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ON BIVARIATE BERNSTEIN–CHLODOWSKY OPERATORS

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Abstract. This work relates to bivariate Bernstein-Chlodowsky operator which is not a tensor product construction. We show that the operator preserves some properties of the original function, for example; function of modulus of continuity, Lipschitz constant, and a kind of monotony.

1. Introduction

Suppose that $(b_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers and that $\lim\limits_{n\to\infty} b_n = \infty$. n-th univariate Bernstein-Chlodowsky operator is defined by

$$C^*_{n,1}(f;x) = \begin{cases} \sum\limits_{k=0}^{n} f\left(\frac{k}{n}b_n\right) P_{n,k}\left(\frac{x}{b_n}\right) & \text{, if } 0 \leqslant x \leqslant b_n \\ f(x) & \text{, if } x > b_n \end{cases}$$

for every $f \in C[0,\infty)$, i.e., the space of all real-valued, continuous functions on $[0,\infty)$ and $x \in [0,\infty)$, where $P_{n,k}\left(\frac{x}{b_n}\right) := \binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$, $n \in \mathbb{N}$, by Chlodowsky [7]. We have from the additial condition $\lim\limits_{n\to\infty}\frac{b_n}{n} = 0$ that $\lim\limits_{n\to\infty} C^*_{n,1}f = f$ in the space $\left\{f \in C[0,\infty) : \frac{|f(x)|}{1+x^2} \text{ is convergent as } x \to \infty\right\}$ with the norm $\|f\| = \sup\limits_{x\geqslant 0}\frac{|f(x)|}{1+x^2}$, and that convergence uniform on compact subsets of $[0,\infty)$. Moreover we have

$$\left|C^*_{n,1}(f;x) - f(x)\right| \leqslant 2\omega\left(f;\sqrt{\frac{x(b_n-x)}{n}}\right)$$

for any function $f \in C_B[0,\infty) := \{f \in C[0,\infty) : f \text{ is bounded}\}$, and $0 \leqslant x \leqslant b_n$, where $\omega(f;\delta)$ is the familiar modulus of continuity of f with argument δ. In addition to above, for the brief history we refer to the book of Altomare and Campiti [2]. Also, let f be a real valued, continuous function defined on $[0,\infty)$. If $f\left(\sum\limits_{i=1}^{n}\alpha_i x_i\right) \geqslant \sum\limits_{i=1}^{n}\alpha_i f(x_i)$, such that $\alpha_i \geqslant 0$ and $\sum\limits_{i=1}^{n}\alpha_i = 1$, then f is called a convex function on $[0,\infty)$. Because of $C^*_{n,1}(t;x) = x$, if f is convex then

$$f(x) = f\left(\sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right)\frac{k}{n}b_n\right) \geqslant \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) f\left(\frac{k}{n}b_n\right) = C^*_{n,1}(f;x).$$

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Now let (a_n) and (b_n), $n\in\mathbb{N}$, be positive, increasing sequences such that $\lim\limits_{n\to\infty} a_n = \infty$, $\lim\limits_{n\to\infty} b_n=\infty$ and $\lim\limits_{n\to\infty}\frac{a_n}{n}=0$, $\lim\limits_{n\to\infty}\frac{b_n}{n}=0$. Let S and S^c denote the following regions

$$S=\left\{(x,y)\in\mathbb{R}^2 : x\geqslant 0,\ y\geqslant 0,\ \frac{x}{a_n}+\frac{y}{b_n}\leqslant 1\right\},$$
$$S^c=\left\{(x,y)\in\mathbb{R}^2 : x\geqslant 0,\ y\geqslant 0,\ \frac{x}{a_n}+\frac{y}{b_n}> 1\right\}.$$

For any continuous, real valued function, defined on $\Delta := S\cup S^c$, i.e., $f\in C(\Delta)$, n-th bivariate Bernstein-Chlodowsky operator is defined by

$$C^*_{n,2}(f;(x,y))=\begin{cases}\sum\limits_{k=0}^{n}\sum\limits_{l=0}^{n-k} f\left(\frac{k}{n}a_n,\frac{l}{n}b_n\right)P_{n,(k,l)}\left(\frac{x}{a_n},\frac{y}{b_n}\right), & \text{if } (x,y)\in S\\ f(x,y), & \text{if } (x,y)\in S^c\end{cases}$$

where $P_{n,(k,l)}\left(\frac{x}{a_n},\frac{y}{b_n}\right):=\binom{n}{k,l}\left(\frac{x}{a_n}\right)^k\left(\frac{y}{b_n}\right)^l\left(1-\frac{x}{a_n}-\frac{y}{b_n}\right)^{n-k-l}$, $(x,y)\in S$, $n\in\mathbb{N}$, and $\binom{n}{k,l}:=\dfrac{n!}{k!\,l!\,(n-k-l)!}$ is the multinomial coefficient. We note here that this two dimensional extension of the Bernstein-Chlodowsky operator, $C^*_{n,2}$, is not a tensor product construction. Some works related to $C^*_{n,1}$ and $C^*_{n,2}$ can be seen in [1, 2, 7, 8, 9, 10].

In this work, we first show that the bivariate Bernstein-Chlodowsky operator $C^*_{n,2}$ preserves some properties of the original function, for example, function of modulus of continuity, Lipschitz constant, and a kind of monotony.

2. Preservation of some properties by $C^*_{n,2}$

Recall that a continuous function f from $D\subset\mathbb{R}^2$ into $\mathbb{R}$ is said to be *Lipschitz continuous of order* μ, $\mu\in(0,1]$, if there exists a constant $A>0$ such that for every (x_1,x_2), $(y_1,y_2)\in D$, f satisfies

$$|f(x_1,x_2)-f(y_1,y_2)|\leqslant A\sum_{i=1}^{2}|x_i-y_i|^{\mu}. \tag{1}$$

The set of Lipschitz continuous functions defined above is denoted by $Lip_A(\mu,D)$.

Also, a continuous non-negative function $\omega(u_1,u_2)$ defined in D is called *the modulus of continuity function*, if it satisfies the following conditions [13]:

1. $\omega(0,0)=0$,
2. $\omega(u_1,u_2)$ is non-decreasing, i.e., $\omega(u_1,u_2)\geqslant\omega(v_1,v_2)$ for $(u_1,u_2)\geqslant(v_1,v_2)$ which means that $u_i\geqslant v_i$, $i=1,2$.
3. $\omega(u_1,u_2)$ is semi-additive, i.e., $\omega((u_1,u_2)+(v_1,v_2))\leqslant\omega(u_1,u_2)+\omega(v_1,v_2)$.

The first property that is preserved by $C^*_{n,2}$ is the general function of modulus of continuity which is related to smoothness. Similar results related to this idea can be found in [5] and [12].

THEOREM 1. *If* $\omega(x,y)$ *is a function of modulus of continuity, then so is* $C^*_{n,2}(\omega;(x,y))$.

Proof. Let (x_1,y_1), $(x_2,y_2)\in S$ and $(x_1,y_1)\leqslant(x_2,y_2)$, then

$$
\begin{aligned}
&C^*_{n,2}(f;(x_2,y_2))\\
&=\sum_{k=0}^{n}\sum_{l=0}^{n-k} f\left(\frac{k}{n}a_n,\frac{l}{n}b_n\right)\binom{n}{k,l}\left(\frac{x_2}{a_n}\right)^k\left(\frac{y_2}{b_n}\right)^l\left(1-\frac{x_2}{a_n}-\frac{y_2}{b_n}\right)^{n-k-l}\\
&=\sum_{k=0}^{n}\sum_{l=0}^{n-k} f\left(\frac{k}{n}a_n,\frac{l}{n}b_n\right)\binom{n}{k,l}\left(\frac{x_1+x_2-x_1}{a_n}\right)^k\left(\frac{y_1+y_2-y_1}{b_n}\right)^l\left(1-\frac{x_2}{a_n}-\frac{y_2}{b_n}\right)^{n-k-l}\\
&=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k,l}\left(1-\frac{x_2}{a_n}-\frac{y_2}{b_n}\right)^{n-k-l}\sum_{i=0}^{k}\binom{k}{i}\left(\frac{x_1}{a_n}\right)^i\left(\frac{x_2-x_1}{a_n}\right)^{k-i}\\
&\quad\times\sum_{j=0}^{l}\binom{l}{j}\left(\frac{y_1}{b_n}\right)^j\left(\frac{y_2-y_1}{b_n}\right)^{l-j} f\left(\frac{k}{n}a_n,\frac{l}{n}b_n\right).
\end{aligned}
$$

Changing the orders of the above summations and taking $k-i=m$, $l-j=\nu$ in the result, then $C^*_{n,2}(f;(x_2,y_2))$ turns into

$$
\begin{aligned}
&C^*_{n,2}(f;(x_2,y_2))\\
&=\sum_{i=0}^{n}\sum_{m=0}^{n-i}\sum_{j=0}^{n-i-m}\sum_{\nu=0}^{n-i-m-j}\frac{n!}{i!j!m!\nu!(n-(i+j+m+\nu))!}\\
&\quad\times\left(1-\frac{x_2}{a_n}-\frac{y_2}{b_n}\right)^{n-(i+j+m+\nu)}\left(\frac{x_1}{a_n}\right)^i\left(\frac{y_1}{b_n}\right)^j\\
&\quad\times\left(\frac{x_2-x_1}{a_n}\right)^m\left(\frac{y_2-y_1}{b_n}\right)^\nu f\left(\frac{i+m}{n}a_n,\frac{j+\nu}{n}b_n\right).
\end{aligned}
\tag{2}
$$

Moreover

$$
\begin{aligned}
&C^*_{n,2}(f;(x_1,y_1))\\
&=\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i,j}\left(\frac{x_1}{a_n}\right)^i\left(\frac{y_1}{b_n}\right)^j f\left(\frac{i}{n}a_n,\frac{j}{n}b_n\right)\\
&\quad\times\left[\left(1-\frac{x_2}{a_n}-\frac{y_2}{b_n}\right)+\left(\frac{x_2}{a_n}+\frac{y_2}{b_n}-\frac{x_1}{a_n}-\frac{y_1}{b_n}\right)\right]^{n-i-j}\\
&=\sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{n!}{i!j!(n-i-j)!}\left(\frac{x_1}{a_n}\right)^i\left(\frac{y_1}{b_n}\right)^j f\left(\frac{i}{n}a_n,\frac{j}{n}b_n\right)\\
&\quad\times\sum_{m=0}^{n-i-j}\sum_{\nu=0}^{n-i-j-m}\binom{n-i-j}{m,\nu}\left(\frac{x_2-x_1}{a_n}\right)^m\left(\frac{y_2-y_1}{b_n}\right)^\nu\left(1-\frac{x_2}{a_n}-\frac{y_2}{b_n}\right)^{n-(i+j+m+\nu)}
\end{aligned}
$$

Interchanging the order of the above summation gives that

$$
\begin{aligned}
&C^*_{n,2}\left(f;(x_1,y_1)\right) \qquad (3)\\
&=\sum_{i=0}^{n}\sum_{m=0}^{n-i}\sum_{j=0}^{n-i-m}\sum_{\nu=0}^{n-i-m-j}\frac{n!}{i!j!m!\nu!\left(n-(i+j+m+\nu)\right)!}f\left(\frac{i}{n}a_n,\frac{j}{n}b_n\right)\\
&\times\left(\frac{x_1}{a_n}\right)^i\left(\frac{y_1}{b_n}\right)^j\left(\frac{x_2-x_1}{a_n}\right)^m\left(\frac{y_2-y_1}{b_n}\right)^\nu\left(1-\frac{x_2}{a_n}-\frac{y_2}{b_n}\right)^{n-(i+j+m+\nu)}.
\end{aligned}
$$

From (2) and (3) we get

$$
\begin{aligned}
&C^*_{n,2}\left(f;(x_2,y_2)\right)-C^*_{n,2}\left(f;(x_1,y_1)\right) \qquad (4)\\
&=\sum_{i=0}^{n}\sum_{m=0}^{n-i}\sum_{j=0}^{n-i-m}\sum_{\nu=0}^{n-i-m-j}\frac{n!}{i!j!m!\nu!\left(n-(i+j+m+\nu)\right)!}\\
&\times\left(\frac{x_1}{a_n}\right)^i\left(\frac{y_1}{b_n}\right)^j\left(\frac{x_2-x_1}{a_n}\right)^m\left(\frac{y_2-y_1}{b_n}\right)^\nu\left(1-\frac{x_2}{a_n}-\frac{y_2}{b_n}\right)^{n-(i+j+m+\nu)}\\
&\times\left\{f\left(\frac{i+m}{n}a_n,\frac{j+\nu}{n}b_n\right)-f\left(\frac{i}{n}a_n,\frac{j}{n}b_n\right)\right\}.
\end{aligned}
$$

Replace f with ω then (4) ensures that $C^*_{n,2}\left(\omega;(x_2,y_2)\right)-C^*_{n,2}\left(\omega;(x_1,y_1)\right)\geqslant 0$ for $(x_2,y_2)\geqslant(x_1,y_1)$. To show the semi-additivity, we take (4) into consideration, then,

$$
\begin{aligned}
&C^*_{n,2}\left(\omega;(x_2,y_2)\right)-C^*_{n,2}\left(\omega;(x_1,y_1)\right) \qquad (5)\\
&\leqslant\sum_{i=0}^{n}\sum_{m=0}^{n-i}\sum_{j=0}^{n-i-m}\sum_{\nu=0}^{n-i-m-j}\frac{n!}{i!j!m!\nu!\left(n-(i+j+m+\nu)\right)!}\omega\left(\frac{m}{n}a_n,\frac{\nu}{n}b_n\right)\\
&\times\left(\frac{x_1}{a_n}\right)^i\left(\frac{y_1}{b_n}\right)^j\left(\frac{x_2-x_1}{a_n}\right)^m\left(\frac{y_2-y_1}{b_n}\right)^\nu\left(1-\frac{x_2}{a_n}-\frac{y_2}{b_n}\right)^{n-(i+j+m+\nu)}\\
&=\sum_{m=0}^{n}\sum_{i=0}^{n-m}\sum_{\nu=0}^{n-m-i}\sum_{j=0}^{n-m-i-\nu}\frac{n!}{i!j!m!\nu!\left(n-(i+j+m+\nu)\right)!}\omega\left(\frac{m}{n}a_n,\frac{\nu}{n}b_n\right)\\
&\times\left(\frac{x_1}{a_n}\right)^i\left(\frac{y_1}{b_n}\right)^j\left(\frac{x_2-x_1}{a_n}\right)^m\left(\frac{y_2-y_1}{b_n}\right)^\nu\left(1-\frac{x_2}{a_n}-\frac{y_2}{b_n}\right)^{n-(i+j+m+\nu)}\\
&=\sum_{m=0}^{n}\sum_{\nu=0}^{n-m}\binom{n}{m,\nu}\left(\frac{x_2-x_1}{a_n}\right)^m\left(\frac{y_2-y_1}{b_n}\right)^\nu\omega\left(\frac{m}{n}a_n,\frac{\nu}{n}b_n\right)\\
&\times\sum_{i=0}^{n-m-\nu}\sum_{j=0}^{n-m-\nu-i}\binom{n-m-\nu}{i,j}\left(\frac{x_1}{a_n}\right)^i\left(\frac{y_1}{b_n}\right)^j\left(1-\frac{x_2}{a_n}-\frac{y_2}{b_n}\right)^{n-(i+j+m+\nu)}\\
&=C^*_{n,2}\left(\omega;(x_2-x_1,y_2-y_1)\right),
\end{aligned}
$$

which shows the semi-additivity. Furthermore it is clear that $C^*_{n,2}\left(\omega;(0,0)\right)=\omega(0,0)=0$. Finally for the case $(x_i,y_i)\in S^c$, $i=1,2$, the proof is obvious. □

Preservation of the Lipschitz constant for some univariate or multivariate operators can be found in [3, 4, 5] and [11]. Same result for $C^*_{n,2}$ will be presented in the following theorem.

THEOREM 2. *If* $f \in Lip_A(\mu;\Delta)$, *then* $C^*_{n,2}f \in Lip_A(\mu;\Delta)$.

Proof. Let (x_1,y_1), $(x_2,y_2) \in S$ and $(x_1,y_1) \leqslant (x_2,y_2)$. From (1) and (4) we reach to

$$\begin{aligned}
&\left|C^*_{n,2}(f;(x_2,y_2)) - C^*_{n,2}(f;(x_1,y_1))\right| \\
&\leqslant \sum_{i=0}^{n}\sum_{m=0}^{n-i}\sum_{j=0}^{n-i-m}\sum_{\nu=0}^{n-i-m-j} \frac{n!}{i!j!m!\nu!(n-(i+j+m+\nu))!} \\
&\quad\times \left(\frac{x_1}{a_n}\right)^i \left(\frac{y_1}{b_n}\right)^j \left(\frac{x_2-x_1}{a_n}\right)^m \left(\frac{y_2-y_1}{b_n}\right)^\nu \left(1-\frac{x_2}{a_n}-\frac{y_2}{b_n}\right)^{n-(i+j+m+\nu)} \\
&\quad\times \left| f\left(\frac{i+m}{n}a_n, \frac{j+\nu}{n}b_n\right) - f\left(\frac{i}{n}a_n, \frac{j}{n}b_n\right)\right| \\
&\leqslant A\sum_{i=0}^{n}\sum_{m=0}^{n-i}\sum_{j=0}^{n-i-m}\sum_{\nu=0}^{n-i-m-j} \frac{n!}{i!j!m!\nu!(n-(i+j+m+\nu))!} \\
&\quad\times \left(\frac{x_1}{a_n}\right)^i \left(\frac{y_1}{b_n}\right)^j \left(\frac{x_2-x_1}{a_n}\right)^m \left(\frac{y_2-y_1}{b_n}\right)^\nu \\
&\quad\times \left(1-\frac{x_2}{a_n}-\frac{y_2}{b_n}\right)^{n-(i+j+m+\nu)} \left\{\left(\frac{m}{n}a_n\right)^\mu + \left(\frac{\nu}{n}b_n\right)^\mu\right\} \\
&= A\left\{C^*_{n,1}\left(t_1^\mu; x_2-x_1\right) + C^*_{n,1}\left(t_2^\mu; y_2-y_1\right)\right\}.
\end{aligned}$$

Since the univariate Bernstein Chlodowsky operators satisfy $C^*_{n,1}\left(t_i^\mu;x_i\right) \leqslant x_i^\mu$, $i=1,2$, by the convexity of the function $f(t)=t^\mu$ then the last inequality reduces to

$$\left|C^*_{n,2}(f;(x_2,y_2)) - C^*_{n,2}(f;(x_1,y_1))\right| \leqslant A\left\{|x_2-x_1|^\mu + |y_2-y_1|^\mu\right\}.$$

In a similar way we can verify the case $(x_1,y_1) \geqslant (x_2,y_2)$. Moreover if $x_1 \geqslant x_2$, $y_1 \leqslant y_2$, then $(x_2,y_1) \in S$. It is easily obtained from the above argument that

$$\begin{aligned}
&\left|C^*_{n,2}(f;(x_1,y_1)) - C^*_{n,2}(f;(x_2,y_2))\right| \\
&\leqslant \left|C^*_{n,2}(f;(x_1,y_1)) - C^*_{n,2}(f;(x_2,y_1))\right| + \left|C^*_{n,2}(f;(x_2,y_2)) - C^*_{n,2}(f;(x_2,y_1))\right| \\
&\leqslant A\left\{|x_2-x_1|^\mu + |y_2-y_1|^\mu\right\}.
\end{aligned}$$

The case $x_1 \leqslant x_2$, $y_1 \geqslant y_2$ can be reached similarly. Furthermore if (x_1,y_1), $(x_2,y_2) \in S^c$, the proof is clear by the definition of the operator. Hence it is found from the above arguments that $C^*_{n,2}f \in Lip_A(\mu;\Delta)$. □

Besides above properties, n-th bivariate Bernstein-Chlodowsky operator $C^*_{n,2}$ preserves a kind of monotony that is given in the following theorem. Also, similar results can be viewed in [5] and [12].

THEOREM 3. *Let $f(x,y) \geqslant 0$. If $x^{-1}f(x,y)$ (or $y^{-1}f(x,y)$) is non-increasing for x (or y) on $(0,\infty)$, then $x^{-1}C^*_{n,2}(f;(x,y))$ (or $y^{-1}C^*_{n,2}(f;(x,y))$) is also non-increasing for x (or y) on $(0,\infty)$.*

Proof. Straightforward computation gives (for example x) that for $n \geqslant 1$ and $x \in (0,a_n]$,

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{C^*_{n,2}(f;(x,y))}{x}\right)\\
&=\sum_{k=2}^{n}\sum_{l=0}^{n-k}\binom{n}{k,l}(k-1)\frac{x^{k-2}}{(a_n)^k}\left(\frac{y}{b_n}\right)^l\left(1-\frac{x}{a_n}-\frac{y}{b_n}\right)^{n-k-l} f\left(\frac{k}{n}a_n,\frac{l}{n}b_n\right)\\
&\quad-\frac{1}{a_n}\sum_{k=1}^{n}\sum_{l=0}^{n-k}\binom{n}{k,l}(n-k-l)\frac{x^{k-1}}{(a_n)^k}\left(\frac{y}{b_n}\right)^l\left(1-\frac{x}{a_n}-\frac{y}{b_n}\right)^{n-k-l-1} f\left(\frac{k}{n}a_n,\frac{l}{n}b_n\right)\\
&\quad-\sum_{l=0}^{n}\binom{n}{l}\left(\frac{y}{b_n}\right)^l f\left(0,\frac{l}{n}b_n\right)\left(\frac{(n-l)\left(1-\frac{x}{a_n}-\frac{y}{b_n}\right)^{n-l-1}\frac{x}{a_n}+\left(1-\frac{x}{a_n}-\frac{y}{b_n}\right)^{n-l}}{x^2}\right)\\
&=\sum_{k=1}^{n-1}\sum_{l=0}^{n-1-k}\frac{n}{(k+1)a_n}\binom{n-1}{k,l}\left(\frac{x}{a_n}\right)^k\left(\frac{y}{b_n}\right)^l\frac{k}{x}\left(1-\frac{x}{a_n}-\frac{y}{b_n}\right)^{n-1-k-l} f\left(\frac{k+1}{n}a_n,\frac{l}{n}b_n\right)\\
&\quad-\sum_{k=1}^{n-1}\sum_{l=0}^{n-k-1}\binom{n-1}{k,l}\frac{n}{ka_n}\left(\frac{x}{a_n}\right)^k\left(\frac{y}{b_n}\right)^l\frac{k}{x}\left(1-\frac{x}{a_n}-\frac{y}{b_n}\right)^{n-1-k-l} f\left(\frac{k}{n}a_n,\frac{l}{n}b_n\right)\\
&\quad-\sum_{l=0}^{n}\binom{n}{l}\left(\frac{y}{b_n}\right)^l f\left(0,\frac{l}{n}b_n\right)\left(\frac{(n-l)\left(1-\frac{x}{a_n}-\frac{y}{b_n}\right)^{n-l-1}\frac{x}{a_n}+\left(1-\frac{x}{a_n}-\frac{y}{b_n}\right)^{n-l}}{x^2}\right)
\end{aligned}$$

Therefore the above equation gives rise to

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{C^*_{n,2}(f;(x,y))}{x}\right)\\
&=\sum_{k=1}^{n-1}\sum_{l=0}^{n-1-k}\left\{\frac{n}{(k+1)a_n}f\left(\frac{k+1}{n}a_n,\frac{l}{n}b_n\right)-\frac{n}{ka_n}f\left(\frac{k}{n}a_n,\frac{l}{n}b_n\right)\right\}\\
&\quad\times\binom{n-1}{k,l}\frac{k}{x}\left(\frac{x}{a_n}\right)^k\left(\frac{y}{b_n}\right)^l\left(1-\frac{x}{a_n}-\frac{y}{b_n}\right)^{n-1-k-l}\\
&\quad-\sum_{l=0}^{n}\binom{n}{l}\left(\frac{y}{b_n}\right)^l\left(1-\frac{x}{a_n}-\frac{y}{b_n}\right)^{n-l}\left(\frac{(n-l)\frac{x}{a_n}+1-\frac{x}{a_n}-\frac{y}{b_n}}{x^2\left(1-\frac{x}{a_n}-\frac{y}{b_n}\right)}\right)f\left(0,\frac{l}{n}b_n\right),
\end{aligned}$$

whose right hand side is non-positive by the hypothesis.

So it is obtained that $\frac{C_{n,2}^{*}(f;(x,y))}{x}$ is non-increasing for $x \in (0, a_n]$. This conclusion is obvious for $x > a_n$ by the definition of $C_{n,2}^{*}$. Hence the proof is completed. $\square$

REMARK 1. The obtained results in this paper are valid for the m-variable Bernstein-Chlodowsky operator which is not a tensor product construction.

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