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GENERALIZED STIELTJES CONSTANTS AND INTEGRALS INVOLVING THE LOG–LOG FUNCTION: KUMMER'S THEOREM IN ACTION

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Abstract. In this note, we recall Kummer's Fourier series expansion of the 1-periodic function that coincides with the logarithm of the Gamma function on the unit interval $(0,1)$, and we use it to find closed forms for some numerical series related to the generalized Stieltjes constants, and some integrals involving the function $x \mapsto \ln\ln(1/x)$.

1. Introduction and notation

The aim of this paper is to present an alternative proof of the reflection principle of the first order generalized Stieltjes constants, and to give an alternative approach to the evaluation of some integrals involving the function $x \mapsto \ln\ln(1/x)$. The basic tool for this investigation is a result of Kummer recalled below (Theorem 1).

The first order generalized Stieltjes constant $\gamma_1(a)$ is defined for $a \in (0,1)$ by

$$\gamma_1(a) = \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{\ln(a+k)}{a+k} - \frac{1}{2}\ln^2(n+a)\right).$$

From this, it is easy to show that

$$\gamma(a) - \gamma(1-a) = \lim_{n\to\infty}\left(\sum_{k=-n}^{n}\frac{\ln|a+k|}{a+k}\right) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}}{}' \frac{\ln|a+n|}{a+n},$$

where the primed sum denotes the "principal value" as shown above. For integers p and q with $0 < p < q$ the difference $\gamma(p/q) - \gamma(1-p/q)$ can be expressed as follows

$$\gamma(p/q) - \gamma(1-p/q) = -\pi\ln(2\pi q e^{\gamma})\cot\left(\frac{p\pi}{q}\right) + 2\pi\sum_{j=1}^{q-1}\sin\left(\frac{2\pi jp}{q}\right)\ln\Gamma\left(\frac{j}{q}\right).$$

The formula is attributed to Almkvist and Meurman who obtained it by calculating the derivative of the functional equation for the Hurwitz zeta function $\zeta(s,v)$ with respect to s at rational v, see [2]. However, it was recently discovered that an equivalent form of this formula was already obtained by Carl Malmsten in 1846 (see [5]). An elementary proof of this formula will be presented in Proposition 2.

In a recent series of articles ([3], [9], [10], [11], [14]), the authors proved some formulas from the *Table of integrals, Series, and Products*, of Gradshteyn and Ryzhik

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[7]. Further, the monographs [12, 13] are devoted to providing proofs for the formulas in [7]. In fact, we are particularly interested in integrals involving the function $x \mapsto \ln\ln(1/x)$. Indeed, entries 4.325 of [7] contain the following evaluations:

$$\int_0^1 \frac{\ln(\ln(1/x))}{1+x^2} = \frac{\pi}{2}\ln\frac{\sqrt{2\pi}\Gamma(3/4)}{\Gamma(1/4)}$$

$$\int_0^1 \frac{\ln(\ln(1/x))}{1+x+x^2} = \frac{\pi}{\sqrt{3}}\ln\frac{\sqrt[3]{2\pi}\Gamma(2/3)}{\Gamma(1/3)}$$

$$\int_0^1 \frac{\ln(\ln(1/x))}{1+2x\cos t+x^2} = \frac{\pi}{2\sin t}\ln\frac{(2\pi)^{t/\pi}\Gamma\left(\frac{1}{2}+\frac{t}{2\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{t}{2\pi}\right)}$$

These integrals can be traced back to [6]. The first of them was the object of a detailed investigation in [14], where the author says that his approach can be adapted to prove also the second one. A general approach that yields the first two integrals, and much more evaluations, can also be found in [2]. This line of investigation was completed by adapting the methods of [14] to obtain general results that include all the above mentioned integrals in [11].

Our aim is to present an alternative approach to the evaluation of these integrals. Our starting point will be Kummer's Fourier expansion of $\operatorname{Log}\Gamma$, (Theorem 1), where Γ is the well-known Eulerian gamma function. This result is attributed to Kummer in (1847), a more accessible reference is [4, Section 1.7]:

THEOREM 1. (Kummer, [8]) *For* $0 < x < 1$,

$$\ln\frac{\Gamma(x)}{\sqrt{2\pi}} = -\frac{\ln(2\sin(\pi x))}{2} + (\gamma+\ln(2\pi))\left(\frac{1}{2}-x\right) + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\ln k}{k}\sin(2\pi kx),$$

where γ *is the Euler-Mascheroni constant.*

2. The reflection formula for the first order generalized Stieltjes constants

As we explained in the introduction, this formula relates the first order generalized Stieltjes constant $\gamma_1(a)$ to its reflected value $\gamma_1(1-a)$ for rational a. The presented proof is different from that of Almkvist and Meurman, and has the advantage of being elementary in the sense that it does not make use of the functional equation of the Hurwitz zeta function.

PROPOSITION 2. *For positive integers* p *and* q *with* $p < q$*, we have*

$$\sum_{n\in\mathbb{Z}}{}' \frac{\ln\left|n+\frac{p}{q}\right|}{n+\frac{p}{q}} = -\pi\ln(2\pi q e^{\gamma})\cot\left(\frac{p\pi}{q}\right) + 2\pi\sum_{j=1}^{q-1}\sin\left(\frac{2\pi jp}{q}\right)\ln\Gamma\left(\frac{j}{q}\right).$$

where the primed sum denotes the "principal value", defined as follows:

$$\sum_{n\in\mathbb{Z}}{}' a_n = \lim_{n\to\infty}\left(\sum_{k=-n}^{n} a_k\right).$$

Proof. The statement of Theorem 1 is written as

$$\sum_{k=1}^{\infty} \frac{\ln k}{k} \sin(2\pi kx) = -\frac{\pi}{2}\ln\pi + \frac{\pi}{2}\ln\sin(\pi x) + \pi\ln(2\pi e^{\gamma})\left(x-\frac{1}{2}\right) + \pi\ln\Gamma(x). \quad (1)$$

Now, consider a positive integer q with $q \geqslant 2$. For $j \in \{1,2,\dots,q-1\}$ we have

$$\sum_{k=1}^{\infty} \frac{\ln k}{k} \sin\left(\frac{2\pi kj}{q}\right) = -\frac{\pi}{2}\ln\pi + \frac{\pi}{2}\ln\sin\left(\frac{\pi j}{q}\right) + \pi\ln(2\pi e^{\gamma})\left(\frac{j}{q}-\frac{1}{2}\right) + \pi\ln\Gamma\left(\frac{j}{q}\right). \quad (2)$$

Multiply both sides of (2) by $\sin\left(\frac{2\pi jp}{q}\right)$, where p is some integer from $\{1,\dots,q-1\}$, and add the resulting equalities for $j = 1,\dots,q-1$, to obtain

$$\sum_{k=1}^{\infty} \frac{\ln k}{k} A_{p,q}(k) = -\frac{\pi\ln\pi}{2} B_{p,q} + \frac{\pi}{2} C_{p,q} + \pi\ln(2\pi e^{\gamma}) D_{p,q} + \pi\sum_{j=1}^{q-1} \sin\left(\frac{2\pi jp}{q}\right)\ln\Gamma\left(\frac{j}{q}\right), \quad (3)$$

where

$$A_{p,q}(k) = \sum_{j=1}^{q-1} \sin\left(\frac{2\pi jp}{q}\right)\sin\left(\frac{2\pi jk}{q}\right)$$

$$B_{p,q} = \sum_{j=1}^{q-1} \sin\left(\frac{2\pi jp}{q}\right)$$

$$C_{p,q} = \sum_{j=1}^{q-1} \sin\left(\frac{2\pi jp}{q}\right)\ln\sin\left(\frac{\pi j}{q}\right)$$

$$D_{p,q} = \sum_{j=1}^{q-1} \left(\frac{j}{q}-\frac{1}{2}\right)\sin\left(\frac{2\pi jp}{q}\right).$$

These sums are now simplified. Let $\omega_q = \exp\left(\frac{2\pi i}{q}\right)$, and use $\sum_{j=0}^{q-1} \omega_q^{nj} = q\chi_q(n)$ where $\chi_q(n) = 1$ if $n \equiv 0 \mod q$ and $\chi_q(n) = 0$ otherwise. The imaginary part of the identity gives

$$B_{p,q} = 0. \quad (4)$$

Also,

$$A_{p,q}(k) = \frac{1}{2}\sum_{j=1}^{q-1}\left(\cos\left(\frac{2\pi j(p-k)}{q}\right) - \cos\left(\frac{2\pi j(p+k)}{q}\right)\right)$$
$$= \frac{1}{2}\Re\left(\sum_{j=0}^{q-1}\omega^{(p-k)j} - \sum_{j=0}^{q-1}\omega^{(p+k)j}\right) = \frac{q}{2}(\chi_q(p-k) - \chi_q(p+k)).$$

That is

$$A_{p,q}(k)=\begin{cases} \frac{q}{2} & \text{if } k\equiv p \mod q,\\ -\frac{q}{2} & \text{if } k\equiv -p \mod q.\end{cases} \tag{5}$$

On the other hand, the change of summation index $j\leftarrow q-j$ in the formula for $C_{p,q}$ shows that

$$C_{p,q}=\sum_{j=1}^{q-1}\sin\left(2\pi p-\frac{2\pi jp}{q}\right)\ln\sin\left(\pi-\frac{\pi j}{q}\right)=-C_{p,q}.$$

Thus,

$$C_{p,q}=0. \tag{6}$$

Finally, use (4) to obtain

$$D_{p,q}=\frac{1}{q}\sum_{j=1}^{q-1} j\sin\left(\frac{2\pi jp}{q}\right).$$

Now for, $0<\theta<\pi$, we have

$$\begin{aligned}1+2\sum_{j=1}^{q-1}\cos(2j\theta)&=\sum_{j=1-q}^{q-1}e^{2ij\theta}=\frac{e^{2iq\theta}-e^{2i(1-q)\theta}}{e^{2i\theta}-1}\\ &=\frac{\sin((2q-1)\theta)}{\sin\theta}=\sin(2q\theta)\cot\theta-\cos(2q\theta).\end{aligned}$$

Taking the derivative with respect to θ and substituting $\theta=\pi p/q$ we get

$$D_{p,q}=-\frac{1}{2}\cot\left(\frac{p\pi}{q}\right). \tag{7}$$

Replacing (4),(5),(6) and (7) in (3) we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\frac{\ln(qk+p)}{k+p/q}-\frac{\ln(qk+q-p)}{k+1-p/q}\right)&=-\pi\ln(2\pi e^{\gamma})\cot\left(\frac{p\pi}{q}\right)\\ &\quad+2\pi\sum_{j=1}^{q-1}\sin\left(\frac{2\pi jp}{q}\right)\ln\Gamma\left(\frac{j}{q}\right).\end{aligned} \tag{8}$$

The final step is to use the well-known cotangent partial fraction expansion:

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\frac{1}{k+p/q}-\frac{1}{k+1-p/q}\right)&=\lim_{N\to\infty}\sum_{k=-N}^{N}\frac{1}{k+p/q}=\frac{q}{p}-\sum_{k=1}^{N}\frac{2p/q}{k^2-(p/q)^2}\\ &=\frac{q}{p}+\sum_{k=1}^{\infty}\frac{2p/q}{(p/q)^2-k^2}=\pi\cot\left(\frac{p\pi}{q}\right).\end{aligned} \tag{9}$$

Thus, subtracting $(\ln q)$ times (9) from (8) we obtain the desired conclusion. □

EXAMPLES. Taking $p=1$ and $q\in\{3,4\}$ we obtain

$$\sum_{n\in\mathbb{Z}}{}' \frac{\ln\left|n+\frac{1}{3}\right|}{n+\frac{1}{3}} = \frac{\pi}{2\sqrt{3}}\ln\left(\frac{3\Gamma^{12}(\frac{1}{3})}{2^8\pi^8 e^{2\gamma}}\right).$$

$$\sum_{n\in\mathbb{Z}}{}' \frac{\ln\left|n+\frac{1}{4}\right|}{n+\frac{1}{4}} = \pi\ln\left(\frac{\Gamma^{4}(\frac{1}{4})}{2^4\pi^3 e^{\gamma}}\right).$$

3. The evaluation of some integrals involving the log-log function

In this section we use Theorem 1, to evaluate some difficult integrals.

PROPOSITION 3. *For $0<x<1$, we have:*

$$\int_0^1 \frac{\ln\ln(1/u)}{u^2-2(\cos 2\pi x)u+1}\,du = \frac{\pi}{2\sin(2\pi x)}\left((1-2x)\ln(2\pi)+\ln\left(\frac{\Gamma(1-x)}{\Gamma(x)}\right)\right).$$

And, taking the limit as x tend to $1/2$, we obtain

$$\int_0^1 \frac{\ln\ln(1/u)}{(u+1)^2}\,du = \ln\sqrt{2\pi}+\frac{\Gamma'(1/2)}{2\Gamma(1/2)} = \ln\sqrt{\frac{\pi}{2}}-\frac{\gamma}{2}.$$

Proof. Indeed, subtracting the corresponding Kummer's Formulas, for $\ln\Gamma(x)$ and $\ln\Gamma(1-x)$ we see that, for $0<x<1$ we have

$$\ln\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right) = (\gamma+\ln(2\pi))(1-2x)+\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\ln k}{k}\sin(2\pi kx), \tag{10}$$

or equivalently,

$$\ln\left(\frac{(2\pi)^x\Gamma(x)}{(2\pi)^{1-x}\Gamma(1-x)}\right) = \gamma(1-2x)+\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\ln k}{k}\sin(2\pi kx). \tag{11}$$

Now, using the fact that for $\Re s>0$ and $k\geqslant 1$ we have $\frac{\Gamma(s)}{k^s}=\int_0^\infty t^{s-1}e^{-kt}\,dt$, we conclude that for $s>0$, and $0<x<1$, we have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{e^{2\pi ikx}}{k^s} &= \frac{1}{\Gamma(s)}\sum_{k=1}^{\infty}\left(\int_0^\infty t^{s-1}e^{-kt}e^{2\pi ikx}dt\right)\\ &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{-t+2\pi ix}}{1-e^{-t+2\pi ix}}t^{s-1}\,dt.\end{aligned}$$

Restricting our attention to the imaginary parts we get

$$\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^s} = \frac{\sin(2\pi x)}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^t+e^{-t}-2\cos(2\pi x)}\,dt. \tag{12}$$

Now, taking the derivative with respect to s at $s=1$ we obtain, for $0<x<1$, the following:

$$\sum_{k=1}^{\infty}\frac{\ln k}{k}\sin(2\pi kx)=\frac{\Gamma'(1)}{\Gamma^2(1)}\int_0^{\infty}\frac{\sin(2\pi x)}{e^t+e^{-t}-2\cos(2\pi x)}\,dt$$
$$-\frac{\sin(2\pi x)}{\Gamma(1)}\int_0^{\infty}\frac{\ln t}{e^t+e^{-t}-2\cos(2\pi x)}\,dt. \tag{13}$$

Taking into account the facts $\Gamma'(1)=-\gamma$, $\Gamma(1)=1$, and

$$\int_0^{\infty}\frac{\sin(2\pi x)}{e^t+e^{-t}-2\cos(2\pi x)}\,dt=\pi\left(\frac{1}{2}-x\right),\quad \text{for } 0<x<1$$

we conclude that

$$\sum_{k=1}^{\infty}\frac{\ln k}{k}\sin(2\pi kx)=-\frac{\gamma\pi}{2}(1-2x)-\sin(2\pi x)\int_0^{\infty}\frac{\ln t}{e^t+e^{-t}-2\cos(2\pi x)}\,dt. \tag{14}$$

The change of variables $t=\ln(1/u)$ yields:

$$\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\ln k}{k}\sin(2\pi kx)+\gamma(1-2x)=-\frac{2\sin(2\pi x)}{\pi}\int_0^1\frac{\ln\ln(1/u)}{u^2-2(\cos 2\pi x)u+1}\,du. \tag{15}$$

Finally, combining (11) and (15) we obtain the desired result. Concerning the limit as x tend to $1/2$, we use the well-known fact that $\frac{\Gamma'(1/2)}{\Gamma(1/2)}=\psi(1/2)=-\gamma-2\ln 2$, (see [1, 6.3.3]). $\square$

EXAMPLES. Taking $x=1/3$, $x=1/4$ and $x=1/6$ we obtain

$$\int_0^1\frac{\ln\ln(1/u)}{u^2+u+1}\,du=\frac{-\pi}{6\sqrt{3}}\ln\left(\frac{3^3}{4^4\pi^8}\Gamma^{12}\left(\frac{1}{3}\right)\right) \tag{16}$$

$$\int_0^1\frac{\ln\ln(1/u)}{u^2+1}\,du=\frac{-\pi}{4}\ln\left(\frac{1}{4\pi^3}\Gamma^4\left(\frac{1}{4}\right)\right) \tag{17}$$

$$\int_0^1\frac{\ln\ln(1/u)}{u^2-u+1}\,du=\frac{-\pi}{3\sqrt{3}}\ln\left(\frac{1}{(2\pi)^5}\Gamma^6\left(\frac{1}{6}\right)\right)=\frac{-\pi}{3\sqrt{3}}\ln\left(\frac{3^3}{2^7\pi^8}\Gamma^{12}\left(\frac{1}{3}\right)\right). \tag{18}$$

where we used freely the duplication, and the reflection formulas for the gamma function [1, 6.1.17 and 6.1.18]. In particular, we used $\Gamma(\frac{1}{6})=\frac{\sqrt{3/\pi}}{\sqrt[3]{2}}\Gamma^2(\frac{1}{3})$ that follows readily from these formulas.

The second degree polynomial in the integrand's denominator in Proposition 3 has negative discriminant. In the next proposition the corresponding denominator has real roots outside the interval $[0,1]$. This case seems to be new to the best knowledge of the author.

PROPOSITION 4. *Let $A_\Gamma:\mathbb{R}\longrightarrow\mathbb{R}$ be the function defined by*

$$A_\Gamma(y)=-\frac{\ln(2\pi)}{2}y+\frac{\sinh(\pi y)}{\pi}\int_0^1\frac{\ln\ln(1/u)}{u^2+2\cosh(\pi y)\,u+1}\,du.$$

Then, for $y \in \mathbb{R}$ *we have*

$$\Gamma\left(\frac{1+iy}{2}\right) = \sqrt{\frac{\pi}{\cosh(\pi y/2)}}\, e^{iA_\Gamma(y)}.$$

Proof. Let us rephrase Proposition 3, by taking $x = \frac{t+1}{2}$ in order to give more symmetric aspect to the formula there:

$$\forall t \in (-1,1), \quad \int_0^1 \frac{\ln\ln(1/u)}{u^2 + 2\cos(\pi t)\,u + 1}\,du = \frac{\pi}{2\sin(\pi t)}\left(\ln(2\pi)\,t + \ln\left(\frac{\Gamma(\frac{1+t}{2})}{\Gamma(\frac{1-t}{2})}\right)\right),$$

or equivalently, for $-1 < t < 1$, we have

$$\exp\left(-\ln(2\pi)t + \frac{2\sin(\pi t)}{\pi}\int_0^1 \frac{\ln\ln(1/u)}{u^2 + 2\cos(\pi t)\,u + 1}\,du\right) = \frac{\Gamma(\frac{1+t}{2})}{\Gamma(\frac{1-t}{2})}.$$

Using analytic continuation we deduce that, for $-1 < \Re z < 1$ we have also

$$\exp\left(-\ln(2\pi)z + \frac{2\sin(\pi z)}{\pi}\int_0^1 \frac{\ln\ln(1/u)}{u^2 + 2\cos(\pi z)\,u + 1}\,du\right) = \frac{\Gamma(\frac{1+z}{2})}{\Gamma(\frac{1-z}{2})}.$$

In particular, setting $z = iy$ with $y \in \mathbb{R}$, we obtain

$$e^{2iA_\Gamma(y)} = \frac{\Gamma(\frac{1+iy}{2})}{\Gamma(\frac{1-iy}{2})}.$$

But, by Euler's reflection formula [1, 6.1.17] we know that

$$\left|\Gamma\left(\frac{1+iy}{2}\right)\right|^2 = \Gamma\left(\frac{1+iy}{2}\right)\overline{\Gamma\left(\frac{1+iy}{2}\right)} = \Gamma\left(\frac{1+iy}{2}\right)\Gamma\left(\frac{1-iy}{2}\right) = \frac{\pi}{\cosh(\pi y/2)},$$

therefore, the square of the continuous function:

$$y \mapsto \sqrt{\frac{\cosh(\pi y/2)}{\pi}}\,\Gamma\left(\frac{1+iy}{2}\right)e^{-iA_\Gamma(y)}$$

is equal to 1 for every $y \in \mathbb{R}$, hence, it must be constant and consequently identical to 1 which is its value for $y = 0$. □

COROLLARY 1. *Let the principal determination of the argument of a nonzero complex number* z *be denoted by* Arg, *and let* α *be defined by the formula*

$$\alpha = \inf\left\{y > 0 : \Gamma\left(\frac{1+iy}{2}\right) = -\sqrt{\frac{\pi}{\cosh(\pi y/2)}}\right\}.$$

Then, for every $y \in (-\alpha, \alpha)$ *we have*

$$\int_0^1 \frac{\ln\ln(1/u)}{u^2 + 2\cosh(\pi y)\,u + 1}\,du = \ln\sqrt{2\pi}\cdot\frac{\pi y}{\sinh(\pi y)} + \frac{\pi}{\sinh(\pi y)}\operatorname{Arg}\Gamma\left(\frac{1+iy}{2}\right).$$

Moreover, using Mathematica [15] *we readily obtain* $\alpha \approx 10.106689535698$.

Proof. The definition of α implies that

$$\forall y \in (-\alpha,\alpha), \quad \Gamma\left(\frac{1+iy}{2}\right) \in \mathbb{C}\setminus((-\infty,0]\times\{0\}).$$

Thus, the function $y \mapsto \mathrm{Arg}(\Gamma(\frac{1+iy}{2})) - A_\Gamma(y)$ is continuous on $(-\alpha,\alpha)$, takes its values in $2\pi\mathbb{Z}$, and is equal to 0 for $y=0$. Therefore, $A_\Gamma(y) = \mathrm{Arg}(\Gamma(\frac{1+iy}{2}))$, for every $y \in (-\alpha,\alpha)$, which is the desired conclusion. □

EXAMPLES.

$$\int_0^1 \frac{\ln\ln(1/u)}{u^2+4u+1}\,du = \frac{\ln(2\pi)}{\sqrt{3}}\ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) + \frac{\pi}{\sqrt{3}}\mathrm{Arg}\,\Gamma\left(\frac{1}{2}+\frac{i}{\pi}\ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)\right).$$

$$\int_0^1 \frac{\ln\ln(1/u)}{u^2+3u+1}\,du = \frac{2\ln(2\pi)}{\sqrt{5}}\ln(\phi) + \frac{2\pi}{\sqrt{5}}\mathrm{Arg}\,\Gamma\left(\frac{1}{2}+\frac{i}{\pi}\ln(\phi)\right).$$

where $\phi = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio.

More generally, for $2 < k < 2\cosh(\alpha\pi) \approx 6.156\times 10^{13}$, the following holds

$$\int_0^1 \frac{\ln\ln(1/u)}{u^2+ku+1}\,du = \frac{2\ln(2\pi)}{\sqrt{k^2-4}}\ln(\phi_k) + \frac{2\pi}{\sqrt{k^2-4}}\mathrm{Arg}\,\Gamma\left(\frac{1}{2}+\frac{i}{\pi}\ln(\phi_k)\right)$$

with $\phi_k = \dfrac{\sqrt{k+2}+\sqrt{k-2}}{2}$.

It is worth mentioning that *Mathematica* [15] gives the results of examples (16), (17) and (18), but it fails to give the results of the previous examples. However, numerical quadrature confirms the results.

In our final proposition we consider the evaluation of another log-log integral. This integral was given in [2] as a corollary of a more difficult evaluation. Our approach is straightforward and simpler.

PROPOSITION 5. ([2]) *For any complex number z with $\Re z > 0$, we have*

$$F(z) \stackrel{\text{def}}{=} \int_0^1 \frac{t^{z-1}\ln(\ln(1/t))}{1+t^z}\,dt = -\frac{\ln 2}{2z}\mathrm{Log}(2z^2)$$

where Log *is the principal branch of the logarithm.*

Proof. We start by evaluating $F(1)$. Note that

$$F(1) = \int_0^1 \frac{\ln(\ln(1/t))}{1+t}\,dt = \int_0^\infty \frac{e^{-x}}{1+e^{-x}}\ln(x)\,dx.$$

So,

$$\left|F(1) - \sum_{k=1}^n (-1)^{k-1}\int_0^\infty e^{-kx}\ln(x)dx\right| \leqslant \int_0^\infty \frac{|\ln x|}{1+e^x}e^{-nx}\,dx.$$

Because $x \mapsto \frac{|\ln x|}{1+e^x}$ is integrable on $(0,+\infty)$, we conclude using Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty}\int_0^\infty \frac{|\ln x|}{1+e^x}e^{-nx}\,dx = 0$$

Thus,

$$F(1) = \sum_{k=1}^{\infty}(-1)^{k-1}\int_0^\infty e^{-kx}\ln(x)dx.$$

A simple change of variables shows that

$$\int_0^\infty e^{-kx}\ln(x)dx = \frac{1}{k}\int_0^\infty e^{-u}(\ln u - \ln k)\,du = -\frac{\gamma}{k} - \frac{\ln k}{k}$$

since $\int_0^\infty \ln(u)e^{-u}\,du = \Gamma'(1) = -\gamma$. It follows that

$$F(1) = -\gamma\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k} + \sum_{k=1}^{\infty}\frac{(-1)^k\ln k}{k} = -\gamma\ln 2 + \sum_{k=1}^{\infty}\frac{(-1)^k\ln k}{k}. \qquad (20)$$

Now, note that

$$\begin{aligned}\ln^2(k+1) - \ln^2 k &= \ln^2 k\left(\left(1+\frac{1}{\ln k}\ln\left(1+\frac{1}{k}\right)\right)^2 - 1\right)\\ &= \ln^2 k\left(\frac{2}{k\ln k} + \mathscr{O}\left(\frac{1}{k^2\ln k}\right)\right)\\ &= \frac{2\ln k}{k} + \mathscr{O}\left(\frac{\ln k}{k^2}\right).\end{aligned}$$

This proves that the series $\sum\left(\ln^2(k+1) - \ln^2 k - 2\frac{\ln k}{k}\right)$ is convergent. Consequently, if we define $G_n = \sum_{k=1}^{n}\frac{\ln k}{k}$ then there is a real number ℓ such that $G_n = \frac{1}{2}\ln^2 n + \ell + o(1)$. But

$$\begin{aligned}\sum_{k=1}^{2n}\frac{(-1)^k\ln(k)}{k} &= \sum_{k=1}^{n}\frac{\ln(2k)}{k} - \sum_{k=1}^{2n}\frac{\ln k}{k} = (\ln 2)H_n + G_n - G_{2n}\\ &= (\ln 2)(\ln n + \gamma) + \frac{1}{2}\left(\ln^2(n) - \ln^2(2n)\right) + o(1)\\ &= -\frac{1}{2}\ln^2 2 + \gamma\ln 2 + o(1),\end{aligned}$$

where we used $H_n = \sum_{k=1}^{n} 1/k = \ln n + \gamma + o(1)$, (see [1, 4.1.32]).

Now, let n tend to $+\infty$ to obtain

$$\sum_{k=1}^{\infty}\frac{(-1)^k\ln(k)}{k} = -\frac{1}{2}\ln^2 2 + \gamma\ln 2.$$

Combining this with (20) we conclude that $F(1) = -\frac{1}{2}\ln^2 2$.

Next, for $z \in (0,+\infty)$ the change of variables $t^z = u$ shows that

$$F(z) = \frac{1}{z}\int_0^1 \frac{\ln\ln(1/u) - \ln z)}{1+u}\,du = \frac{F(1)}{z} - \frac{\ln(z)\ln(2)}{z} = -\frac{\ln(2z^2)\ln(2)}{2z},$$

and the desired conclusion follows by analytic continuation. □

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