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SOME STABILITY RESULTS RELATED TO SOME FIXED POINT THEOREMS

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Abstract. In this paper, we introduce two types of stability and we investigate some fixed point theorems, as Schauder Theorem, Borsuk Theorem and Knaster Lemma, from this viewpoint.

1. Introduction

The starting point of a wide range of stability notions was the following question:

”*When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation ?*”

The author of previous question was S. M. Ulam(see [21]). The first mathematician which gave an important answer to Ulam's question was D. H. Hyers (see [8]). In fact, he proved the following theorem:

THEOREM. *Let* $f : U \longrightarrow V$ *be a function between two Banach spaces and* $\varepsilon > 0$. *If*

$$\|f(x+y) - f(x) - f(y)\| < \varepsilon,$$

for all $x, y \in U$, *then exists a unique additive function* $A : U \longrightarrow V$ *such that*

$$\|A(x) - f(x)\| < \varepsilon,$$

for all $x \in U$.

This result opened a new direction in mathematics and a new mathematical concept was born. Now it is said that the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, satisfies the Hyers–Ulam stability. The stability concept has influenced a number of mathematicians studying the stability problems of functional equations (a large collection of results can be found in [9]). Today, the terminologies Hyers–Ulam stability is also applicable to the case of other mathematical objects: differential equation (*e.g.* [1] or [20]), dynamical equations (*e.g.* [2]), linear recurrences (*e.g.* [11] or [13]), convexity (*e.g.* [14] or [15]), mean value point (*e.g.* [6] or [12]) or fixed points (*e.g.* [4]). In this context, we find more ways to obtain the Hyers–Ulam stability. For

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example, V. Radu ([17]) used the fixed point methods to study the functional equation or I. A. Rus ([18]) introduced an operatorial approach for many types of problems.

In this paper, we want to investigate the stability of the fixed point equations and of the fixed points. An important collection of some recent results on this topic can be found in the paper [5], but we want to show another viewpoint. We present two types of stability and we investigate some fixed point theorems from this viewpoint. The definitions and some comments are introduced in second section of this paper. We mention that our definitions are less restrictive as the classic definition of Hyers-Ulam stability (for example, see [19]), but our viewpoint leads us to some fixed point theorems which were less studied.

The third section is reserved to the Contraction Principle. Nowadays a lot of stability results are known about this theorem, some of them are stronger than our results(*e.g.* [3] or [16]), but it is a good example of the applicability of our ideas. The following sections are about Schauder Theorem, Borsuk Theorem and Knaster Lemma. We present some conditions for both types of stability.

2. The definitions and some counterexamples

Let (X,d) be a metric space. For any function $f : X \longrightarrow X$, we denote $Fix(f)$, its fixed points set. Also, we denote $FP(X)$, the set of all functions $f : X \longrightarrow X$ with property $Fix(f) \neq \varnothing$. Let $M(X)$ be a nonempty subset of $FP(X)$. We introduce two definitions involving the stability of equation $f(x) = x$, respectively the solution of this equation.

DEFINITION 1. *Let $f \in FP(X)$. We say that the equation $f(x) = x$ is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $x \in X$ satisfying the property*

$$d(f(x),x) < \delta,$$

there exists a fixed point $u \in Fix(f)$ with

$$d(x,u) < \varepsilon.$$

DEFINITION 2. *Let $f \in M(X)$. We say that the point $u \in Fix(f)$ is stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that any function $g \in M(X)$, which is satisfying the property*

$$d(g(x), f(x)) < \delta,$$

for any $x \in X$, admits a fixed point $v \in X$ such that

$$d(v,u) < \varepsilon.$$

We can see that not every fixed point equation and not every fixed point is stable according to the previous definitions.. First, we consider the function

$$h : \mathbb{R} \longrightarrow \mathbb{R}, h(x) = \frac{x^3}{x^2+1}.$$

Then $Fix(h) = \{0\}$. Further

$$\lim_{x\longrightarrow\infty}(h(x)-x) = \lim_{x\longrightarrow\infty}\frac{-x}{x^2+1} = 0.$$

Then, for any $\delta > 0$, we find $\gamma > 0$ such that for any $x \in (\gamma, \infty)$ we have

$$|h(x)-x| < \delta.$$

Now, for any $\varepsilon > 0$ we find $c \in (\gamma, \infty)$ and $c \notin (-\varepsilon, \varepsilon)$ such that $|h(c)-c| < \delta$. This means that the equation $h(x) = x$ is not stable in the sense of Definition 1.

For the second definition, we reload an example which can be found in [12]. We consider the function

$$h : \mathbb{R} \longrightarrow \mathbb{R}, h(x) = x^2 - x + 1.$$

Then $Fix(h) = \{1\}$. Let $\delta > 0$ and we consider the function

$$g : \mathbb{R} \longrightarrow \mathbb{R}, g(x) = x^2 - x + 1 + \frac{\delta}{2}.$$

For any $x \in X$, we have

$$|g(x) - h(x)| < \delta,$$

but $Fix(g) = \varnothing$. This means that the fixed point of h is not stable.

The conclusion is that in order to assure these types of stability we need supplementary conditions.

3. The Contraction Principle

We start with some easy example. Let X be a complete metric space. Let $f : X \longrightarrow X$ be a contraction. We denote by k, the Lipschitz constant of f, so

$$d(f(x), f(y)) \leqslant kd(x,y),$$

for any $x, y \in X$, where $k \in [0,1)$. The Contraction Principle (*e.g.* Theorem 1.1, pag. 9 from [7]) give us a unique point $u \in X$ such that $f(u) = u$. First, we prove that the equation $f(x) = x$ is stable is sense of Definition 1.

PROPOSITION 1. *Let $f : X \to X$ be a contraction and u, its fixed point. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in X$ satisfying the property*

$$d(f(x), x) < \delta,$$

we have $d(x,u) < \varepsilon$.

Proof. For some $\delta > 0$ and for any $x \in X$ such that $d(f(x), x) < \delta$, we have

$$\begin{aligned} d(x,u) &\leqslant d(x, f(x)) + d(f(x), u) \\ &= d(x, f(x)) + d(f(x), f(u)) \\ &\leqslant \delta + kd(x,u). \end{aligned}$$

Then

$$(1-k)\,d\,(x,u) \leqslant \delta. \tag{1}$$

Now, for any $\varepsilon > 0,$ we choose $\delta > 0$ such that

$$\delta < (1-k)\,\varepsilon.$$

From (1), we obtain

$$(1-k)\,d\,(x,u) < (1-k)\,\varepsilon$$

and the conclusion follows now. $\square$

For the second type of stability, we observe that, in this case, $M(X)$ represents the class of all contraction defined to X on itself. The following proposition says that the only fixed point of any $f \in M(X)$ is stable.

PROPOSITION 2. *Let $f : X \to X$ be a contraction and u, its fixed point. For any $\varepsilon > 0$, there exists $\delta > 0$ such any contraction $g : X \longrightarrow X$ satisfying the property*

$$d\,(g\,(x)\,, f\,(x)) < \delta,$$

for any $x \in X$, admits a fixed point $v \in X$ such that

$$d\,(v,u) < \varepsilon.$$

Proof. The existence of v is due to Contraction Principle. For some $\delta > 0,$ we have

$$\begin{aligned} d\,(v,u) &= d\,(g\,(v)\,, f\,(u)) \\ &\leqslant d\,(g\,(v)\,, f\,(v)) + d\,(f\,(v)\,, f\,(u)) \\ &\leqslant \delta + kd\,(v,u)\,, \end{aligned}$$

which it goes to

$$(1-k)\,d\,(v,u) \leqslant \delta. \tag{2}$$

Now, for any $\varepsilon > 0,$ we choose $\delta > 0$ such that $\delta < (1-k)\,\varepsilon$. Then, (2) becomes

$$(1-k)\,d\,(v,u) < (1-k)\,\varepsilon$$

and we conclude the proof. $\square$

4. Schauder and Borsuk Theorems

In this section, we consider $(Y, \|\cdot\|)$ a linear normed space. We denote K, a convex and compact subset of Y. The fixed point theorem of Schauder said that any continuous function on K to itself admits a fixed point. (*e.g.* Theorem 3.2, pag. 119 from [7]). We prove that the equation $f(x) = x$ is stable in the sense of Definition 1 and the stability of the fixed point holds if this point is unique.

PROPOSITION 3. *Let $f : K \longrightarrow K$ be a continuous function. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in X$ which it is satisfying the property*

$$\|f(x) - x\| < \delta,$$

there exists $u \in Fix(f)$ such

$$\|x - u\| < \varepsilon.$$

Proof. We prove by contradiction. We claim that there exists $\varepsilon > 0$ with following property: for any $\delta > 0$ there exists $x \in K$ such that $\|f(x) - x\| < \delta$ and for any $u \in Fix(f)$ we have

$$\|x - u\| \geqslant \varepsilon.$$

We choose $\delta = \dfrac{1}{n}$. For any $n \in \mathbb{N}, n \geqslant 1,$ we find $x_n \in K$ such that

$$\|f(x_n) - x_n\| < \frac{1}{n} \text{ and } \|x_n - u\| \geqslant \varepsilon.$$

The set K is compact, so the sequence $(x_n)_{n \geqslant 1}$ has a convergent sub-sequence $\left(x_{n_k}\right)_{k \geqslant 1}$. We denote v, its limit. The function f is continuous, so

$$f\left(x_{n_k}\right) \underset{k \longrightarrow \infty}{\longrightarrow} f(v).$$

Then

$$\begin{aligned} \|f(v) - v\| &= \left\|f(v) - f\left(x_{n_k}\right) + f\left(x_{n_k}\right) - x_{n_k} + x_{n_k} - v\right\| \\ &\leqslant \left\|f(v) - f\left(x_{n_k}\right)\right\| + \left\|f\left(x_{n_k}\right) - x_{n_k}\right\| + \left\|x_{n_k} - v\right\| \\ &\leqslant \left\|f(v) - f\left(x_{n_k}\right)\right\| + \left\|x_{n_k} - v\right\| + \frac{1}{n_k}. \end{aligned}$$

The condition $k \longrightarrow \infty$ leads to $f(v) = v$.

Since $x_{n_k} \longrightarrow v$, then, for any $\varepsilon > 0$, we find k_0 such that for any $k > k_0$, we have

$$\left\|x_{n_k} - v\right\| < \varepsilon.$$

This result contradicts our claim and concludes the proof. $\square$

For the second type of stability, we present two examples to show that the hypothesis from the following proposition cannot be weakened in general.

First, we consider the function $f : [0,1] \longrightarrow [0,1]$ defined by

$$f(x) = \begin{cases} \dfrac{1}{3}, & x \in \left[0, \dfrac{1}{3}\right] \\ x, & x \in \left(\dfrac{1}{3}, \dfrac{2}{3}\right) \\ \dfrac{2}{3}, & x \in \left[\dfrac{2}{3}, 1\right] \end{cases}.$$

It is a continuous function and $Fix(f) = \left[\frac{1}{3}, \frac{2}{3}\right]$. For any $\delta > 0$, we choose $\nu > 0$ such that $\nu < \min\left\{\delta, \frac{1}{3}\right\}$. We define the function

$$g : [0,1] \longrightarrow [0,1],\ g(x) = f(x) + \nu.$$

For any $x \in [0,1]$, we have $|g(x) - f(x)| < \delta$. More, we obtain $Fix(g) = \left\{\frac{2}{3} + \nu\right\}$. Now, we consider $\alpha = \frac{1}{3} \in Fix(f)$ and $\varepsilon = \frac{1}{3}$. We observe that $(\alpha - \varepsilon, \alpha + \varepsilon) \cap Fix(g) = \varnothing$ and the point α is not stable in the sense of the Definition 2.

The problem is nonuniqueness of the fixed point. The following example investigates the case when the fixed point is unique on a neighborhood.

We consider the function $f : [0,1] \longrightarrow [0,1]$ defined by

$$f(x) = \begin{cases} \frac{1}{3}, & x \in \left[0, \frac{1}{3}\right] \\ \frac{4}{3}x - \frac{1}{9}, & x \in \left(\frac{1}{3}, \frac{2}{3}\right) \\ \frac{7}{9}, & x \in \left[\frac{2}{3}, 1\right] \end{cases}.$$

It is a continuous function and $Fix(f) = \left\{\frac{1}{3}, \frac{7}{9}\right\}$. Now, for any $\delta > 0$, we choose $\nu > 0$ such that $\nu < \min\left\{\delta, \frac{2}{9}\right\}$. We define the function

$$g : [0,1] \longrightarrow [0,1],\ g(x) = f(x) + \nu.$$

For any $x \in [0,1]$, we have $|g(x) - f(x)| < \delta$. More, we obtain $Fix(g) = \left\{\frac{7}{9} + \nu\right\}$. Now, we consider $\varepsilon = \frac{4}{9}$ and $\alpha = \frac{1}{3} \in Fix(f)$. We observe that $(\alpha - \varepsilon, \alpha + \varepsilon) \cap Fix(g) = \varnothing$ and the point α is not stable in the sense of the Definition 2.

The previous examples lead to the following result.

PROPOSITION 4. *Let $f : K \longrightarrow K$ be a continuous function which admits a unique fixed point $u \in K$. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that any continuous function $g : K \longrightarrow K$, which is satisfying the property*

$$\|g(x) - f(x)\| < \delta,$$

for any $x \in K$, admits a fixed point $v \in X$ such that

$$\|v - u\| < \varepsilon.$$

To prove Proposition 4, first we need the following lemma.

LEMMA 1. *Let* $f : K \longrightarrow K$ *be a continuous function. Then, for any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that for any continuous function* $g : K \longrightarrow K$ *that is satisfying the property*

$$\|g(x) - f(x)\| < \delta$$

for any $x \in K$, *and for any* $v \in Fix(g)$ *there exists* $u \in Fix(f)$ *such that*

$$\|v - u\| < \varepsilon.$$

Proof. We prove by contradiction. If we suppose the contrary, then there exists $\varepsilon > 0$ such that for any $\delta > 0$, there exist a continuous function $g : K \longrightarrow K$, which is satisfying the property $\|g(x) - f(x)\| < \delta$, for any $x \in K$, and a fixed point $v \in Fix(g)$ such that for any $w \in Fix(f)$, we have $\|v - w\| \geqslant \varepsilon$. We choose $\delta = \frac{1}{n}$, $n \in \mathbb{N}$, $n \geqslant 1$. We obtain a sequence of a continuous functions $(g_n)_{n\geqslant 1}$ such $\|g_n(x) - f(x)\| < \frac{1}{n}$, for any $x \in K$. Also, we obtain a sequence $(v_n)_{n\geqslant 1}$ such that $g_n(v_n) = v_n$, for any $n \geqslant 1$. Moreover,

$$\|v_n - w\| \geqslant \varepsilon, \tag{3}$$

for any $n \geqslant 1$ and $w \in Fix(f)$.

The set K is compact, so the sequence $(v_n)_{n\geqslant 1}$ has a convegent sub-sequence $(v_{n_k})_{k\geqslant 1}$. Denote u, its limit. Then $f(v_{n_k}) \underset{k\longrightarrow\infty}{\longrightarrow} f(u)$.

We have

$$\begin{aligned} \|f(u) - u\| &\leqslant \left\|f(u) - f(v_{n_k})\right\| + \left\|f(v_{n_k}) - g_{n_k}(v_{n_k})\right\| \\ &\quad + \left\|g_{n_k}(v_{n_k}) - v_{n_k}\right\| + \left\|v_{n_k} - u\right\| \\ &\leqslant \left\|f(u) - f(v_{n_k})\right\| + \frac{1}{n_k} + \left\|v_{n_k} - u\right\|. \end{aligned}$$

The condition $k \longrightarrow \infty$ shows us that u is a fixed point of f. From $v_{n_k} \longrightarrow u$, we obtain that for any $\varepsilon > 0$, we find $k > 0$ such that $\left\|v_{n_k} - u\right\| < \varepsilon$, which it contradicts (3). This concludes our proof. $\square$

Proof. [Proof of Proposition 4] The conclusion can be obtained from the previous lemma and from the uniqueness of the fixed point of f. $\square$

Further, we investigate a similar result. Let B be the closed unit ball from $\mathbb{R}^n$. We denote $\mathcal{F}$, the following set:

$$\mathcal{F} = \{f : B \longrightarrow \mathbb{R}^n \,|\, f \text{ is continuous and } f(-x) = -f(x), \text{ for any } x \in \partial B\}.$$

Borsuk Theorem said that any $f \in \mathcal{F}$ has at least one fixed point (*e.g* Theorem 3.3, pag. 119 from [7]).

PROPOSITION 5. *Let* $f \in \mathcal{F}$. *Then, for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any* $x \in X$ *which it is satisfying the property*

$$\|f(x) - x\| < \delta,$$

there exists $u \in Fix(f)$ *such*

$$\|x-u\| < \varepsilon.$$

Proof. Hence the closed ball B is compact, then the proof is similar to the proof of Proposition 3. $\square$

The stability in the sense of Definition 2 fails in the hypothesis of Borsuk Theorem. First, we observe that if $u \in Fix(f) \cap \partial B$, then $f(-u) = -f(u) = -u$, so $-u \in Fix(g)$. On the other hand, the function $f : [-1,1] \longrightarrow \mathbb{R},\ f(x) = x^3$ has $Fix(f) = \{-1,0,1\}$. The fixed points $x = 1$ and $x = -1$ are not stable. For any $\delta > 0$, we define the function $g : [-1,1] \longrightarrow \mathbb{R},\ g(x) = x^3 - \frac{\delta}{2}x$. Then $Fix(g) = \{0\}$. For any $x \in [-1,1]$, we have

$$|g(x) - f(x)| = \left|\frac{\delta}{2}x\right| \leqslant \frac{\delta}{2} < \delta.$$

If we choose $\varepsilon = \frac{1}{2}$, we cannot find a fixed point of g on the interval $\left(-1-\frac{1}{2}, -1+\frac{1}{2}\right)$ or $\left(1-\frac{1}{2}, 1+\frac{1}{2}\right)$.

This example leads to the following result related to the stability, in the sense of Definition 2, of the fixed point obtained from Borsuk Theorem.

PROPOSITION 6. *Let* $f \in \mathcal{F}$. *We suppose that it admits a unique fixed point* $u \in B$. *Then, for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that any function* $g \in \mathcal{F}$, *which is satisfying the condition*

$$\|g(x) - f(x)\| < \delta, (x \in B)$$

admits a fixed point $v \in B$ *such that*

$$\|v-u\| < \varepsilon.$$

Proof. The proof is similar with the proof of Proposition 4. $\square$

5. Knaster Lemma on the real axis

Knaster Lemma on real axis said that $Fix(f) \neq \varnothing$, for any nondecreasing function $f : [a,b] \longrightarrow [a,b]$. In fact, it is a particular case of Knaster–Tarski Theorem. (*see* Theorem 1.1, pag 26 from [7]).

Unfortunately, the hypothesis of Knaster Lemma is too weak to obtain a stability in the sense of Definition 1. For example, the function $f : [0,1] \longrightarrow [0,1]$, defined by

$$f(x) = \begin{cases} \frac{1}{2}, & x \in \left[0, \frac{1}{2}\right) \\ 1, & x \in \left[\frac{1}{2}, 1\right] \end{cases},$$

has $Fix(f) = \{1\}$. We choose $\varepsilon = \frac{1}{2}$. The equality

$$\lim_{x \nearrow \frac{1}{2}} f(x) = \frac{1}{2}$$

and the inequality

$$|f(x) - x| \leqslant \left| f(x) - \frac{1}{2} \right| + \left| x - \frac{1}{2} \right|$$

show that for any $\delta > 0$, we find $x_0 < \frac{1}{2}$ such that

$$|f(x_0) - x_0| < \delta \quad \text{and} \quad |x_0 - 1| > \frac{1}{2}.$$

The stability in the sense of Definition 2 must be studied only in the cases of uniqueness of the fixed point. The examples from the previous section are conclusive.

PROPOSITION 7. *Let $f : [a,b] \longrightarrow [a,b]$ be a nondecreasing function which it admits a unique fixed point $c \in [a,b]$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that any nondecreasing function $g : [a,b] \longrightarrow [a,b]$, which is satisfying the property*

$$|g(x) - f(x)| < \delta,$$

for any $x \in [a,b]$, admits a fixed point $d \in [a,b]$ such that $|d - c| < \varepsilon$.

Proof. We have three cases. First, we suppose that $a < c < b$. Then, we claim that $f(x) > x$, for any $x \in [a,c)$ and $f(x) < x$, for any $x \in (c,b]$. We admit by contradiction that there exists $u \in [a,c)$ such that $f(u) < u$. For any $x \in [a,u]$, we have $f(a) \leqslant f(x) \leqslant f(u)$. The conditions $f(a) \geqslant a$ and $f(u) < u$ goes to conclusion $f(x) \in [a,u]$, for any $x \in [a,u]$. Then the restriction $f : [a,u] \longrightarrow [a,u]$ admits a fixed point on the interval $[a,u]$. This contradicts the uniqueness of the point c. A similar reasoning shows us that $f(x) < x$, for any $x \in (c,b]$.

Now, for an arbitrary $\varepsilon > 0$, we consider $y \in (c,b) \cap (c, c+\varepsilon)$ and $z \in (a,c) \cap (c-\varepsilon, c)$. We have $f(y) < y$ and $f(z) > z$. We choose $\delta > 0$ such that $f(y) + \delta < y - \delta$ and $f(z) - \delta > z + \delta$. We consider a nondecreasing function $g : [a,b] \longrightarrow [a,b]$ such that $|g(x) - f(x)| < \delta$, for any $x \in [a,b]$. We have

$$g(y) < f(y) + \delta < y - \delta < y.$$

Further,

$$g(z) > f(z) - \delta > z + \delta > z.$$

The function g is nondecreasing. Then, for any $x \in [z,y]$ we have

$$g(z) \leqslant g(x) \leqslant g(y).$$

The restriction $g:[z,y]\longrightarrow[z,y]$ is satisfying the hypothesis of Knaster Lemma. Then, the function g admits a fixed point $d\in[z,y]\subset(c-\varepsilon,c+\varepsilon)$.

For the second case, we assume that $c=a$. From the previous case, we obtain that $f(x)<x$, for any $x\in(a,b]$.

For an arbitary $\varepsilon>0$, we consider $y\in(a,b)\cap(a,a+\varepsilon)$. We have $f(y)<y$. We choose $\delta>0$ such that $f(y)+\delta<y-\delta$. We consider a nondecreasing function $g:[a,b]\longrightarrow[a,b]$ such that $|g(x)-f(x)|<\delta$, for any $x\in[a,b]$. We have

$$g(y)<f(y)+\delta<y-\delta+\delta=y.$$

Further,

$$g(a)\geqslant a.$$

The function g is nondecreasing. Then, for any $x\in[a,y]$ we have

$$g(a)\leqslant g(x)\leqslant g(y).$$

and the restriction $g:[a,y]\longrightarrow[a,y]$ is satisfying the hypothesis of Knaster Lemma. Then the function g admits a fixed point $d\in[a,y]\subset[a,a+\varepsilon)$.

For the last case, we assume that $c=b$. Then $f(x)>x$, for any $x\in[a,b)$. For an arbitary $\varepsilon>0$, we consider $y\in(a,b)\cap(b-\varepsilon,b)$. We have $f(y)>y$. We choose $\delta>0$ such that $f(y)-\delta>y+\delta$.

We consider a nondecreasing function $g:[a,b]\longrightarrow[a,b]$ such that $|g(x)-f(x)|<\delta$, for any $x\in[a,b]$. We have

$$g(y)>f(y)-\delta>y+\delta>y.$$

Further,

$$g(b)\leqslant b.$$

The function g is nondecreasing. Then, for any $x\in[y,b]$ we have

$$g(y)\leqslant g(x)\leqslant g(b).$$

and the restriction $g:[y,b]\longrightarrow[y,b]$ is satisfying the hypothesis of Knaster Lemma. Then the function g admits a fixed point $d\in[y,b]\subset(b-\varepsilon,b]$. Now the proof is complete. □

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REFERENCES

[1] C ALSINA AND R. GER, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl., **2** (1998), 373–380.

[2] SZ. ANDRÁS AND A. R. MÉZÁROS, *Ulam-Hyers stability of dynamic equations on time scales via Picard operators,* Appl. Math. Comput., **219**, 9 (2013), 4853–4864.

[3] V. BERINDE, *On the stability of fixed point procedure*, Bul. Ştiinţ. Univ. Baia Mare, Ser. B, Matematică-Informatică, **18**, 1 (2002), 7–14.

[4] M. BOTA, E. KARAPINAR AND O. MLEŞNIŢE, *Ulam-Hyers stability results for fixed point problems via α-ψ-contractive mapping in (b)-metric space*, Abstr. Appl. Anal., **2013** (2013), Article ID 825293.
[5] J. BRZDĘK, L. CĂDARIU AND K. CIEPLIŃSKI, *Fixed Point Theory and the Ulam Stability*, J. of Func. Spaces, **2014** (2014), Article ID 829419, http://dx.doi.org/10.1155/2014/829419.
[6] P. GĂVRUŢA, S.-M. JUNG AND Y. LI, *Hyers–Ulam stability of mean value points*, Ann. Funct. Anal., **1**, 2 (2010), 68–74.
[7] A. GRANAS AND J. DUGUNDJI, *Fixed Point Theory*, Springer-Verlag, New York, 2003.
[8] D.H. HYERS, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA, **27** (1941), 222–224.
[9] S. -M. JUNG, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.
[10] Y. LI, *Hyers-Ulam Stability of Linear Differential Equations $y'' = \lambda^2 y$*, Thai Journal of Mathematics, **8** (2010), 215–219.
[11] C. MORTICI, M. TH. RASSIAS AND S.-M. JUNG, *On the stability of a functional equation associated with the Fibonacci numbers*, Abstr. Appl. Anal., **2014** (2014), Article ID 546046.
[12] C. MORTICI, M. MONEA AND D. Ş. MARINESCU, *The stability of some points arising from continuous, differential and integral expressions*, Monatsh. Math., **180**, 1 (2015), 101–122, DOI: 10.1007/s00605-015-0779-7.
[13] M. S. MOSLEHIAN AND D. POPA, *On the stability of the first-order linear recurrence in topological vector spaces*, Non. Anal: Theory, Meth. & Appl., **73**, 9 (2010), 2792–2799.
[14] C. T. NGA AND K. NIKODEM, *On approximately convex functions*, Proc. Amer. Math. Soc., **118** (1993), 103–108.
[15] J. C. PARNAMI AND H. L. VASUDEVA, *On the stability of almost convex functions*, Proc. Amer. Math. Soc., **97** (1986), 67–70.
[16] B. E. RHOADES, *Fixed point theorems and stability results for fixed point iteration procedures*, Indian J. Pure Appl. Math., **21**, 1 (1990), 1–9.
[17] V. RADU, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, **4**, 1 (2003), 91–96.
[18] I. A. RUS, *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory, **10**, 2 (2009), 305–320.
[19] I. A. RUS, *Ulam Stability of the Operatorial Equations*, included in book: TH. M. RASSIAS, J. BRZDÉK (EDS.), *Functional Equations in Mathematical Analysis*, Springer, 2010, pp. 287–306.
[20] S.-E. TAKAHASI, T. MIURA AND S. MIYAJIMA, *On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$*, Bull. Korean Math. Soc., **39** (2002), 309–315.
[21] S. M. ULAM, *Problems in Modern Mathematics*, Chap. VI, Science Editions, Wiley, New York, 1964.

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