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POINTWISE ESTIMATE FOR LINEAR COMBINATIONS OF PHILLIPS OPERATORS

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Abstract. For pointwise approximation of bounded continuous functions by linear combinations of Phillips operators we represent equivalent relation by means of Ditzian-Totik modulus of smoothness. The rate of approximation is better compared with similar estimates, proved in the past for other Szász-type operators.

1. Introduction

The aim of this paper is to prove pointwise estimate for approximation of bounded continuous functions $f(x)$ defined on $[0,\infty)$ by linear combinations of Phillips operators, which for $n \in \Re, n > 0$ are given by

$$(\widetilde{S}_n f)(x) := s_{n,0} f(0) + \sum_{k=1}^{\infty} s_{n,k}(x) n \int_0^{\infty} s_{n,k-1}(t) f(t) dt, \tag{1.1}$$

where

$$s_{n,k}(x) = \frac{(nx)^k}{k!} e^{-nx}, \quad k \in \mathbb{N}_0, \ \ n > \alpha, x \in [0,\infty),$$

for every function f, for which the right-hand side of (1.1) makes sense. For $n > \alpha$ this is the case for real valued continuous functions on $[0,\infty)$ satisfying an exponential growth condition, i.e.

$$f \in C_\alpha[0,\infty) = \{ f \in C[0,\infty) : |f(t)| \leqslant M e^{\alpha t}, \quad t \in [0,\infty)\}.$$

For $\alpha = 0$ we use the following notation for bounded continuous functions, i.e.

$$f \in C_B[0,\infty) = \{ f \in C[0,\infty) : |f(t)| \leqslant M, t \in [0,\infty)\}.$$

The operators $\widetilde{S}_n$ were first considered in a paper by Phillips in 1954 in [22]. In 2011 in a joint paper with M. Heilmann – [18] we proved direct and strong converse result of type A in terminology of Ditzian and Ivanov [15] which we formulate here as (see Theorems 5.1 and 5.2 in [18]).

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THEOREM A. *For every $f \in C_B[0,\infty)$ and $n > 0$ there holds*

$$\|\widetilde{S}_n f - f\|_{C_B[0,\infty)} \leqslant 2K_\varphi^2\left(f, \frac{1}{n}\right), \tag{1.2}$$

$$K_\varphi^2\left(f, \frac{1}{n}\right) \leqslant 92.16\|\widetilde{S}_n f - f\|_{C_B[0,\infty)}, \tag{1.3}$$

where the definition of the K-functional $K_\varphi^2(f,\cdot)$ is given some lines below.

We choose the step-weight $\varphi(x) = \sqrt{x}$ and assume $t > 0$ sufficiently small to define for $1 \leqslant p \leqslant \infty$:

$$\omega_\varphi^r(f,t)_p = \sup_{0<h\leqslant t} \|\Delta_{h\varphi}^r f\|_p,$$

where the symmetric difference is given by

$$\Delta_{h\varphi(x)}^r f(x) = \sum_{k=0}^r (-1)^k \binom{r}{k} f\left(x + \left(\frac{r}{2} - k\right) h\varphi(x)\right), \tag{1.4}$$

whenever the arguments of the function f are contained in the cooresponding interval. In [14], (see Chapters 2,3 6.1) Ditzian and Totik proved that these moduli are equivalent to the K-functional:

$$K_\varphi^r(f,t^r)_p = \inf\{\|f-g\|_p + t^r\|\varphi^r g^{(r)}\|_p\}, \tag{1.5}$$

where the inf is taken over all functions g, such that $g, \varphi^r g^{(r)} \in L_p[0,\infty)$, $1 \leqslant p \leqslant \infty$. For bounded continuous functions f as usual we consider the supremum norm $\|\cdot\|_{C_B[0,\infty)}\|$ instead of L_p norm and further we omit the symbol ∞ in $K_\varphi^r(f,t)_\infty$. As a corollary from Theorem A we get the following equivalence

$$\|\widetilde{S}_n f - f\|_{C_B[0,\infty)} \approx \omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)_\infty.$$

The last implies the characterization of the rate of the approximation by $\widetilde{S}_n$ in terms of the smoothness of the approximated function, i.e. for $0 < \alpha \leqslant 1$ we have

$$\|\widetilde{S}_n f - f\|_{C_B[0,\infty)} = O(n^{-\alpha}) \Longleftrightarrow \omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)_\infty = O(n^{-\alpha}). \tag{1.6}$$

To increase the order of approximation by Phillips operators, C. May was the first who studied in [20] the linear combinations of $\widetilde{S}_n$. These linear combinations were introduced by Butzer – [5] in order to improve the degree of approximation by Bernstein polynomials. The combinations in [20] are generalized by Agrawal and Gupta in [6, 1, 2, 3, 7, 8] where iterative combinations are also considered. It was proved in [19] that all of the above mentioned combinations suit into the following general approach, introduced in the book of Ditzian-Totik – [14]. We consider the linear combinations of

$$\widetilde{S}_{n,r} = \sum_{i=0}^r \alpha_i(n)\widetilde{S}_{n_i}, \tag{1.7}$$

where in general the coefficients $\alpha_i(n)$ may depend on n. We determine the coefficients $\alpha_i(n)$ in (1.7) such that all polynomials of degree at most $r+1$ are reproduced, i.e. $\widetilde{S}_{n,r}p = p \ \forall p \in \mathscr{P}_{r+1}$. This seems to be natural as the operators $\widetilde{S}_n$ preserve linear functions. The requirement that each polynomial of degree at most $r+1$ should be reproduced leads to a linear system of equations

$$\sum_{i=0}^{r} \alpha_i(n) = 1, \tag{1.8}$$

$$\sum_{i=0}^{r} n_i^{-l}\alpha_i(n) = 0, \quad 1 \leqslant l \leqslant r, \tag{1.9}$$

which has the unique solution

$$\alpha_i(n) = n_i^r \prod_{\substack{k=0\\k\neq i}}^{r} \frac{1}{n_i - n_k}.$$

We note that $\widetilde{S}_{n,0} = \widetilde{S}_n$. It was shown in [19] that all three types of linear combinations mentioned above are special cases of this general form of linear combinations.

For the proofs of our theorem we need two additional assumptions for the coefficients. The first condition is

$$an \leqslant n_0 < n_1 < \ldots < n_r \leqslant An, \tag{1.10}$$

where a, A denote positive constants, independent of n. Secondly we assume that

$$\sum_{i=0}^{r} |\alpha_i(n)| \leqslant C, \tag{1.11}$$

with a constant C independent of n. This condition is due to the fact that the linear combinations are no longer positive operators. Especially for the proof of the direct result in [18, 23] this assumption is important. Combinations of the type, satisfying the four conditions $(8,9,10,11)$ were already used by Ditzian [13] to achieve for $\alpha < 2r$ and $\varphi^2(x) = x(1-x)$ that

$$\|B_{n,r}f - f\|_{C[0,1]} = O\left(n^{-\frac{\alpha}{2}}\right) \Longleftrightarrow \omega_{\varphi}^{2r}(f,h)_{\infty} = O\left(h^{\alpha}\right). \tag{1.12}$$

The case of pointwise approximations by linear combinations of Bernstein operators $B_{n,r}f$ was recently studied by S. Guo and coauthors in [10, 11] and by L. Xie in [26, 27]. The pointwise approximation by linear combinations of Bernstein-Kantorovich operators was considered in [12, 25]. For the sake fo completeness we cite the main result in [11] (see Theorem 1 there):

THEOREM B. *For* $f \in C[0,1]$, $r \in \mathbb{N}$, $0 < \alpha < 2r$, $1 - \frac{1}{r} \leqslant \lambda \leqslant 1$, *we have*

$$B_{n,r}(f,x) - f(x) = O\left((n^{-\frac{1}{n}}\phi^{1-\lambda}(x))^{\alpha}\right) \Longleftrightarrow \omega_{\phi^{\lambda}}^{2r}(f,t) = O(t^{\alpha}),$$

with $\phi^2(x) = x(1-x)$. *For* $0 \leqslant \lambda < 1 - \frac{1}{r}$ *the last equivalence is not true.*

The first result for pointwise approximation by linear combinations of Durrmeyer variant of Szász-Mirakyan operator (SMD) was proved by D. X. Zhou in [28]. The equivalent relation in [28] was given in terms of ordinary moduli of smoothness $\omega^r(f,\delta)$. We recall that the SMD operator is defined as

$$\bar{S}_n(f,x) = \sum_{k=0}^{\infty} n s_{n,k}(x) \cdot \int_0^{\infty} s_{n,k}(t) f(t)dt. \tag{1.13}$$

Later in 1998 S. Guo and coauthors in [9] extended this result by the Ditzian-Totik modulus of smoothness $\omega^r_{\varphi^\lambda}(f,t)$ with parameter $0 \leqslant \lambda \leqslant 1$, where the definition of $\omega^r_{\varphi^\lambda}(f,t)$ and related $K-$ functional $K^r_{\varphi^\lambda}(f,t^r)$ are the same as in (1.4) and (1.5) with φ replaced by φ^λ. It is known that for $\lambda = 1$ we obtain the Ditzian-Totik modulus and for $\lambda = 0$ – the ordinary moduli of smoothness $\omega^r(f,t)$. If with $L_{n,r}$ we denote the linear combinations of SMD operator $\bar{S}_n(f,x)$ then the result of Guo et. al. in [9] states the following:

THEOREM C. *If* $f \in C_B[0,\infty)$, $r \in \mathbb{N}$, $0 < \alpha < r$, $0 \leqslant \lambda \leqslant 1$, *then the following statements are equivalent*

$$|L_{n,r}(f,x) - f(x)| = O\left((n^{-\frac{1}{2}} \cdot \delta_n^{1-\lambda}(x))^\alpha\right), \tag{1.14}$$

$$\omega^r_{\varphi^\lambda}(f,t) = O(t^\alpha), \tag{1.15}$$

where $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}} \approx \max\{\varphi(x), \frac{1}{\sqrt{n}}\}$.

As the authors in [9] mentioned, the Zhou's inverse estimate did not (and could not) cover the range between r and $2r$, the same follows in [9]. On the other hand for $\lambda = 1$ $2r$ can replace r and the same equivalence (1.14) $\Longleftrightarrow$ (1.15) holds true. The Phillips operator known also as genuine SMD operator is very close to the Szász-Mirakyan-Durrmeyer (SMD) operator. The both operators are commutative and preserve linear functions. But Phillips operator interpolates f at $x = 0$, which property is missing for the SMD operator. Also close to 0 the moments of Phillips operator contain $x = \varphi^2(x)$ as a multiplyer, which is essential to prove our direct pointwise estimate. Now we formulate our main result:

THEOREM 1. *Let* $f \in C_B[0,\infty)$, $r \in \mathbb{N}$, $0 < \alpha < 2r+2$. *Then,*
(i) for $x \in [0,1]$, $1 - \frac{1}{r} < \lambda \leqslant 1$ *the following direct pointwise estimate holds true*

$$\left|\widetilde{S}_{n,r}(f,x) - f(x)\right|$$
$$\leqslant C(r)\left\{\omega^{2r+2}_{\varphi^\lambda}\left(f, n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)\right) + \left(n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)\right)^{2(r+1)} \|f\|_{L_\infty[0,\infty)}\right\};$$

(ii) for $x \in (1,\infty)$ *the following direct pointwise estimate holds true*

$$\left|\widetilde{S}_{n,r}(f,x) - f(x)\right| \leqslant C(r)\left\{\omega^{2r+2}_{\varphi}(f, \frac{1}{\sqrt{n}}) + n^{-(r+1)}\|f\|_{L_\infty[0,\infty)}\right\};$$

(iii) for $x \in [0,\infty)$ the following equivalence holds true

$$\left|\widetilde{S}_{n,r}(f,x) - f(x)\right| = O\left(n^{-\frac{\alpha}{2}}\right) \Longleftrightarrow \omega_{\varphi}^{2r+2}(f,t) = O(t^{\alpha}).$$

For $r=0$ the linear combinations $\widetilde{S}_{n,r}$ reduces to the single Phillips operator of degree n. In this case we can unify the cases $x \in [0,1]$ and $x \in (1,\infty)$ in our second result:

THEOREM 2. *Let $f \in C_B[0,\infty)$, $0 \leqslant \lambda \leqslant 1$. Then for all $x \in [0,\infty)$ we have the pointwise estimate*

$$\left|\widetilde{S}_{n}(f,x) - f(x)\right| \leqslant C \cdot \omega_{\varphi^{\lambda}}^{2}\left(f, n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)\right).$$

REMARK 1. If we compare Theorem 1 with Theorem C, we see that in our main result the range of the parameter α is $(0,2r+2)$, while for the SMD operator in Theorem C the range is $(0,r)$. The next advantage in Theorem 1 is that here in (i) when $x \in [0,1]$ we have $\varphi(x)$ instead of $\delta_n(x)$ in (1.14).

REMARK 2. To prove our main result we use essentially the results proved recently in [18, 19, 23, 24]. Some of the estimates, obtained in the last three papers will be given as auxilliary results in the next section. This paper may be considered as natural continuation of the study for approximation by linear combinations of Phillips operators, but in the pointwise form. We point out also that in the papers of May,Gupta, Agrawal etc. mentioned above the question to obtain equivalence relation for approximation by $\widetilde{S}_{n,r}$ in a pointwise form was not considered.

REMARK 3. If we compare Theorem 2 with Theorem 1 in [4] (see (11) on p. 1498) we see that the same direct pointwise estimate holds true for the case of Phillips operator. The difference is that while in [4] this result is established in weighted norm with weight function $\rho(x) = (1+x)^m$, $m \in \mathbb{N}$ for unbounded functions with polynomial growth, in our case we consider bounded continuous functions.

The paper is organized as follows. In Section 2 we give some auxilliary results and lemmas. The proofs of Theorem 1 and Theorem 2 are given in Section 3. Note that throughout this paper C always denotes a positive constant not necessarily the same at each occurence and $C(r)$ is a constant, dependent only on r.

2. Auxilliary results

Recently in [23] the following direct pointwise estimate was proved:

THEOREM D. *With $\varphi(x) = \sqrt{x}$, $x \in [0,\infty)$-fixed we have*

$$|\widetilde{S}_{n,r}(f,x) - f(x)| \leqslant C \cdot A(r,x) \cdot \left\{ n^{-(r+1)} \|f\|_{C_B[0,\infty)} + \omega_{\varphi}^{2(r+1)}\left(f, \frac{1}{\sqrt{n}}\right)_{\infty} \right\}, \quad (2.1)$$

where $A(r,x)$ is constant, dependent only on r,x and C is a constant from the second condition (1.11), imposed on $\alpha_i(n)$.

If we follow step by step the proof of this result, we confirm that the constant $A(r,x)$ do not depend on x. Actually for the case $x\in(1,\infty)$ we prove the same pointwise estimate, but in a quite different and simpler method. Based on this direct pointwise estimate very recently the following equivalence relation in $C[0,\infty)$ was established in [24]:

THEOREM E. *Let $f\in C_B[0,\infty)$, $r\in\mathbb{N}_0$, $k,n\in\mathbb{N}$. Then we have for $\alpha<2r+2$*

$$K_{\varphi}^{2r+2}\left(f,n^{-r-1}\right)\leqslant\|\widetilde{S}_{k,r}f-f\|_{C_B[0,\infty}+M\left(\frac{k}{n}\right)^{r+1}\cdot K_{\varphi}^{2r+2}\left(f,k^{-r-1}\right),$$

$$\|\widetilde{S}_{n,r}f-f\|_{C_B[0,\infty)}=O\left(n^{-\frac{\alpha}{2}}\right)\Longleftrightarrow\omega_{\varphi}^{2r+2}(f,h)_{\infty}=O\left(h^{\alpha}\right). \tag{2.2}$$

For the proof of our main statement we need explicitly representation of the moments for the Phillips operator $\widetilde{S}_n$ – [18] (Lemma 2.1) and for the linear combinations, established in [19] (see Lemma 5.2 there):

LEMMA 1. *For $f_{\mu,x}(t)=(t-x)^{\mu}$, $\mu\in\mathbb{N}_0$ we have*

$$(\widetilde{S}_{n,r}f_{0,x})(x)=1,\quad(\widetilde{S}_{n,r}f_{\mu,x})(x)=0,1\leqslant\mu\leqslant r+1, \tag{2.3}$$

$$(\widetilde{S}_{n,r}f_{\mu,x})(x)=\sum_{j=1}^{\mu-(r+1)}\binom{\mu-j-1}{j-1}\frac{\mu!}{j!}x^j\sum_{i=0}^{r}n_i^{j-\mu}\alpha_i(n), \tag{2.4}$$

for $r+2\leqslant\mu\leqslant 2r+2$,

$$(\widetilde{S}_{n,r}f_{\mu,x})(x)=\sum_{j=1}^{\left[\frac{\mu}{2}\right]}\binom{\mu-j-1}{j-1}\frac{\mu!}{j!}x^j\sum_{i=0}^{r}n_i^{j-\mu}\alpha_i(n), \tag{2.5}$$

for $\mu\geqslant 2r+2$,

$$(\widetilde{S}_{n}f_{\mu,x})(x)=\sum_{j=1}^{\left[\frac{\mu}{2}\right]}\binom{\mu-j-1}{j-1}\frac{\mu!}{j!}x^jn^{j-\mu}, \tag{2.6}$$

for $\mu\geqslant 2$,

Our next auxilliary result is the following estimate for the weighted norms of intermediate derivatives (see the proof of Theorem 9.5.3, inequalities (b) and (c) in [14]).

LEMMA 2. *For $\varphi(x)=\sqrt{x}$, $f\in L_p[0,\infty)$, $1\leqslant p\leqslant\infty$, $\varphi^{2r}f^{(2r)}\in L_p[0,\infty)$, then for all $i<r$ the following holds true*

$$\|\varphi^{2r-2i}f^{(2r-i)}\|_{L_p[0,\infty)}\leqslant C\left(\|\varphi^{2r}f^{(2r)}\|_{L_p[0,\infty)}+\|f\|_{L_p[0,\infty)}\right). \tag{2.7}$$

If we follow step by step the proof of (2.7) given in the book of Ditzian-Totik (see the proof of Theorem 9.5.3, inequalities (b) and (c) in [14]) we may replace the weight function φ by φ^{λ} to deliver the following inequality, needed to prove (i) in our main result:

LEMMA 3. *For* $f(x)\in C[0,\infty)$, $r\geqslant 1$, $f^{(2r-1)}(x)\in A.C.loc$, *when* $1-\frac{1}{r}<\lambda\leqslant 1$, $m=1,2,\ldots,r-1$ *we have*

$$\|\varphi^{2r\lambda-2m}f^{(2r-m)}\|\leqslant C(\|f\|+\|\varphi^{2r\lambda}f^{(2r)}\|), \tag{2.8}$$

where the norm $\|\cdot\|:=\|\cdot\|_{L_\infty[0,\infty)}$.

3. Proofs of main statements

We start with the proof of Theorem 1.

Proof.
Proof of (i). We divide the proof of (i) in Theorem 1 into two subcases:

1. *case* $0\leqslant x\leqslant \frac{1}{n}$.

If $x=0$ the left-hand side of (i) is 0 and the proof is obvious. So let $x\in(0,\frac{1}{n}]$. Let $g\in C_B[0,\infty)$, $g^{(2r+1)}\in A.C.loc$ and $\|\varphi^{(2r+2)}g^{(2r+2)}\|_{L_\infty[0,\infty)}<\infty$. Then, recalling the propery (2.3) in Lemma 1 we apply the following form of remainder in Taylor formula

$$\left|\widetilde{S}_{n,r}(g,x)-g(x)\right|\leqslant\left|\widetilde{S}_{n,r}\left(\int_x^t(t-u)^{r+1}g^{(r+2)}(u)du,x\right)\right|$$

$$\leqslant\|\varphi^{2(r+1)\lambda-2r}\cdot g^{(r+2)}\|_{L_\infty[0,\infty)}\cdot\sum_{i=0}^{r}|\alpha_i(n)|\left|\widetilde{S}_{n_i}\left(\int_x^t\frac{(t-u)^{r+1}}{\varphi^{2(r+1)\lambda-2r}(u)}du x\right)\right|. \tag{3.1}$$

For each $u=t+\tau(x-t)$, $\tau,t\in[0,1]$ we can estimate (using the concavity of the function $\varphi^{2(r+1)\lambda-2r}(x)$):

$$\frac{|t-u|^{r+1}}{\varphi^{2(r+1)\lambda-2r}(u)}=\frac{\tau^{r+1}\cdot|x-t|^{r+1}}{\varphi^{2(r+1)\lambda-2r}(t+\tau(x-t))}$$

$$\leqslant\frac{\tau^{r+1}\cdot|x-t|^{r+1}}{\varphi^{2(r+1)\lambda-2r}(t)+\tau\left(\varphi^{2(r+1)\lambda-2r}(x)-\varphi^{2(r+1)\lambda-2r}(t)\right)}\leqslant\frac{|x-t|^{r+1}}{\varphi^{2(r+1)\lambda-2r}(x)}. \tag{3.2}$$

Following the estimates (3.1) and (3.2) and the conditions (1.10)–(1.11) we write

$$\left|\widetilde{S}_{n,r}(g,x)-g(x)\right|$$

$$\leqslant C\|\varphi^{2(r+1)\lambda-2r}\cdot g^{(r+2)}\|_{L_\infty[0,\infty)}\cdot\varphi^{2r-2(r+1)\lambda}(x)\cdot\widetilde{S}_n(|t-x|^{r+2},x). \tag{3.3}$$

The Cauchy-Schwarz inequality implies

$$\widetilde{S}_n(|t-x|^{r+2},x)\leqslant\sqrt{\widetilde{S}_n((t-x)^{2r+2},x)}\sqrt{\widetilde{S}_n((t-x)^2,x)}.$$

It is easy to observe, that for $x \leqslant \frac{1}{n}$ the dominant summand in (2.6) is for $j=1$. Therefore from (2.6) we get

$$\widetilde{S}_n((t-x)^{2r+2},x) \leqslant C(r)n^{-2(r+1)} \cdot xn, \widetilde{S}_n((t-x)^2,x) = \frac{2x}{n}.$$

Consequently

$$\widetilde{S}_n(|t-x|^{r+2},x) \leqslant C(r)xn^{-(r+1)}. \tag{3.4}$$

Hence (3.3) and (3.4) yield

$$\left|\widetilde{S}_{n,r}(g,x)-g(x)\right|$$

$$\leqslant C(r)n^{-(r+1)} \cdot \varphi^{2(r+1)(1-\lambda)}(x) \cdot \|\varphi^{2(r+1)\lambda-2r} \cdot g^{(r+2)}\|_{L_\infty[0,\infty)}. \tag{3.5}$$

By the standart decomposition using the auxilliary function g we write for $f \in C_B[0,\infty)$

$$\left|\widetilde{S}_{n,r}(f,x)-f(x)\right|$$

$$\leqslant \left|\widetilde{S}_{n,r}(f,x)-\widetilde{S}_{n,r}(g,x)\right| + \left|\widetilde{S}_{n,r}(g,x)-g(x)\right| + |g(x)-f(x)|$$

$$\leqslant C(r)\left\{\|f-g\|_{L_\infty[0,\infty)} + \left(n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)\right)^{2(r+1)} \cdot \|\varphi^{2(r+1)\lambda-2r} \cdot g^{(r+2)}\|_{L_\infty[0,\infty)}\right\}. \tag{3.6}$$

In the last upper bound we apply Lemma 3, estimate $\|g\| \leqslant \|f\| + \|f-g\|$, and take the inf over all auxilliary functions g to arrive at the following direct pointwise estimate

$$\left|\widetilde{S}_{n,r}(f,x)-f(x)\right|$$

$$\leqslant C(r)\left\{\omega_{\varphi^\lambda}^{2r+2}\left(f,n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)\right) + \left(n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)\right)^{2(r+1)} \|f\|_{L_\infty[0,\infty)}\right\}. \tag{3.7}$$

In the last inequality we have used the equivalence between $K_{\varphi^\lambda}^{2r+2}(f,t^{2r+2})$ and the moduli $\omega_{\varphi^\lambda}^{2r+2}(f,t)$. In the case $\frac{1}{n} < x \leqslant 1$ our goal is to establish the same upper bound (3.7).

2. *case* $\frac{1}{n} < x \leqslant 1$.

In this subcase we apply another form of Taylor formula and according to (2.3) we have

$$\left|\widetilde{S}_{n,r}(g,x)-g(x)\right| \leqslant \sum_{i=1}^{r} \frac{|g^{(2r+2-i)}(x)|}{(2r+2-i)!} \cdot \left|\widetilde{S}_{n,r}((t-x)^{2r+2-i},x)\right|$$

$$+ \left|\widetilde{S}_{n,r}\left(\int_x^t (t-u)^{2r+1} g^{(2r+2)}(u)du,\, x\right)\right| := I_1 + I_2. \tag{3.8}$$

From (2.4) in Lemma 1 and the corresponding upper bound for these moments (see Corollary 5.3 in [19]) we verify that

$$\left|\widetilde{S}_{n,r}((t-x)^{2r+2-i},x)\right| \leqslant C(r)n^{-(r+1)}\cdot\varphi^{2(r+1)-2i}(x).$$

Therefore

$$I_1 \leqslant C(r)\sum_{i=1}^{r}\|\varphi^{2(r+1)\lambda-2i}g^{(2r+2-i)}\|_{L_\infty)[0,\infty)}\cdot\varphi^{2(r+1)(1-\lambda)}(x)\cdot n^{-r-1}. \tag{3.9}$$

Using Lemma 3 we arrive at

$$I_1 \leqslant C(r)\left(n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)\right)^{2(r+1)}\left\{\|g\|+\|\varphi^{(2r+2)\lambda}g^{(2r+2)}\|\right\}. \tag{3.10}$$

In a similar way as in the first case, according to [9] – see (2.5) on p. 162, we observe that for u between t and x we have

$$\frac{|t-u|^{2r+1}}{\varphi^{2(r+1)\lambda}(u)} \leqslant \frac{|t-x|^{2r+1}}{\varphi^{2(r+1)\lambda}(x)}. \tag{3.11}$$

From (2.4) in Lemma 1 and the corresponding upper bound for these moments (see Corollary 5.3 in [19]) we verify that for $\frac{1}{n}<x\leqslant 1]$

$$\left|\widetilde{S}_{n,r}((t-x)^{2r+2},x)\right| \leqslant C(r)n^{-(r+1)}\cdot\varphi^{2(r+1)}(x). \tag{3.12}$$

Hence (3.11) and (3.12) imply

$$I_2 \leqslant C(r)\left(n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)\right)^{2(r+1)}\cdot\|\varphi^{(2r+2)\lambda}g^{(2r+2)}\|. \tag{3.13}$$

Lastly having the upper bounds (3.10) and (3.13) we proceed in the same way as in the first case to establish the validity of (3.7) for all $x\in[0,1]$.

Proof of (ii). If now $x\in(1,\infty)$ the proof follows in the same way as in 1. case $(x\in[0,\frac{1}{n}])$, but for $\lambda=1$. We only need to observe that here the estimate (3.4) follows from Corollary 5.3 in [19].

Proof of (iii). Here the proof is a simple corollary from recently established equivalence relation (2.2) in Theorem E. □

We continue with the proof of Theorem 2.

Proof. If $x=0$ the left-hand side is 0 and the proof is obvious. So let $x\in(0,\infty)$. Let $g\in C_B[0,\infty)$, $g'\in A.C.loc$ and $\|\varphi^2 g''\|_{L_\infty[0,\infty)}<\infty$. We repeat step by step the proof of (i) in Theorem 1 to verify the validity of (3.5) and (3.6) for $r=0$. We needed only the representation for the second moment of Phillips operator. Now taking the inf over all auxilliary functions g we get the proof of Theorem 2. □

REMARK 4.. If we compare Theorem 2 with Theorem 1 we see that the pointwise estimate in Theorem 2 is better than that of (i) in Theorem 1 for $r=0$, due to the absence of the second summand in the right-hand side. On the other hand if n is fixed the argument in the second Ditzian-Totik modulus in Theorem 2 can be unbounded for $x\in(0,\infty)$, while in (ii) in Theorem 1 the argument of the modulus is bounded for all $x\in(0,\infty)$.

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