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GENERALIZATION ON UNIQUENESS OF MEROMORPHIC FUNCTIONS OF A CERTAIN DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper by introducing the notion of multiplicity we study the uniqueness of meromorphic functions concerning differential polynomials and obtain some results. The results of the paper improve and generalize some results due to Jin-Dong Li [7].

1. Introduction

In this paper, a meromorphic function means meromorphic in the open complex plane. We shall adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as the characteristic function $T(r,f)$, the counting function of the poles $N(r,f)$, and the proximity function $m(r,f)$ (see [10], [4]).

Let f and g be two nonconstant meromorphic functions and let $a \in \mathbb{C} \cup \{\infty\}$. We say that f and g share a CM if $f-a$ and $g-a$ have the same zeros, with the same multiplicities. Similarly, we say that f and g share a IM if $f-a$ and $g-a$ have the same zeros, ignoring multiplicities. When $a=\infty$ the zeros of $f-a$ means the poles of f. A meromorphic function $a(\not\equiv \infty)$ is called a small function with respect to f provided that $T(r,a) = S(r,f)$.

Let p be a positive integer. We use $N_{p)}(r,\frac{1}{f-a})$ to denote the counting function of the zeros of $f-a$ whose multiplicities are not greater than p, $N_{(p}(r,\frac{1}{f-a})$ to denote the counting function of the zeros of $f-a$ whose multiplicities are not less than p. And $\overline{N}_{p)}(r,\frac{1}{f-a})$ $\overline{N}_{(p}(r,\frac{1}{f-a})$ are their reduced functions, respectively. We also use $N_p(r,\frac{1}{f-a})$ to denote the counting function of the zeros of $f-a$ where a zero with multiplicity m is counted m times if $m \leqslant p$ and p times if $m > p$. Clearly, $N_1(r,\frac{1}{f-a}) = \overline{N}(r,\frac{1}{f-a})$.
Define

$$\delta_p(a,f) = 1 - \limsup_{r\to\infty} \frac{N_p(r,\frac{1}{f-a})}{T(r,f)}$$

Obviously,

$$1 \geqslant \Theta(a,f) \geqslant \delta_p(a,f) \geqslant \delta(a,f) \geqslant 0.$$

Hayman [3] and Clunie[1] proved the following result:

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THEOREM A. *Let $f(z)$ be a transcendental entire function and let $n \geqslant 1$ be a positive integer. Then $f^n f' = 1$ has infinitely many solutions.*

In 1997, Yang and Hua [9] obtained a unicity theorem corresponding to the above result and proved the following result:

THEOREM B. *Let $f(z)$ and $g(z)$ be two transcendental entire functions and let $n \geqslant 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$ or $f(z) = c_2 e^{-cz}$ and $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

THEOREM C. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and let $n \geqslant 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$ or $f(z) = c_2 e^{-cz}$ and $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

In 2011, R. S. Dyavnal [2] have improved these results and obtained the following results:

THEOREM D. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions whose zeros and poles have multiplicities not smaller than s, where s is a positive integer. Let $n \geqslant 2$ be a positive integer satisfying the inequality $(n+1)s \geqslant 12$. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$ or $f(z) = c_2 e^{-cz}$ and $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying the equality $(c_1 c_2)^{n+1} c^2 = -1$.*

THEOREM E. *Let $f(z)$ and $g(z)$ be two transcendental entire functions whose zeros have multiplicities not smaller than s, where s is a positive integer.Let n be a positive integer satisfying the inequality $(n+1)s \geqslant 7$. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$ or $f(z) = c_2 e^{-cz}$ and $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying the equality $(c_1 c_2)^{n+1} c^2 = -1$.*

In 2008, Li [6] proved the following result:

THEOREM F. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and let $n \geqslant 23$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 IM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$ or $f(z) = c_2 e^{-cz}$ and $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying the equality $(c_1 c_2)^{n+1} c^2 = -1$.*

In 2015, Jin Dong Li [7] proved the following two theorems.

THEOREM G. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions whose zeros and poles have multiplicities not smaller than s, where s is a positive integer. Let $n \geqslant 2$ be a positive integer satisfying the inequality $(n+1)s \geqslant 24$. If $f^n f'$ and $g^n g'$ share 1 IM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$ or $f(z) = c_2 e^{-cz}$ and $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying the equality $(c_1 c_2)^{n+1} c^2 = -1$.*

THEOREM H. *Let $f(z)$ and $g(z)$ be two transcendental entire functions whose zeros have multiplicities not smaller than s, where s ia a positive integer. Let n be a positive integer satisfying the inequality $(n+1)s \geqslant 13$. If $f^n f'$ and $g^n g'$ share 1 IM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$ or $f(z) = c_2 e^{-cz}$ and $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying the equality $(c_1 c_2)^{n+1} c^2 = -1$.*

In this paper, we partially extend Theorem G and Theorem H to a certain differential polynomials and obtain the following results.

THEOREM 1. *Let f and g be two nonconstant meromorphic functions and let n, k and m be three positive integers with $s(n+m) > 12k + 2m + 19$ and max$\{\chi_1, \chi_2\} < 0$, where*

$$\chi_1 = \frac{2m}{n+m-2k} + \frac{2}{(n+m)s+2k} + \frac{2k+1}{(n+m)s+k} + 1 - \Theta_{k)}(1,f) + \Theta_{k-1)}(1,f) \quad (1)$$

and

$$\chi_2 = \frac{2m}{n+m-2k} + \frac{2}{(n+m)s+2k} + \frac{2k+1}{(n+m)s+k} + 1 - \Theta_{k)}(1,g) + \Theta_{k-1)}(1,g) \quad (2)$$

Let $\{f^n P(f)\}^{(k)}$ and $\{g^n P(g)\}^k$ share 1 IM, then

(i) *when $P(\omega) = a_m \omega^m + a_{m-1}\omega^{m-1} + \ldots + a_0$, one of the following two cases holds.*

$(i1)$ *$f \equiv tg$, for a constant t such that $t^d = 1$, where $d = \{n+m, \ldots, n+m-i, \ldots, n\}$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.*

$(i2)$ *f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$.*

(ii) *when $P(\omega) = c_0$, $f(z) = tg(z)$ for a constant t such that $t^n = 1$.*

THEOREM 2. *Let f and g be two nonconstant entire functions and let n, k and m be three positive integers with $s(n+m) > 1$ and max$\{\chi_1, \chi_2\} < 0$, where*

$$\chi_1 = \frac{m}{n+m-2k} + \frac{1}{(n+m)s+2k} - \Theta_{k)}(1,f) \quad (3)$$

and

$$\chi_2 = \frac{m}{n+m-2k} + \frac{1}{(n+m)s+2k} - \Theta_{k)}(1,g) \quad (4)$$

Let $\{f^n P(f)\}^{(k)}$ $\{g^n P(g)\}^{(k)}$ share 1 IM, then

(i) *when $P(\omega) = a_m \omega^m + a_{m-1}\omega^{m-1} + \ldots + a_0$, one of the following two cases holds.*

$(i1)$ *$f \equiv tg$, for a constant t such that $t^d = 1$, where $d = \{n+m, \ldots, n+m-i, \ldots, n\}$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.*

$(i2)$ *f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$.*

(ii) *when $P(\omega) = c_0$, $f(z) = tg(z)$ for a constant t such that $t^n = 1$.*

2. Some Lemmas

To prove our result, we need following lemmas:

LEMMA 1. [7] *Let $f(z)$ and $g(z)$ be two meromorphuc functions and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 IM and*

$$\triangle = (2k+3)\Theta(\infty,f)+(2k+4)\Theta(\infty,g)+(k+3)\Theta(0,f)+(2k+3)\Theta(0,g)$$
$$+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)>7k+13$$

then either $f^{(k)}g^{(k)}\equiv 1$ or $f\equiv g$.

LEMMA 2. [8] *Let h be a nonconstant meromorphic function that is not a polynomial with its degree $\leqslant k-1$.Then*

$$N_0(r,\frac{1}{h^{(k)}})\leqslant N_k(r,\frac{1}{h})+k\overline{N}(r,h)+S(r,h)$$

where $k(\geqslant 1)$ is a positive integer, and $N_0(r,\frac{1}{h^{(k)}})$ denotes the counting function of those zeros of $h^{(k)}$ that are not the zeros of h.

3. Proof of Theorem 1

Proof. Let

$$F=f^nP(f) \quad and \quad G=g^nP(g) \tag{5}$$

Consider,

$$\overline{N}(r,\frac{1}{F})=\overline{N}(r,\frac{1}{f^nP(f)})\leqslant\frac{1}{s(n+m)}N(r,\frac{1}{F})\leqslant\frac{1}{s(n+m)}[T(r,F)+o(1)],$$

then we have

$$\Theta(0,F)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f})}{T(r,F)}\geqslant 1-\frac{2}{s(n+m)} \tag{6}$$

and

$$\begin{aligned}\delta_{k+1}(0,F)&=1-\limsup_{r\to\infty}\frac{N_{k+1}(r,\frac{1}{F})}{T(r,F)}\\&=1-\limsup_{r\to\infty}\frac{(k+m+1)\overline{N}(r,\frac{1}{F})}{T(r,F)}\\&\geqslant 1-\frac{k+m+1}{s(n+m)}\end{aligned} \tag{7}$$

$$\Theta(\infty,F)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,F)}{T(r,F)}\geqslant 1-\frac{1}{s(n+m)} \tag{8}$$

Similarly,

$$\Theta(0,G) \geqslant 1-\frac{2}{s(n+m)} \tag{9}$$

$$\delta_{k+1}(0,G) \geqslant 1-\frac{k+m+1}{s(n+m)} \tag{10}$$

$$\Theta(\infty,G) \geqslant 1-\frac{1}{s(n+m)} \tag{11}$$

Next, we have from Lemma 1 ,(5)–(11) and the condition $s(n+m) > 12k+2m+19$, we get

$$\triangle > 7k+13. \tag{12}$$

From (12), Lemma 1 and the condition that $F^{(k)}$ and $G^{(k)}$ share 0 IM we know that F and G are meromorphic functions such that $F^{(k)}G^{(k)} = 1$ or $F = G$.

We discuss the following two cases.

Case 1. Suppose that $F^{(k)}G^{(k)} = 1$. Then it follows from (5) that

$$\{f^nP(f)\}^{(k)}\{g^nP(g)\}^{(k)} = 1 \tag{13}$$

Let $z_0 \notin \{z : P(z) = 0\}$ be a zero of f of order p. Then it follows from (13) that z_0 is a pole of g. Suppose that z_0 is a pole of g of order q, then we have $np-k = (n+m)q+k$, i.e., $n(p-q) = mq+2k$ which implies that $p \geqslant q+1$ and $q \geqslant \frac{n-2k}{m}$, so we have

$$p \geqslant \frac{n+m-2k}{m} \tag{14}$$

Let $z_1 \notin \{z : P(z) = 0\}$ be a zero of $P(f)$ of order $p_1 \geqslant k+1$, then it follows from (13) that z_1 is a pole of g. Suppose that z_1 is a pole of g of order q_1.

Then from (13) we have $p_1 - k = (n+m)q_1 + k$.

From this we get

$$p_1 \geqslant (n+m)s+2k. \tag{15}$$

Let $z_2 \notin \{z : P(z) = 0\}$ be a zero of $\{f^nP(f)\}^{(k)}$ of order p_2 that is not a zero of $fP(f)$. Then from (13) we see that z_2 is a pole of g. Suppose that z_2 is a pole of g of order q_2, then $p_2 = (n+m)q_2 + k$. Thus

$$p_2 \geqslant (n+m)s+k. \tag{16}$$

Let $z_3 \notin \{z : P(z) = 0\} \cup \{z : f(z)P(f) = 0\}$ be a zero of $\{f^nP(f)\}^{(k)} = 1$ of multiplicity p_3. Then from (13) we deduce that z_3 is a pole of g of multiplicity q_3, say. Hence $p_3 = (n+m)q_3 + k \geqslant (n+m)s+k$.

This together with (14)–(16) and Lemma 2 gives

$$\begin{aligned}
\overline{N}(r,f) &\leqslant \overline{N}(r,\frac{1}{g})+\overline{N}_{k-1)}(r,\frac{1}{g-1})+\overline{N}(r,\frac{1}{g-1})\\
&+\frac{1}{(n+m)s+k}N_0(r,\frac{1}{\{g^nP(g)\}^{(k)}})+o(logr)\\
&\leqslant \frac{1}{p}N(r,\frac{1}{g})+\overline{N}_{k-1)}(r,\frac{1}{g-1})+\frac{1}{p_1}N(r,\frac{1}{g-1})\\
&+\frac{1}{(n+m)s+k}N_0(r,\frac{1}{\{g^nP(g)\}^{(k)}})+o(logr)\\
&\leqslant \frac{1}{\frac{n+m-2k}{m}}N(r,\frac{1}{g})+\overline{N}_{k-1)}(r,\frac{1}{g-1})+\frac{1}{(n+m)s+2k}N(r,\frac{1}{g-1})\\
&+\frac{1}{(n+m)s+k}\{k\overline{N}(r,g)+k\overline{N}(r,\frac{1}{g})+N_k(r,\frac{1}{g-1})\}+o(logr)+S(r,g).
\end{aligned}$$

$$\begin{aligned}
\overline{N}(r,f) &\leqslant \{\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2k+1}{(n+m)s+k}+1-\Theta_{k-1)}(1,g)\\
&+\varepsilon\}T(r,g)+S(r,g)
\end{aligned} \tag{17}$$

By (17), the above analysis and the second fundamental theorem we get

$$\begin{aligned}
T(r,f) &\leqslant \overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f-1})+S(r,f)\\
&\leqslant \{\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2k+1}{(n+m)s+k}\\
&+1-\Theta_{k-1)}(1,g)+\varepsilon\}T(r,g)+\frac{m}{n+m-2k}N(r,\frac{1}{f})\\
&+\overline{N}_{k)}(r,\frac{1}{f-1})+\frac{1}{(n+m)s+2k}N(r,\frac{1}{f-1})+S(r,f)+S(r,g)
\end{aligned} \tag{18}$$

$$\begin{aligned}
T(r,f) &\leqslant \{\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2k+1}{(n+m)s+k}\\
&+1-\Theta_{k-1)}(1,g)+\varepsilon\}T(r,g)+\{\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}\\
&+1-\Theta_{k)}(1,g)+\varepsilon\}T(r,g)+S(r,f)+S(r,g)
\end{aligned}$$

Similarly,

$$\begin{aligned}
T(r,g) &\leqslant \{\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2k+1}{(n+m)s+k}\\
&+1-\Theta_{k-1)}(1,f)+\varepsilon\}T(r,f)+\{\frac{m}{n+m-2k}\\
&\frac{1}{(n+m)s+2k}+1-\Theta_{k)}(1,g)+\varepsilon\}T(r,g)+S(r,f)+S(r,g)
\end{aligned} \tag{19}$$

from (18) and (19) we get

$$(-\chi_1 - 2\varepsilon)T(r,f) + (-\chi_2 - 2\varepsilon)T(r,g) \leqslant S(r,f) + S(r,g) \tag{20}$$

where χ_1 and χ_2 are defined as in and (2) respectively. From (20) and the condition $\max\{\chi_1, \chi_2\} < 0$ we get a contradiction.

Case 2. Suppose that $F \equiv G$. Then from (5) we get

$$f^n P(f) \equiv g^n P(g)$$

i.e.,

$$f^n(a_m f^m + ... + a_0) = g^n(a_m g^m + ... + a_0) \tag{21}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f \equiv gh$ into (21) we deduce $a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + ... a_0 g^n(h^n - 1) = 0$ which implies $h^d = 1$, where $d = (n+m, ..., n+m-i, ..., n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, .., m$. Thus $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = (n+m, ..., n+m+i, ...n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, .., m$. If h is not a constant, then we know by (21) that f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m \omega_1^m + a_{m-1}\omega_1^{m-1} + ... + a_0) - \omega_2^n(a_m \omega_2^m + a_{m-1}\omega_2^{m-1} + ... + a_0)$.

This proves the Theorem 1.

4. Proof of Theorem 2

Proof. Since f and g are entire functions, we have $N(r,f) = N(r,g) = 0$. Proceeding as in the proof Theorem 1 and applying Lemma 1, we complete the proof of Theorem 2.

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