mu_p(f)-2\varepsilon)(1+o(1))\log M\left(\frac{r_m}{4^{n-1}}, g\right)\right\} \\ &\geqslant \exp^{[\frac{n}{2}(p+q)]}\{\log(r_m)^{\mu_q(g)-2\varepsilon}\}. \end{aligned} \tag{4.8}$$

From (4.6) and (4.8), we have

$$\frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q]} T(r_m, f_{n,g})}{T(r_m, f)} \geqslant \frac{\exp^{[p-1]}(r_m^{\mu_q(g)-2\varepsilon})}{\exp^{[p-1]}(r_m^{\mu_p(f)+\varepsilon})}$$

and hence the result. □

THEOREM 4.6. *Let $f(z)$, $g(z)$ be entire functions having positive iterated lower order and finite iterated order of $f(z)$ such that $\rho_p(f) < \mu_q(g)$.*

(i) If n is odd, then

$$\lim_{r\to\infty} \frac{\log^{[\frac{n-1}{2}(p+q)-1]} T(r, f_{n,g})}{T(r, f)} = \infty$$

and

$$\lim_{r\to\infty} \frac{\log^{[\frac{n-1}{2}(p+q)]} M(r, f_{n,g})}{\log M(r, f)} = \infty$$

and

(ii) if n is even, then

$$\lim_{r\to\infty} \frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q]} T(r, f_{n,g})}{T(r, f)} = \infty$$

and

$$\lim_{r\to\infty} \frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q+1]} M(r, f_{n,g})}{\log M(r, f)} = \infty.$$

Proof. Case (i). Suppose n is odd. Then by (4.1) and (4.7), we have for chosen $\varepsilon[0 < 2\varepsilon < \mu_p(f)]$ and for sufficiently large r

$$\frac{\log^{[\frac{n-1}{2}(p+q)-1]} T(r, f_{n,g})}{T(r, f)} \geqslant \frac{\exp^{[p]}(r^{\mu_p(f)-2\varepsilon})}{\exp^{[p-1]}(r^{\rho_p(f)+\varepsilon})}.$$

So,

$$\lim_{r\to\infty} \frac{\log^{[\frac{n-1}{2}(p+q)-1]} T(r, f_{n,g})}{T(r, f)} = \infty.$$

Case (ii). Suppose n is even. Then from (4.8) and (4.1) for sufficiently large r and chosen ε $[0 < 3\varepsilon < \mu_q(g) - \rho_p(f)]$, we have

$$\frac{\log^{[(\frac{n}{2}-1)p+\frac{n}{2}q]} T(r, f_{n,g})}{T(r, f)} \geqslant \frac{\exp^{[p-1]}(r^{\mu_q(g)-2\varepsilon})}{\exp^{[p-1]}(r^{\rho_p(f)+\varepsilon})}$$

and the result follows. □

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