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A SUBCLASS OF ANALYTIC FUNCTIONS RELATED WITH CONIC DOMAIN

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Abstract. In this paper, we introduce a new subclass of analytic functions by using the concept of conic domain. We prove inclusion relations, a characterization theorem, coefficient inequalities, a distortion theorem, a covering theorem, and the radii of close-to-convexity, startlikeness and convexity for this class of functions.

1. Introduction and preliminaries

Let $\mathscr{A}$ be the class of all analytic functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1}$$

defined in the open unit disc $E=\{z:|z|<1\}$. We denote by $\mathscr{S}$ the subclass of $\mathscr{A}$ consisting of functions which are univalent in E and normalized by the conditions $f(0)=f'(0)-1=0$.

For $0\leqslant\eta<1$, $\mathscr{S}^*(\eta)$ and $\mathscr{C}(\eta)$ denote the classes of functions in $\mathscr{S}$ which are, respectively, starlike and convex of order η in E, (see [7, 2, 16]).

Let $\mathscr{T}$ denote the subclass of $\mathscr{S}$ consisting of functions given by

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n, \qquad a_n\geqslant 0, \tag{2}$$

with negative coefficients. Silverman [20] introduced and investigated the following subclasses of the function class $\mathscr{T}$.

$$\mathscr{T}^*(\eta):=\mathscr{S}^*(\eta)\cap\mathscr{T} \text{ and } K(\eta):=\mathscr{C}(\eta)\cap\mathscr{T}, \quad (0\leqslant\eta<1).$$

For two functions f, $g\in\mathscr{A}$, the convolution or Hadamard product is given by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n,$$

where f is given by (1) and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$.

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A function f is said to be subordinate to a function g, written as $f \prec g$ if there exists a Schwarz function w analytic in E and satisfies Schwarz lemma (i.e, $w(z) = 0$ and $|w(z)| < 1$) such that

$$f(z) = g(w(z)), \quad z \in E.$$

In particular if g is univalent in E, then $f(0) = g(0)$ and $f(E) \subset g(E)$.

Uniformly convex and starlike functions were first introduced by Goodman [5, 6], and were studied subsequently by Rønning [17], Ma and Minda [13] and Kanas and Sugawa [11].

Kanas and Wisniowska [8, 9] studied the class of k-uniformly convex functions, denoted by $k-UCV$ and the corresponding class of $k-ST$ related by the Alexander type relation. For $k \geqslant 0$ define the conic domain Ω_k as follows, see [10].

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}.$$

For fixed k, Ω_k represents the conic region bounded, successively, by the imaginary axis $(k = 0)$, the right branch of hyperbola $(0 < k < 1)$, a parabola $(k = 1)$ and an ellipse $(k > 1)$, see also Noor [14, 15].

Noor et-al. [14] defined the generalized conic domain $\Omega_{k,\rho}$, as,

$$\Omega_{k,\eta} = (1-\eta)\Omega_k + \eta, \qquad (0 \leqslant \eta < 1).$$

The functions which play the role of extremal functions for these conic regions are given as:

$$q_{k,\eta}(z) = \begin{cases} \frac{1+(1-2\eta)z}{1-z}, & k = 0, \\ 1 + \frac{2(1-\eta)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ 1 + \frac{2(1-\eta)}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right\}, & 0 < k < 1, \\ 1 + \frac{1-\eta}{k^2-1}\sin\left(\frac{\pi}{2K(t)}\int_0^{\frac{\mu(z)}{t}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(xt)^2}}dx\right) + \frac{1-\eta}{k^2-1}, & k > 1. \end{cases} \tag{3}$$

A function h such that $h(0) = 1$ is said to be in the class $P(q_{k,\rho})$ if it is subordinate to $q_{k,\eta}(z)$ with $z \in E$, that is, $p(E) \subset q_{k,\eta}(E) = \Omega_{k,\eta}$.

The classes $k-ST(\eta)$ and $k-UCV(\eta)$, the class of k-uniformly starlike functions and k-uniformly convex functions of order η respectively are defined as follows.

$$k-ST(\eta) = \left\{ f \in \mathscr{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \eta,\ k \geqslant 0 \right\},$$

and

$$k-UCV(\eta) = \left\{ f \in \mathscr{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| + \eta,\ k \geqslant 0 \right\},$$

or equivalently

$$k-ST(\eta)=\left\{f\in\mathscr{A}:\frac{zf'(z)}{f(z)}\prec q_{k,\eta}(z)\right\},\tag{5}$$

and

$$k-UCV(\eta)=\left\{f\in\mathscr{A}:1+\frac{zf''(z)}{f'(z)}\prec q_{k,\eta}(z)\right\},$$

where $q_{k,\eta}(z)$ is given by (3). For detail study about these classes, see [18].

By virtue of (3), (5) and the properties of the domains $\Omega_{k,\eta}$, we have

$$\operatorname{Re}(p(z))>\operatorname{Re}(q_{k,\eta}(z))>\frac{k+\eta}{k+1}.\tag{6}$$

For $\alpha_j\in\mathbb{C}$ $(j=1,2,3,\ldots,l)$ and $\beta_j\in\mathbb{C}$ $(\beta_j\neq 0,-1,-2,\ldots;\ j=1,2,3,\ldots,m)$, the generalized hypergeometric functions ${}_lF_m(z)$ is defined by

$${}_lF_m={}_lF_m(\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m;z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_l)_n}{(\beta_1)_n\ldots(\beta_m)_n}\frac{z^n}{n!},$$

$$(l\leqslant m+1:l,\ m=0,1,2,\ldots,\ z\in E),$$

and $(\xi)_n$ is the Pochhammer symbol defined by

$$(\xi)_n=\begin{cases}1 & n=0,\\ \xi(\xi+1)(\xi+2)\ldots(\xi+n-1) & n\in N.\end{cases}$$

Corresponding to the function

$$h(\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m)=z\,{}_lF_m(\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m;z),$$

the Dziok-Srivastava operator [3] $H_m^l(\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m)$ is defined by

$$\begin{aligned}H_m^l(\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m)f(z)&=h(\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m)*f(z)\\&=z+\sum_{n=2}^{\infty}\Gamma_n a_n z^n,\end{aligned}\tag{7}$$

where

$$\Gamma_n=\frac{(\alpha_1)_{n-1}\ldots(\alpha_l)_{n-1}}{(\beta_1)_{n-1\ldots}(\beta_m)_{n-1}}\frac{1}{(n-1)!}.\tag{8}$$

It is well known [3] that

$$\begin{aligned}&\alpha_1 H_m^l(\alpha_1+1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m)f(z)\\&=z\left(H_m^l(\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m)f(z)\right)'\\&\quad+(\alpha_1-1)H_m^l(\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m)f(z).\end{aligned}\tag{9}$$

To make the notation simple, we write

$$H_m^l[\alpha_1]f(z) = H_m^l(\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m)f(z).$$

For special values of $\alpha's$ and $\beta's$, this operator contains many operators which were extensively studied by several authors, see [1, 3, 12].

Using the operator defined in (7), we define a new class $k-\mathscr{U}\mathscr{S}\mathscr{T}(\alpha_1,\lambda,\eta,\mu)$ as;

DEFINITION 1. A function f given in (2) is said to be in the class $k-\mathscr{U}\mathscr{S}\mathscr{T}(\alpha_1,\lambda,\eta,\mu)$ if it satisfies the following inequality:

$$\operatorname{Re}\left(\frac{z\left(J_m^l[\alpha_1,\lambda,\mu]f\right)'}{J_m^l[\alpha_1,\lambda,\mu]f}\right) > k\left|\frac{z\left(J_m^l[\alpha_1,\lambda,\mu]f\right)'}{J_m^l[\alpha_1,\lambda,\mu]f}-1\right|+\eta, \tag{10}$$

where $(0\leqslant\mu\leqslant\lambda\leqslant 1;\ 0\leqslant\eta<1;\ k\geqslant 0)$ and

$$J_m^l[\alpha_1,\lambda,\mu]f = \lambda\mu z^2[H_m^l(\alpha_1)f]'' + (\lambda-\mu)z[H_m^l(\alpha_1)f]' + (1-\lambda+\mu)H_m^l f. \tag{11}$$

An inequality (10) can be written equivalently as

$$\frac{z\left(J_m^l[\alpha_1,\lambda,\mu]f\right)'}{J_m^l[\alpha_1,\lambda,\mu]f} \prec q_{k,\eta}(z), \tag{12}$$

where $q_{k,\eta}(z)$ is given by (3).

For convenience, we write $J_m^l[\alpha_1]f = J_m^l[\alpha_1,\lambda,\mu]f$.

LEMMA 1. [19] *Let q be convex in E and* $\operatorname{Re}(\mu_1 q(z)+\mu_2)>0$, *where* $\mu_1,\mu_2\in\mathbb{C}\setminus\{0\}$, $z\in E$. *If $h(z)$ is analytic in E with $q(0)=h(0)$ and*

$$h(z)+\frac{zh'(z)}{\mu_1 h(z)+\mu_2} \prec q(z), \quad z\in E,$$

then $h(z)\prec q(z)$.

2. Main results

In this section, we will prove our main results.

THEOREM 1. *Let* $R(\alpha_1) > \frac{1-\eta}{k+1}$, *and* $f\in\mathscr{A}$. *Then*

$$k-\mathscr{U}\mathscr{S}\mathscr{T}[\alpha_1+1,\lambda,\eta,\mu] \subset k-\mathscr{U}\mathscr{S}\mathscr{T}[\alpha_1,\lambda,\eta,\mu],$$

where $(0\leqslant\mu\leqslant\lambda\leqslant 1;\ 0\leqslant\eta<1;\ k\geqslant 0)$.

Proof. Setting

$$\frac{z\left(J_m^l[\alpha_1]f\right)'}{J_m^l[\alpha_1]f} = H(z),$$

using the identity (9), we have

$$\frac{\alpha_1 J_m^l[\alpha_1+1]f}{J_m^l[\alpha_1]f} = \frac{z(J_m^l[\alpha_1 1]f)'}{J_m^l[\alpha_1 1]f} + (\alpha_1 - 1) = H(z) + (\alpha_1 - 1). \tag{13}$$

Differentiating (13), yields

$$\frac{z(J_m^l[\alpha_1+1]f)'}{J_m^l[\alpha_1+1]f} = H(z) + \frac{zH'(z)}{H(z)+(\alpha_1-1)}. \tag{14}$$

From this and argument given by (12), we may write

$$H(z) + \frac{zH'(z)}{H(z)+(\alpha_1-1)} \prec q_{k,\eta}(z).$$

Therefore the theorem follows by Lemma 1 and the condition (6), because $q_{k,\eta}$ is univalent and convex in E and $\mathrm{Re}\,(q_{k,\eta}) > \frac{1-\eta}{k+1}$. □

THEOREM 2. *A function $f \in T$ given by (2) belongs to the class $k-\mathscr{U}\mathscr{S}\mathscr{T}[\alpha_1,\lambda,\eta,\mu]$ if and only if*

$$\sum_{n=2}^{\infty} \{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}|\Gamma_n|\, a_n \leqslant 1-\eta, \tag{15}$$

where $(0 \leqslant \mu \leqslant \lambda \leqslant 1;\ 0 \leqslant \eta < 1;\ k \geqslant 0)$.

Proof. Since $f \in \mathscr{U}\mathscr{S}\mathscr{T}[\alpha_1,\lambda,\eta,\mu]$ if and only if it satisfies the condition (10). Since it is easily verified that

$$\mathrm{Re}\,(w) > k|w-1| + \eta \Leftrightarrow \mathrm{Re}\,(w(1+ke^{i\theta}) - ke^{i\theta}) > \eta$$
$$(-\pi \leqslant \theta < \pi;\ 0 \leqslant \eta < 1;\ k \geqslant 0).$$

The inequality (10) may be written in the following form:

$$\mathrm{Re}\left(\frac{z\left(J_m^l[\alpha_1]f\right)'}{J_m^l[\alpha_1]f}(1+ke^{i\theta}) - ke^{i\theta}\right) > \eta, \tag{16}$$

or, equivalently,

$$\mathrm{Re}\left(\frac{z\left(J_m^l[\alpha_1]f\right)'(1+ke^{i\theta}) - \left(J_m^l[\alpha_1]f\right)ke^{i\theta}}{J_m^l[\alpha_1]f}\right) > \eta. \tag{17}$$

Now, by setting

$$G(z) = z\left(J_m^l[\alpha_1]f\right)'(1+ke^{i\theta}) - \left(J_m^l[\alpha_1]f\right)ke^{i\theta}. \tag{18}$$

The inequality (17) becomes

$$\left|G(z)+(1-\eta)J_m^l[\alpha_1]f\right| > \left|G(z)-(1+\eta)J_m^l[\alpha_1]f\right| \quad (0 \leqslant \eta < 1), \tag{19}$$

where $J_m^l[\alpha_1]f$ and $G(z)$ are defined by (11) and (18), respectively. We thus observe that

$$\begin{aligned}
&\left|G(z)+(1-\eta)J_m^l[\alpha_1]f\right| \\
&\geqslant |(2-\eta)z| - \left|\sum_{n=2}^{\infty}(n+1-\eta)\{(n-1)(n\lambda\mu+\lambda-\mu)\ +1\}\Gamma_n a_n z^n\right| \\
&\quad + \left|-ke^{i\theta}\sum_{n=2}^{\infty}(n-1)\{(n\lambda\mu+\lambda-\mu)(n-1)+1\}\Gamma_n a_n z^n\right| \\
&\geqslant (2-\eta)\,|z| - \sum_{n=2}^{\infty}(n+1-\eta)\{(n-1)(n\lambda\mu+\lambda-\mu)\ +1\}a_n|\Gamma_n|\,|z^n| \\
&\quad -k\sum_{n=2}^{\infty}(n-1)\{(n\lambda\mu+\lambda-\mu)(n-1)+1\}a_n|\Gamma_n|\,|z^n| \\
&\geqslant (2-\eta)\,|z| - \sum_{n=2}^{\infty}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}\{n(k+1)-(k+\eta)+1\} \\
&\quad \times a_n|\Gamma_n|\,|z^n|.
\end{aligned}$$

Similarly we get,

$$\begin{aligned}
&\left|G(z)-(1+\eta)J_m^l[\alpha_1]f\right| \\
&\leqslant \eta\,|z| + \sum_{n=2}^{\infty}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}\{n(k+1)-(k+\eta)-1\}a_n|\Gamma_n|\,|z^n|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left|G(z)+(1-\eta)J_m^l[\alpha_1]f\right| - \left|G(z)-(1+\eta)J_m^l[\alpha_1]f\right| \\
&\geqslant 2(1-\eta)\,|z| - 2\sum_{n=2}^{\infty}\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}a_n|\Gamma_n|\,|z^n| \\
&\geqslant 0.
\end{aligned}$$

Which implies the inequality (15) asserted by Theorem 2.

Conversely, setting $0 \leqslant |z| = r < 1$, and choosing the values of z on the positive real axis, the inequality (16) reduces to the following form:

$$\mathrm{Re}\left[(1-\eta) - \sum_{n=2}^{\infty}(n-\eta)\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}\Gamma_n a_n r^{n-1} \right.$$
$$\left. -ke^{i\theta}\sum_{n=2}^{\infty}(n-1)\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}\Gamma_n a_n r^{n-1}\right] \geqslant 0.$$

Which is in view of the elementary identity $\mathrm{Re}\left(-e^{-i\theta}\right) \geqslant -\left|e^{i\theta}\right| \geqslant -1$, becomes

$$\mathrm{Re}\left[(1-\eta) - \sum_{n=2}^{\infty}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}\{n-\eta+kn-k\}\Gamma_n a_n r^{n-1}\right] \geqslant 0. \quad (20)$$

Finally letting $r \to 1$ in (20), we get the desired result. □

When $l = \alpha_1 = \lambda = 1$, $m = k = \mu = 0$, then we have the following known result proved by Silverman et-al. [20].

COROLLARY 1. *A function $f \in T$ given by (2) belongs to the class $0 - \mathscr{U}\mathscr{S}\mathscr{T}[1,1,\eta,0] = C(\eta)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\eta)a_n \leqslant 1-\eta.$$

THEOREM 3. *If $f \in k - \mathscr{U}\mathscr{S}\mathscr{T}(\alpha_1,\lambda,\eta,\mu)$ is given by (2), then*

$$a_n \leqslant \frac{(1-\eta)}{\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}}\frac{1}{|\Gamma_n|}; \quad n \geqslant 2, \qquad (21)$$

where $(0 \leqslant \mu \leqslant \lambda \leqslant 1;\ 0 \leqslant \eta < 1;\ k \geqslant 0)$.

Proof. Since $f \in k - \mathscr{U}\mathscr{S}\mathscr{T}(\alpha_1,\lambda,\eta,\mu)$, so by Theorem 2, we have

$$\sum_{n=2}^{\infty}\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}a_n|\Gamma_n| \leqslant (1-\eta).$$

This implies that

$$a_n \leqslant \frac{(1-\eta)}{\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}}\frac{1}{|\Gamma_n|}; \qquad (n \geqslant 2).$$

This completes the proof. □

When $l = \alpha_1 = \lambda = 1$, $m = k = \mu = 0$, then we have the following known result proved by Silverman et-al. [20].

COROLLARY 2. *A function $f \in T$ given by (2) belongs to the class $0-\mathscr{U}\mathscr{S}\mathscr{T}[1,1,\eta,0]=C(\eta)$ if and only if*

$$a_n \leqslant \frac{1-\eta}{n(n-\eta)}; \quad n \geqslant 2.$$

THEOREM 4. *If $f \in k-\mathscr{U}\mathscr{S}\mathscr{T}(\alpha_1,\lambda,\eta,\mu)$, then,*

$$r-\frac{(1-\eta)}{(2+k-\eta)(2\lambda\mu+\lambda-\mu)}\frac{1}{|\Gamma_n|}r^2 \leqslant |f(z)| \leqslant r+\frac{(1-\eta)}{(2+k-\eta)(2\lambda\mu+\lambda-\mu)}\frac{1}{|\Gamma_n|}r^2.$$
$$(|z|=r<1)$$

Proof. Since $f \in k-\mathscr{U}\mathscr{S}\mathscr{T}(\alpha_1,\lambda,\eta,\mu)$, so by Theorem 2, we get

$$\begin{aligned}
&(2+k-\eta)(2\lambda\mu+\lambda-\mu+1)\sum_{n=2}^{\infty} a_n|\Gamma_n| \\
&= \sum_{n=2}^{\infty}(2+k-\eta)(2\lambda\mu+\lambda-\mu+1)a_n|\Gamma_n| \\
&\leqslant \sum_{n=2}^{\infty}\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}a_n|\Gamma_n| \\
&\leqslant (1-\eta),
\end{aligned}$$

which gives the following inequality

$$a_n \leqslant \frac{1}{|\Gamma_n|}\frac{(1-\eta)}{(2+k-\eta)(2\lambda\mu+\lambda-\mu+1)}.$$

Now, from (2) and (21), we have

$$\begin{aligned}
|f(z)| &= \left|z-\sum_{n=2}^{\infty}a_n z^n\right| \leqslant |z|+|z|^2\sum_{n=2}^{\infty}a_n \\
&\leqslant r+r^2\frac{(1-\eta)}{(2+k-\eta)(2\lambda\mu+\lambda-\mu+1)}\frac{1}{|\Gamma_n|}. \qquad (22)
\end{aligned}$$

Similarly we can find

$$|f(z)| \geqslant r-r^2\frac{(1-\eta)}{(2+k-\eta)(2\lambda\mu+\lambda-\mu+1)}\frac{1}{|\Gamma_n|}. \qquad (23)$$

From (22), (23), we get the required result. □

THEOREM 5. *If $f \in k-\mathscr{U}\mathscr{S}\mathscr{T}(\alpha_1,\lambda,\eta,\mu)$ $(|z|=r<1)$, then*

$$1-\frac{2(1-\eta)}{(2+k-\eta)(2\lambda\mu+\lambda-\mu)}\frac{1}{|\Gamma_n|}r \leqslant \left|f'(z)\right| \leqslant 1+\frac{2(1-\eta)}{(2+k-\eta)(2\lambda\mu+\lambda-\mu)}\frac{1}{|\Gamma_n|}r. \qquad (24)$$

Proof. It is easily verified from the inequality (2) that

$$\left|f'(z)\right| \leqslant 1+\sum_{n=2}^{\infty} na_n \left|z\right|^{n-1} \leqslant 1+r\sum_{n=2}^{\infty} na_n, \tag{25}$$

and

$$\left|f'(z)\right| \geqslant 1-\sum_{n=2}^{\infty} na_n \left|z\right|^{n-1} \geqslant 1-r\sum_{n=2}^{\infty} na_n. \tag{26}$$

The assertion (24) would now from (25) and (26) by means of rather simple consequence of (21) given by

$$\sum_{n=2}^{\infty} na_n \leqslant \frac{(1-\eta)}{(2+k-\eta)(2\lambda\mu+\lambda-\mu+1)}\frac{1}{|\Gamma_n|},$$

which completes the proof. □

THEOREM 6. *The class $k-\mathcal{U}\mathcal{S}\mathcal{T}(\alpha_1,\lambda,\eta,\mu)$ is a convex set.*

Proof. Suppose that each of function f_j $(j=1,2)$ given by

$$f_j(z) = z-\sum_{n=2}^{\infty} a_{n,j}\left|\Gamma_n\right| z^n, \;\; a_{n,j}\geqslant 0; \;\; j=1,2,$$

is in the class $k-\mathcal{U}\mathcal{S}\mathcal{T}(\alpha_1,\lambda,\eta,\mu)$.

It is sufficient to show that the function g defined by

$$g(z) = \varepsilon f_1(z)+(1-\varepsilon)f_2(z), \quad (0\leqslant\varepsilon<1),$$

is also in the class $k-\mathcal{U}\mathcal{S}\mathcal{T}(\alpha_1,\lambda,\eta,\mu)$.

Since

$$g(z) = z-\sum_{n=2}^{\infty}(\varepsilon a_{n,1}+(1-\varepsilon)a_{n,2})\Gamma_n z^n.$$

With the aid of Theorem 2, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}\left|\Gamma_n\right|(\varepsilon a_{n,1}+(1-\varepsilon)a_{n,2})\\
\leqslant\; &\varepsilon\sum_{n=2}^{\infty}\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}a_{n,1}\left|\Gamma_n\right|\\
&+(1-\varepsilon)\sum_{n=2}^{\infty}\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}a_{n,2}\left|\Gamma_n\right|.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}\left|\Gamma_n\right|(\varepsilon a_{n,1}+(1-\varepsilon)a_{n,2})\\
\leqslant\; &\varepsilon(1-\eta)+(1-\varepsilon)(1-\eta)\leqslant\varepsilon-\varepsilon\eta+1-\eta-\varepsilon+\varepsilon\eta\leqslant 1-\eta,
\end{aligned}$$

which implies that $g\in k-\mathcal{U}\mathcal{S}\mathcal{T}(\alpha_1,\lambda,\eta,\mu)$. Hence $k-\mathcal{U}\mathcal{S}\mathcal{T}(\alpha_1,\lambda,\eta,\mu)$ is indeed a convex set. □

THEOREM 7. *Let the function f defined by (2) be in the class $k-\mathcal{USF}(\alpha_1,\lambda,\eta,\mu)$. Then f is close-to-convex of order ρ $(0 \leqslant \rho < 1)$, $n \geqslant 2$ in $|z| < r_1(\alpha_1,\lambda,\eta,\rho,\mu)$, where*

$$r_1(\alpha_1,\lambda,\eta,\rho,\beta,\mu) = \inf\left(\frac{(1-\rho)\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}|\Gamma_n|}{n(1-\eta)}\right)^{\frac{1}{n-1}}.$$

Proof. It is sufficient to show that

$$\left|f'(z)-1\right| \leqslant 1-\rho, \ \ (0 \leqslant \rho < 1;\ |z| < r_1(\alpha_1,\lambda,\eta,\rho,\mu)).$$

Since

$$\left|f'(z)-1\right| = \left|-\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \leqslant \sum_{n=2}^{\infty} n a_n \left|z^{n-1}\right|.$$

We have

$$\left|f'(z)-1\right| \leqslant 1-\rho \ \ (0 \leqslant \rho < 1).$$

If

$$\sum_{n=2}^{\infty}\left(\frac{1}{1-\rho}\right) a_n \left|z^{n-1}\right| \leqslant 1. \tag{27}$$

Hence by Theorem 2, inequality (27) will hold true if

$$\left(\frac{n}{1-\rho}\right)|z|^{n-1} \leqslant \frac{\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}|\Gamma_n|}{(1-\eta)}, \ \ (n \geqslant 2),$$

that is, if

$$|z| \leqslant \left(\frac{(1-\rho)\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}|\Gamma_n|}{n(1-\eta)}\right). \tag{28}$$

The assertion of Theorem 7 would now follow easily from (28). □

THEOREM 8. *Let the function f defined by (2) be in the class $k-\mathcal{USF}(\alpha_1,\lambda,\eta,\mu)$. Then f is starlike of order ρ $(0 \leqslant \rho < 1)$, $n \geqslant 2$ in $|z| < r_2(\alpha_1,\lambda,\eta,\rho,\mu)$, where*

$$r_2(\alpha_1,\lambda,\eta,\rho,\mu) = \inf\left(\frac{(1-\rho)\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}|\Gamma_n|}{(n-\rho)(1-\eta)}\right)^{\frac{1}{n-1}}.$$

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leqslant 1-\rho \ \ \ (0 \leqslant \rho < 1;\ |z| < r_2(\alpha_1,\lambda,\eta,\rho,\mu)).$$

Since

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leqslant \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n z^{n-1}},$$

we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leqslant 1-\rho \quad (0 \leqslant \rho < 1).$$

If

$$\sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)a_n|z|^{n-1} \leqslant 1. \tag{29}$$

Hence by Theorem 2, inequality (29) will hold true if

$$\left(\frac{n-\rho}{1-\rho}\right)|z|^{n-1} \leqslant \frac{\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}|\Gamma_n|}{(1-\eta)},$$

that is, if

$$|z|^{n-1} \leqslant \frac{(1-\rho)\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}|\Gamma_n|}{(n-\rho)(1-\eta)}. \tag{30}$$

The assertion of Theorem 8 would now follow easily from (30). □

THEOREM 9. *Let the function f defined by (2) be in the class $k-\mathcal{USJ}(\alpha_1,\lambda,\eta,\mu)$. Then f is convex of order ρ $(0 \leqslant \rho < 1)$, $n \geqslant 2$ in $|z| < r_3(\alpha_1,\lambda,\eta,\rho,\mu)$, where*

$$r_3(\alpha_1,\lambda,\eta,\rho,\mu) = \inf\left(\frac{(1-\rho)\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}|\Gamma_n|}{n(n-\rho)(1-\eta)}\right)^{\frac{1}{n-1}}.$$

Proof. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqslant 1-\rho \quad (0 \leqslant \rho < 1;\ \ |z| < r_3(\alpha_1,\lambda,\eta,\rho,\mu)).$$

Since

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqslant \frac{\sum_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}na_n|z|^{n-1}},$$

we have

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqslant 1-\rho \qquad (0 \leqslant \rho < 1).$$

If

$$\sum_{n=2}^{\infty}\left(\frac{n(n-\rho)}{1-\rho}\right)a_n|z|^{n-1}\leqslant 1. \tag{31}$$

Hence by Theorem 2, inequality (31) will hold true if

$$\left(\frac{n(n-\rho)}{1-\rho}\right)|z|^{n-1}\leqslant\frac{\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}|\Gamma_n|}{(1-\eta)},$$

that is, if

$$|z|^{n-1}\leqslant\frac{(1-\rho)\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}|\Gamma_n|}{n(n-\rho)(1-\eta)}. \tag{32}$$

The assertion of Theorem 9 would now follow easily from (32). □

THEOREM 10. *Let* $f\in k-\mathscr{U}\mathscr{S}\mathscr{T}(\alpha_1,\lambda,\eta,\mu)$, *and*

$$g(z)=z+\sum_{n=2}^{\infty}g_nz^n,\ \ (0\leqslant b_n\leqslant 1).$$

Then $f*g\in k-\mathscr{U}\mathscr{S}\mathscr{T}(\alpha_1,\lambda,\eta,\beta,\mu)$.

Proof. Since $0\leqslant b_n\leqslant 1$, $n\geqslant 2$,

$$\sum_{n=2}^{\infty}\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}a_nb_n|\Gamma_n|$$
$$\leqslant\sum_{n=2}^{\infty}\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}a_n|\Gamma_n|.$$

Thus by Theorem 2, we have

$$\sum_{n=2}^{\infty}\{n(k+1)-(k+\eta)\}\{(n-1)(n\lambda\mu+\lambda-\mu)+1\}a_ng_n|\Gamma_n|\leqslant(1-\eta).$$

This completes the proof of Theorem 10. □

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