Journal of Classical Analysis
Volume 9, Number 2 (2016), 117–125
doi:10.7153/jca-09-12

# NEW RESULTS CONTAINING QUADRATIC HARMONIC NUMBERS

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*Dedicated to Daniel Manca*

*Abstract.* In this paper we give a combinatorial proof of the quadratic harmonic series $\sum_{n\geqslant 1}\frac{H_n^2}{n^{2q+1}}$ in terms of zeta functions and then extend the result to express $\sum_{n\geqslant 1}\frac{H_n^2}{(n+r)^{2q+1}}, (q,r)\in\mathbb{N}$, in closed form in terms of zeta functions.

## 1. Introduction and Preliminaries

Let $\mathbb{C}$ and $\mathbb{R}$ denote respectively the set of complex numbers and the set of real numbers. The Riemann zeta function is defined, for $s\in\mathbb{C}$ with $\Re(s)>1$ by $\zeta(s)=\sum_{n\geqslant 1}\frac{1}{n^s}$. For $p\in\mathbb{N}:=\{1,2,3,\ldots\}$ we define the generalized harmonic number of order $m$ as $H_p^{(m)}=\zeta_p(m)=\sum_{j=1}^{p}\frac{1}{j^m}$. We define the $n^{th}$ harmonic number by

$$H_n=\zeta_n(1)=\sum_{j=1}^{n}\frac{1}{j}=\sum_{j=1}^{\infty}\frac{n}{j(j+n)}=\int_0^1\frac{1-x^n}{1-x}\,dx,\quad H_0:=0. \tag{1}$$

The Psi (or Digamma), $\psi(z)$ function is defined by

$$\psi(z):=\frac{d}{dz}\{\log\Gamma(z)\}=\frac{\Gamma'(z)}{\Gamma(z)}$$

where the Euler gamma function

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt,\quad \Re(z)>0$$

and the Beta function

$$B(a,b)=B(b,a)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 t^{a-1}(1-t)^{b-1}\,dt,$$

*Mathematics subject classification* (2010): 05A10, 05A19, 33C20, 11B65, 11B83, 11M06.
*Keywords and phrases*: Combinatorial series identities, summation formulae, partial fraction approach, harmonic numbers, integral identity, Hurwitz-Lerch transcendent.


Paper JCA-09-12

where $\Re(a) > 0, \Re(b) > 0$. The polygamma functions of order $m \in \mathbb{N} \cup \{0\}$,

$$\psi^{(m)}(z) := \frac{d^m}{dz^m}\{\psi(z)\} = \frac{d^{m+1}}{dz^{m+1}}\{\log\Gamma(z)\}$$

and therefore we may connect the generalized harmonic numbers to the polygamma functions, see [14] by,

$$H_\rho^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\,\psi^{(m)}(\rho+1)$$

where $\rho \in \mathbb{R} \setminus \{-1,-2,-3,\cdots\}$; $m \in \mathbb{N}$ and $\zeta(m+1)$ is the Riemann zeta function. The Lerch transcendent $\Phi(z,t,a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^t}$ is defined for $|z| < 1$, $\Re(a) > 0$ and satisfies the recurrence

$$\Phi(z,t,a) = z\Phi(z,t,a+1) + a^{-t}.$$

In the case when $z = 1$, we have the Hurwitz zeta function

$$\Phi(1,t,a) = \zeta(t,a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^t}.$$

In this paper we will develop identities, new families of closed form representations of quadratic harmonic numbers and reciprocal powers of $n$, series of the form:

$$T(q) = 3\sum_{n=1}^{\infty} \frac{H_n^2}{n^{2q+1}}, \tag{2}$$

and

$$X(q,r) = \sum_{n=1}^{\infty} \frac{H_n^2}{(n+r)^{2q+1}} \tag{3}$$

for $q \in \mathbb{N}$ and $r \in \mathbb{N}$, in terms of zeta functions.

Some results regarding quadratic harmonic number sums related to (2) exist in the current literature. The classical result, conjectured by Au-Yeung and proved by Borwein et. al., see [16] and [15] is $\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{17}{4}\zeta(4)$. Flajolet and Salvy [4] give some specific results for $\sum_{n=1}^{\infty} \frac{H_n^2}{n^m}$ when $m = 2,3,4,5,7$, and a complete representation for $m$ an odd integer was originally given by Euler [3] and later proved by Borwein et. al., see [1] and [2]. The result,

$$\sum_{n=1}^{\infty} \frac{H_n^2}{\binom{n+k}{k}} = \frac{k}{(k-1)}\left(\zeta(2) + \frac{2}{(k-1)^2} - H_{k-1}^{(2)}\right), \quad k \geqslant 2$$

was given in [6], see also [10] and generalized further to

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n\binom{n+k}{k}^m}, \quad \text{for} \quad m = 1 \text{ and } 2,$$

see [13]. An identity for

$$\sum_{n=1}^{\infty} \frac{\left(H_n^2 + H_n^{(2)}\right)(k+2n)}{n^2 \binom{n+k}{k}^2},$$

was given in [9]. The finite case

$$3\sum_{j=1}^{n} \frac{H_j H_{j-1}}{j} = H_n^3 - H_n^{(3)}$$

was given by [5], and for $(x,b) \in \mathbb{N}$

$$\sum_{j=1}^{n} \frac{j\left(H_j^2 + H_j^{(2)}\right) + 2(j+b)^x H_j H_{j+b-1}^{(x)}}{j(j+b)^x} = H_{n+b}^{(x)}\left(H_n^2 + H_n^{(2)}\right)$$

was proved in [7].

The following known results listed below will be useful in the development of the main theorems. Euler [3], obtained the following result

LEMMA 1. *For* $m \in \mathbb{N} \setminus \{1\}$. *Then:*

$$\begin{aligned} B(m) &= \sum_{n=1}^{\infty} \frac{H_n}{n^m} \\ &= \left(1 + \frac{m}{2}\right)\zeta(m+1) - \frac{1}{2}\sum_{j=1}^{m-2} \zeta(j+1)\zeta(m-j). \end{aligned} \tag{4}$$

The proof of the following lemma is detailed in [12].

LEMMA 2. *For* $x > 0$ *and* $m \in \mathbb{N} \setminus \{1\}$, *in terms of polygamma functions,* $\psi^{(m)}(z)$ *we have,*

$$\begin{aligned} S(m,x) &= \sum_{n=1}^{\infty} \frac{H_n}{(n+x)^{2m+1}} \\ &= \frac{(-1)^m}{(m-1)!} \left[ \begin{array}{c} (\gamma + \psi(x))\psi^{(m-1)}(x) - \frac{1}{2}\psi^{(m)}(x) \\ + \sum_{j=1}^{m-2} \binom{m-2}{j} \psi^{(j)}(x)\psi^{(m-1-j)}(x) \end{array} \right], \end{aligned} \tag{5}$$

*where* $\gamma = .5772\ldots$ *is the Euler-Mascheroni constant. For* $x = 0$, *then (5) reduces to (4).*

The following result is deduced from the work of Flajolet and Salvy [4].

LEMMA 3. *For $m \in \mathbb{N}$ then*

$$A(m) := \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^{2m+1}}$$
$$= (2m+1)\zeta(2)\zeta(2m+1) - \frac{1}{2}(m+2)(2m+1)\zeta(2m+3)$$
$$+2\sum_{j=1}^{m-1} j\zeta(2j+1)\zeta(2m+2-2j). \tag{6}$$

## 2. Closed form and integral identities

The next two theorems relate the main results of this investigation, namely the closed form and integral representation of (2) and (3).

THEOREM 1. *Let $q \in \mathbb{N}$, then we have,*

$$T(q) = 3\sum_{n=1}^{\infty} \frac{H_n^2}{n^{2q+1}}$$
$$= 2B(2q+2) - A(q)$$
$$+2\sum_{j=1}^{2q-1} (-1)^{j+1} \zeta(j+1)\left(B(2q+1-j) - \zeta(2q+2-j)\right), \tag{7}$$

*where $A(q)$ is given by (6) and $B(2q+2)$ is given by (4).*

*Proof.* We begin with the known generating function

$$\ln^2(1-x) = 2\sum_{j \geqslant 1} \frac{x^{j+1} H_j}{j+1}, \quad x \in [-1,1)$$

and from [8]

$$\int_0^1 x^{k-1} \ln^2(1-x)\, dx = \frac{H_k^2 + H_k^{(2)}}{k}.$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{n^{2q}} \left( \frac{H_n^2 + H_n^{(2)}}{n} \right) = \sum_{k=1}^{\infty} \frac{1}{k^{2q}} \int_0^1 x^{k-1} \ln^2(1-x)\, dx$$
$$= 2\sum_{k=1}^{\infty} \frac{1}{k^{2q}} \sum_{n=1}^{\infty} \frac{H_n}{n+1} \int_0^1 x^{k+n}\, dx$$
$$= 2\sum_{n=1}^{\infty} \frac{H_n}{n+1} \sum_{k=1}^{\infty} \frac{1}{k^{2q}(k+n+1)}. \tag{8}$$

By a partial fraction decomposition of (8), we can write

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^{2q}}\left(\frac{H_n^2+H_n^{(2)}}{n}\right) &= 2\sum_{n=1}^{\infty}\frac{H_n}{n+1}\sum_{k=1}^{\infty}\left[\begin{array}{c} -\frac{1}{(n+1)^{2q-1}k(k+n+1)} \\ +\sum_{j=1}^{2q-1}\frac{(-1)^{j+1}}{(n+1)^{2q-j}k^{j+1}}\end{array}\right] \\ &= 2\sum_{n=1}^{\infty}\frac{H_n}{n+1}\left[-\frac{H_{n+1}}{(n+1)^{2q}}+\sum_{j=1}^{2q-1}\frac{(-1)^{j+1}}{(n+1)^{2q-j}}\zeta(j+1)\right] \\ &= -2\sum_{n=1}^{\infty}\frac{H_{n+1}H_n}{(n+1)^{2q+1}}+2\sum_{j=1}^{2q-1}(-1)^{j+1}\zeta(j+1)\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^{2q+1-j}}\end{aligned}$$

and making a change of summation index, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^{2q}}\left(\frac{H_n^2+H_n^{(2)}}{n}\right) &= 2\sum_{n=1}^{\infty}\frac{H_n}{n^{2q+2}}-\sum_{n=1}^{\infty}\frac{H_n^2}{n^{2q+1}} \\ &\quad +2\sum_{j=1}^{2q-1}(-1)^{j+1}\zeta(j+1)\left\{\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^{2q+1-j}}-\zeta(2q+2-j)\right\}.\end{aligned}$$

By rearrangement

$$\begin{aligned}3\sum_{n=1}^{\infty}\frac{H_n^2}{n^{2q+1}} &= 2\sum_{n=1}^{\infty}\frac{H_n}{n^{2q+2}}-\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^{2q+1}} \\ &\quad +2\sum_{j=1}^{2q-1}(-1)^{j+1}\zeta(j+1)\left\{\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^{2q+1-j}}-\zeta(2q+2-j)\right\} \\ &= T(q). \quad \square\end{aligned}$$

It is possible to represent $T(q)$ in integral form and this relationship is given in the next lemma

LEMMA 4. *The integral identity*

$$T(q)=3\sum_{n=1}^{\infty}\frac{H_n^2}{n^{2q+1}}=3\int_0^1\int_0^1\frac{\ln(1-x)\ln(1-y)Li_{2q-1}(xy)}{xy}dxdy$$

*where*

$$Li_m(z):=\sum_{n=1}^{\infty}\frac{z^n}{n^m},\ \ m\in\mathbb{C}\ \ when\ \ |z|<1; \qquad \Re(m)>1\ \ when\ \ |z|=1$$

*is the Polylogarithm, or de Jonquière's function.*

*Proof.* From [11] we have the general representation

$$(-1)^m \frac{H_n^m}{n^m} = \int_0^1 \cdots \int_0^1 \left(\prod_{j=1}^m x_j\right)^{n-1} \prod_{j=1}^m \ln(1-x_j)\, dx_j$$

where $\int_0^1 \cdots \int_0^1$ is a $m$- fold integration procedure, for $m=2$,

$$\frac{H_n^2}{n^2} = \int_0^1 \int_0^1 (x_1x_2)^{n-1} \ln(1-x_1)\ln(1-x_2)\, dx_1 dx_2.$$

Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^2}{n^{2q+1}} &= \int_0^1 \int_0^1 \ln(1-x)\ln(1-y) \sum_{n=1}^{\infty} \frac{(xy)^{n-1}}{n^{2q-1}} dxdy \\
&= \int_0^1 \int_0^1 \frac{\ln(1-x)\ln(1-y)\operatorname{Li}_{2q-1}(xy)}{xy} dxdy. \quad \square
\end{aligned}$$

EXAMPLE 1. For $q=3$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^2}{n^7} &= \int_0^1 \int_0^1 \frac{\ln(1-x)\ln(1-y)\operatorname{Li}_5(xy)}{xy} dxdy \\
&= \frac{55}{6}\zeta(9) - \zeta(2)\zeta(7) - \frac{5}{2}\zeta(4)\zeta(5) + \frac{1}{3}(\zeta(3))^3 - \frac{7}{2}\zeta(3)\zeta(6).
\end{aligned}$$

THEOREM 2. *For $r \in \mathbb{N}$ and $q \in \mathbb{N}$, we have,*

$$\begin{aligned}
X(q,r) = \sum_{n=1}^{\infty} \frac{H_n^2}{(n+r)^{2q+1}} &= X(q,0) - 2B(2q+2) + \zeta(2q+3) \\
&- \zeta(2) H_{r-1}^{(2q+1)} - \sum_{k=1}^{r-1} \left( \frac{(2q+1)H_k}{k^{2q+2}} + \frac{H_{k-1}^2 + H_{k-1}^{(2)}}{k^{2q+1}} \right) \\
&+ \sum_{j=1}^{2q} \sum_{k=1}^{r-1} \frac{1}{k^j} \left\{ \frac{j}{k} \left( \zeta(2q+2-j) - H_k^{(2q+2-j)} \right) + S(2q+2-j,k) \right\} \qquad (9)
\end{aligned}$$

*where $S(m,r)$ is given by (5) and $B(m)$ is given by (4). In the case of $r=0, X(q,0) = \frac{1}{3}T(q)$.*

*Proof.* Consider $X(q,r)=\sum_{n=1}^{\infty}\frac{H_n^2}{(n+r)^{2q+1}}$ then by a change of summation index

$$\begin{aligned}
X(q,r) &= \sum_{n=1}^{\infty}\frac{H_{n-1}^2}{(n+r-1)^{2q+1}}\\
&= X(q,r-1)-2\sum_{n=1}^{\infty}\frac{H_n}{n(n+r-1)^{2q+1}}+\sum_{n=1}^{\infty}\frac{1}{n^2(n+r-1)^{2q+1}}\\
&= X(q,r-1)-2\sum_{n=1}^{\infty}\left(\begin{array}{c}\frac{H_n}{(r-1)^{2q}n(n+r-1)}\\ -\sum_{j=1}^{2q}\frac{H_n}{(r-1)^j(n+r-1)^{2q+2-j}}\end{array}\right)\\
&\quad+\sum_{n=1}^{\infty}\left(\begin{array}{c}\frac{1}{(r-1)^{2q+1}n^2}-\frac{2q+1}{(r-1)^{2q+1}n(n+r-1)}\\ +\sum_{j=1}^{2q}\frac{j}{(r-1)^{j+1}(n+r-1)^{2q+2-j}}\end{array}\right)\\
&= X(q,r-1)-\frac{\zeta(2)}{(r-1)^{2q+1}}-\frac{(2q+1)H_{r-1}}{(r-1)^{2q+2}}-\frac{H_{r-2}^2+H_{r-2}^{(2)}}{(r-1)^{2q+1}}\\
&\quad+\sum_{j=1}^{2q}\frac{j}{(r-1)^{j+1}}\left(\zeta(2q+2-j)-H_{r-1}^{(2q+2-j)}\right)\\
&\quad+\sum_{j=1}^{2q}\frac{1}{(r-1)^j}S(2q+2-j,r-1),
\end{aligned}$$

and this is the recurrence relation for $X(q,r)$, which may be solved by the subsequent reduction of the $X(q,r),X(q,r-1),\ldots,X(q,2),X(q,1)$ terms finally arriving at the relation (9). □

It is also possible to represent $X(q,r)$ in integral form as follows.

PROPOSITION 1. *Let* $\{q,r\}\in\mathbb{N}$,

$$\begin{aligned}
X(q,r) &= \sum_{n=1}^{\infty}\frac{H_n^2}{(n+r)^{2q+1}}\\
&= \int_0^1\int_0^1 \ln(1-x)\ln(1-y)\left\{\begin{array}{c}\Phi(xy,2q-1,r+1)\\ -2r\Phi(xy,2q,r+1)\\ +r^2\Phi(xy,2q+1,r+1)\end{array}\right\}dxdy, \qquad (10)
\end{aligned}$$

*where* $\Phi(z,t,a)$ *is the Lerch transcendent.*

*Proof.* We consider, for $\{j,k\} \in \mathbb{R}^+$ and $|t| \leqslant 1$,

$$\sum_{n=1}^{\infty} \frac{t^n}{(n+r)^{2q+1} \binom{n+j}{j} \binom{n+k}{k}}$$
$$= \sum_{n=1}^{\infty} \frac{t^n n^2 \Gamma(n) \Gamma(j+1) \Gamma(n) \Gamma(k+1)}{(n+r)^{2q+1} \Gamma(n+j+1) \Gamma(n+k+1)}$$
$$= \sum_{n=1}^{\infty} \frac{t^n n^2 B(j+1,n) B(k+1,n)}{(n+r)^{2q+1}}$$

where $B(\cdot,\cdot)$ is the Beta function. Then

$$\sum_{n=1}^{\infty} \frac{t^n n^2 B(j+1,n) B(k+1,n)}{(n+r)^{2q+1}}$$
$$= \int_0^1 \int_0^1 \frac{(1-x)^j (1-y)^k}{xy} \sum_{n=1}^{\infty} \frac{n^2 (txy)^n}{(n+r)^{2q+1}} dxdy$$
$$= \int_0^1 \int_0^1 (1-x)^j (1-y)^k \left\{ \begin{array}{c} \Phi(xy, 2q-1, r+1) \\ -2r\Phi(xy, 2q, r+1) \\ +r^2 \Phi(xy, 2q+1, r+1) \end{array} \right\} dxdy.$$

Now the procedure is to differentiate both sides with respect to $j$ and $k$ respectively and then put $j$ and $k$ to zero with $t = 1$ so that (10) follows. The case of $r = 0$ reduces to (6). $\square$

EXAMPLE 2. In the case $q = 2$, $r = 3$ :

$$\begin{aligned} X(2,3) &= \sum_{n=1}^{\infty} \frac{H_n^2}{(n+3)^5} \\ &= -\frac{3081}{128} + \frac{159}{32}\zeta(2) + \frac{113}{16}\zeta(3) + \frac{69}{16}\zeta(4) + \frac{29}{4}\zeta(5) - \frac{3}{2}\zeta^2(3) \\ &\quad + \frac{189}{84}\zeta(6) - \zeta(7) - \frac{5}{2}\zeta(2)\zeta(3) - \frac{1}{2}\zeta(4)\zeta(3) + \zeta(2)\zeta(5). \\ &= \lim_{(\varepsilon,\delta)\to 0} \int_\varepsilon^1 \int_\delta^1 \frac{\ln(1-x)\ln(1-y)}{x^4 y^4} \left( \begin{array}{c} 9\mathrm{Li}_5(xy) - 6\mathrm{Li}_4(xy) + \mathrm{Li}_3(xy) \\ -\frac{y^2x^2}{32} - 4yx \end{array} \right) dxdy. \end{aligned}$$

*Acknowledgement.* The author expresses his gratitude to a referee for the careful reading of the manuscript.

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(Received May 11, 2016)

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