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EVALUATION OF A CUBIC EULER SUM

RAMYA DUTTA

Abstract. In this paper we calculate the cubic series

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3$$

and two related Euler Sums of weight 6 by a technique involving only the manipulation of series. We also provide a second approach of the computation involving special logarithmic integrals.

Introduction

The quadratic series of Enrico Au-Yeung $\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2$ appeared as Problem 4305 in The American Mathematical Monthly [1948, pp. 431] proposed by H. F. Sandham and a solution due to Martin Kneser involving manipulation of series and the use of logarithmic integral appeared in AMM [1950, pp. 267]. A recent paper [6] gave a new proof based on the calculation of a quadratic logarithmic integral combined with Abel's summation formula.

In this paper we provide two approaches of evaluating *the cubic Euler sum* $\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3$. The first is by a technique involving only the manipulation of series and the second is similar to the technique involving the logarithmic integral in [6].

The closed form of linear Euler sums:

$$E(q,1) := \sum_{n=1}^{\infty}\frac{H_n}{n^q} = \left(1+\frac{q}{2}\right)\zeta(q+1) - \frac{1}{2}\sum_{j=1}^{q-2}\zeta(j+1)\zeta(q-j), \quad \text{for } q \geqslant 2$$

is well known. We consider non-linear Euler sums involving the generalized Harmonic numbers $H_n^{(m)} = \sum_{k=1}^{n}\frac{1}{k^m}$ of the form $\sum_{n=1}^{\infty}\frac{H_n^{(m_1)}\cdots H_n^{(m_k)}}{n^m}$, where $m_1,\ldots,m_k$ and m are positive integers. $W = m_1+\cdots+m_k+m$ is known as the weight of the non-linear Euler sum. In this paper we restrict our attention to non-linear Euler sums of weight 6 that are involved in the evaluation of *the cubic Euler sum*.

We mention that our results are not new and they exist in the mathematical literature. *The first Euler sum* in Theorem 2.2. can be evaluated by results proved in [3] and *the second Euler sum* Theorem 2.3. can be evaluated by the methods in [1] and [2].

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1. The two main lemmas

LEMMA 1.1. *The following equality holds:*

$$\sum_{k=1}^{\infty} \frac{H_k}{k(n+k)} = \frac{1}{n}\left(\frac{1}{2}H_n^2 + \frac{1}{2}H_n^{(2)} + \zeta(2) - \frac{H_n}{n}\right).$$

Proof. We have

$$\sum_{k=1}^{\infty} \frac{H_k}{k(n+k)} = \sum_{k=1}^{\infty}\sum_{j=1}^{k} \frac{1}{jk(n+k)} = \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} \frac{1}{jk(n+k)} \tag{1.1}$$

$$= \sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty} \frac{1}{jk(n+k)} + \sum_{j=1}^{\infty} \frac{1}{j^2(n+j)} \tag{1.2}$$

$$= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{j(k+j)(n+k+j)} + \sum_{j=1}^{\infty} \frac{1}{j^2(n+j)} \tag{1.3}$$

$$= \frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{jk(n+k+j)} + \sum_{j=1}^{\infty} \frac{1}{j^2(n+j)} \tag{1.4}$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} \frac{H_{n+k}}{k(n+k)} + \sum_{j=1}^{\infty} \frac{1}{j^2(n+j)} \tag{1.5}$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} \frac{H_{n+k}}{k(n+k)} + \frac{1}{n}\left(\zeta(2) - \frac{H_n}{n}\right). \tag{1.6}$$

The explanations in the previous calculations are as follows:
(1.1) Order of summation interchanged.
(1.3) The change in variable $k \mapsto k+j$ was made.
(1.4) Used the symmetry of the summation with respect to k and j,

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{j(k+j)(n+k+j)} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k(k+j)(n+k+j)}$$

$$= \frac{1}{2}\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{j(k+j)(n+k+j)} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k(k+j)(n+k+j)}\right)$$

$$= \frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{jk(n+k+j)}.$$

(1.5) We used the identity $\dfrac{H_m}{m} = \dfrac{1}{m}\sum_{j=1}^{\infty}\left(\dfrac{1}{j} - \dfrac{1}{m+j}\right) = \sum_{j=1}^{\infty} \dfrac{1}{j(m+j)}$. (H)

(1.6) $\sum_{j=1}^{\infty} \dfrac{1}{j^2(n+j)} = \sum_{j=1}^{\infty}\left(\dfrac{1}{nj^2} - \dfrac{1}{nj(n+j)}\right) = \dfrac{1}{n}\left(\zeta(2) - \dfrac{H_n}{n}\right)$.

Again, the first series in (1.6) can be written as follows

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H_{n+k}}{k(n+k)} &= \frac{1}{n}\sum_{k=1}^{\infty}\left(\frac{H_k}{k}-\frac{H_{n+k}}{n+k}\right)+\frac{1}{n}\sum_{k=1}^{\infty}\left(\frac{H_{n+k}-H_k}{k}\right)\\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{H_k}{k}+\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{j=1}^{n}\frac{1}{k+j}\right)\\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{H_k}{k}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+j}\right)\\ &= \frac{2}{n}\sum_{k=1}^{n}\frac{H_k}{k}=\frac{H_n^2+H_n^{(2)}}{n}. \end{aligned} \tag{1.7}$$

Thus, combining (1.6) and (1.7) the lemma is proved. □

LEMMA 1.2. *The following equality holds:*

$$\sum_{k=1}^{n-1}\left(\frac{1}{k(n-k)}\right)^2=\frac{2}{n^2}\left(H_n^{(2)}+\frac{2H_n}{n}-\frac{3}{n^2}\right),$$

where it is understood that the sum is nil when $n=1$.

Proof. Straightforward from partial fraction decomposition. □

2. Two Euler sum of weight 6

LEMMA 2.1. *The following equality holds:*

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}=\frac{19}{6}\zeta(6)-2\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\zeta^2(3)-\frac{1}{3}\zeta(6).$$

Proof. Divide both sides of the identity in Lemma 1.2. by n^2 and take the summation over $n\geqslant 1$ and we have,

$$\begin{aligned}2\sum_{n=1}^{\infty}\frac{1}{n^4}\left(H_n^{(2)}+\frac{2H_n}{n}-\frac{3}{n^2}\right) &= \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{1}{n^2k^2(n-k)^2}\\ &= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2m^2(k+m)^2},\end{aligned}$$

where in the later step we made the change of variable $n=m+k$ in the double summation. The double summation can be evaluated in the following manner:

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2m^2(k+m)^2}\\
&=\frac{1}{3}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{(m+k)^3-m^3-k^3}{k^3m^3(k+m)^3}\\
&=\frac{1}{3}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^3m^3}-\frac{1}{3}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^3(k+m)^3}-\frac{1}{3}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{m^3(k+m)^3}\\
&=\frac{1}{3}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^3m^3}-\frac{1}{3}\sum_{m>k\geqslant 1}\frac{1}{k^3m^3}-\frac{1}{3}\sum_{k>m\geqslant 1}^{\infty}\frac{1}{m^3k^3}\\
&=\frac{1}{3}\sum_{k=m=1}^{\infty}\frac{1}{m^3k^3}=\frac{1}{3}\sum_{k=1}^{\infty}\frac{1}{k^6}=\frac{1}{3}\zeta(6).
\end{aligned}
$$

These types of double summations are known in literature. It is a special case of the Tornheim double summations appearing in [4], [5].

Thus,

$$
\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n^4}+\frac{2H_n}{n^5}-\frac{3}{n^6}\right)=\frac{1}{6}\zeta(6).
$$

Rearranging the terms and substituting the value of $E(5,1)$ we get that

$$
\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}=\frac{19}{6}\zeta(6)-2\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\zeta^2(3)-\frac{1}{3}\zeta(6).
$$

This completes the proof. □

THEOREM 2.2. *The first Euler sum. The following equality holds:*

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{H_k^2}{k^4}&=2\zeta(2)\zeta(4)-2\zeta(3)\sum_{k=1}^{\infty}\frac{H_k}{k^2}+2\zeta(2)\sum_{k=1}^{\infty}\frac{H_k}{k^3}-4\sum_{k=1}^{\infty}\frac{H_k}{k^5}+\frac{19}{6}\zeta(6)\\
&=\frac{97}{24}\zeta(6)-2\zeta^2(3).
\end{aligned}
$$

Proof. Dividing both sides of the identity in Lemma 1.1. with n^3 and taking the summation over $n\geqslant 1$ we have,

$$
\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2+H_n^{(2)}}{n^4}+\zeta(2)\zeta(4)-\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k}{kn^3(n+k)}. \tag{1}
$$

Using the partial fraction $\dfrac{1}{n^3(n+k)} = \dfrac{1}{n^3k} - \dfrac{1}{n^2k^2} + \dfrac{1}{nk^2(n+k)}$ we have:

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k}{kn^3(n+k)} &= \sum_{k=1}^{\infty}\frac{H_k}{k}\sum_{n=1}^{\infty}\frac{1}{n^3(n+k)}\\ &= \sum_{k=1}^{\infty}\frac{H_k}{k}\left(\frac{\zeta(3)}{k} - \frac{\zeta(2)}{k^2} + \frac{H_k}{k^3}\right)\\ &= \zeta(3)\sum_{k=1}^{\infty}\frac{H_k}{k^2} - \zeta(2)\sum_{k=1}^{\infty}\frac{H_k}{k^3} + \sum_{k=1}^{\infty}\frac{H_k^2}{k^4}. \qquad (2)\end{aligned}$$

Substituting the value from (2) in (1) and rearranging,

$$\sum_{k=1}^{\infty}\frac{H_k^2}{k^4} = 2\zeta(2)\zeta(4) - 2\zeta(3)\sum_{k=1}^{\infty}\frac{H_k}{k^2} + 2\zeta(2)\sum_{k=1}^{\infty}\frac{H_k}{k^3} - 2\sum_{k=1}^{\infty}\frac{H_k}{k^5} + \sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4}$$

and using the result from Lemma 2.1. along with the values of $E(2,1), E(3,1)$ and $E(5,1)$ we get the desired closed form. $\square$

THEOREM 2.3. *The second Euler sum. The following equality holds:*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3} &= \frac{1}{6}\zeta(2)^3 - 4\zeta(2)\zeta(4) + 4\zeta(3)\sum_{n=1}^{\infty}\frac{H_n}{n^2} - 4\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^3} + 5\sum_{n=1}^{\infty}\frac{H_n}{n^5} - \frac{19}{3}\zeta(6)\\ &= -\frac{97}{12}\zeta(6) + \frac{7}{4}\zeta(4)\zeta(2) + \frac{5}{2}\zeta(3)^2 + \frac{2}{3}\zeta(2)^3.\end{aligned}$$

Proof. Multiplying both sides of Lemma 1.2. with $\dfrac{H_n}{n}$ and taking the summation over $n \geqslant 1$,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2}{n^3}\left(H_nH_n^{(2)} + \frac{2H_n^2}{n} - \frac{3H_n}{n^2}\right) &= \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{H_n}{nk^2(n-k)^2}\\ &= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_{m+k}}{k^2m^2(m+k)} \qquad (2.1)\\ &= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{k^2m^2j(m+k+j)} \qquad (2.2)\\ &= \frac{1}{3}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{m+k+j}{k^2m^2j^2(m+k+j)} \qquad (2.3)\\ &= \frac{1}{3}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{k^2m^2j^2} = \frac{1}{3}\zeta(2)^3.\end{aligned}$$

The explanations in the previous calculations are as follows:
(2.1) The change of variable $n = m + k$ was made.

(2.2) Used the identity (H).
(2.3) Used the symmetry of the summation with respect to m,k and j.

Thus, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3} = \frac{1}{6}\zeta(2)^3 - 2\sum_{n=1}^{\infty} \frac{H_n^2}{n^4} + 3\sum_{n=1}^{\infty} \frac{H_n}{n^5}$ and substituting the value of $E(5,1)$ and *the first Euler sum* from Theorem 2.2. completes the calculation. □

3. Proof of the main theorem

We are now ready to prove our main theorem.

THEOREM 3.1. *The cubic Euler sum. The following equality holds:*

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^3 = \frac{93}{16}\zeta(6) - \frac{5}{2}\zeta(3)^2.$$

Proof. Multiplying both sides of the identity in Lemma 1.1. by $\frac{H_n}{n^2}$ and taking the summation over $n \geqslant 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^3}\left(\frac{1}{2}H_n^3 + \frac{1}{2}H_n H_n^{(2)} + \zeta(2)H_n - \frac{H_n^2}{n}\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{H_k H_n}{kn^2(n+k)}.$$

Due to the symmetry with respect to n,k in the double summation in the right hand side we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{H_k H_n}{kn^2(n+k)} &= \frac{1}{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{H_k H_n}{kn^2(n+k)} + \frac{H_k H_n}{k^2 n(n+k)}\right) \\
&= \frac{1}{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{H_k H_n}{k^2 n^2}\right) = \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{H_n}{n^2}\right)^2.
\end{aligned}$$

Thus, rearranging the terms we have

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^3 = \left(\sum_{n=1}^{\infty} \frac{H_n}{n^2}\right)^2 - \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3} - 2\zeta(2)\sum_{n=1}^{\infty} \frac{H_n}{n^3} + 2\sum_{n=1}^{\infty} \frac{H_n^2}{n^4},$$

and substituting the values of *the first Euler sum* from Theorem 2.2. and *the second Euler sum* from Theorem 2.3. along with the value of $E(2,1), E(3,1)$ we get the desired closed form. □

4. A second approach

LEMMA 4.1. (A logarithmic integral) *For integers $n \geqslant 1$ the following equality holds:*

$$\int_0^1 x^{n-1} \log^3(1-x)\, dx = -\frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{n}.$$

Proof. We consider the Beta Function given by $\mathrm{B}(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\,dx=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ defined over positive reals $a,b>0$. Thus, when $a=n\geqslant 1$ is a positive integer we calculate the third order partial derivative with respect to b of $\mathrm{B}(n,b)$ at $b=1$. Using differentiation under the integration sign we have,

$$\left[\frac{\partial^3}{\partial b^3}\mathrm{B}(n,b)\right]_{b=1}=\left[\int_0^1 x^{n-1}\frac{\partial^3}{\partial b^3}(1-x)^{b-1}\,dx\right]_{b=1}=\int_0^1 x^{n-1}\log^3(1-x)\,dx.$$

On the other hand, $\mathrm{B}(n,b)=\dfrac{\Gamma(n)\Gamma(b)}{\Gamma(n+b)}=\dfrac{(n-1)!}{b(b+1)\cdots(b+n-1)}$.

Taking three successive logarithmic derivatives of $\mathrm{B}(n,b)$ with respect to b,

$$\frac{\partial}{\partial b}\mathrm{B}(n,b)=-\mathrm{B}(n,b)\Big(\sum_{j=0}^{n-1}\frac{1}{b+j}\Big),$$

$$\frac{\partial^2}{\partial b^2}\mathrm{B}(n,b)=\mathrm{B}(n,b)\Big(\sum_{j=0}^{n-1}\frac{1}{(b+j)^2}+\Big(\sum_{j=0}^{n-1}\frac{1}{b+j}\Big)^2\Big),$$

$$\frac{\partial^3}{\partial b^3}\mathrm{B}(n,b)=-\mathrm{B}(n,b)\Big(2\sum_{j=0}^{n-1}\frac{1}{(b+j)^3}+3\Big(\sum_{j=0}^{n-1}\frac{1}{b+j}\Big)\Big(\sum_{j=0}^{n-1}\frac{1}{(b+j)^2}\Big)+\Big(\sum_{j=0}^{n-1}\frac{1}{b+j}\Big)^3\Big).$$

When $b=1$ we have

$$\left[\frac{\partial^3}{\partial b^3}\mathrm{B}(n,b)\right]_{b=1}=\int_0^1 x^{n-1}\log^3(1-x)\,dx=-\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{n}. \qquad \square$$

The second proof of Theorem 3.1.

Proof. Dividing both sides of the identity in Lemma 4.1. by n^2 and taking the summation over $n\geqslant 1$,

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3+\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3}+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}&=-\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^{n-1}\log^3(1-x)\,dx\\&=-\int_0^1\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}\right)\frac{\log^3(1-x)}{x}\,dx\\&=-\int_0^1\frac{\mathrm{Li}_2(x)\log^3(1-x)}{x}\,dx,\end{aligned}\tag{3}$$

where $\mathrm{Li}_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}$ for, $x\in(-1,1)$, is the Polylogarithmic function. Interchange of integration and summation can be justified by Tonelli's theorem.

The Dilogarithm function admits the following reflection formula:

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\zeta(2)-\log x\log(1-x).\tag{4}$$

Using (4) in the integral in (3) we have

$$\begin{aligned}
I &= \int_0^1 \frac{\mathrm{Li}_2(x)\log^3(1-x)}{x}\,dx \\
&= -\int_0^1 \frac{\mathrm{Li}_2(1-x)\log^3(1-x)}{x}\,dx + \zeta(2)\int_0^1 \frac{\log^3(1-x)}{x}\,dx - \int_0^1 \frac{\log x\log^4(1-x)}{x}\,dx \\
&= -I_1 + \zeta(2)I_2 - I_3.
\end{aligned}$$

The first integral: Using $\dfrac{\mathrm{Li}_2(x)}{1-x} = \sum_{n=1}^{\infty} H_n^{(2)} x^n$ we have

$$\begin{aligned}
I_1 &= \int_0^1 \frac{\mathrm{Li}_2(1-x)\log^3(1-x)}{x}\,dx = \int_0^1 \frac{\mathrm{Li}_2(x)\log^3 x}{1-x}\,dx \\
&= \sum_{n=1}^{\infty} H_n^{(2)} \int_0^1 x^n \log^3 x\,dx = -6\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(n+1)^4} = -6\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^4} + 6\zeta(6).
\end{aligned}$$

Thus, substituting the value from Lemma 2.1. we get, $I_1 = -6\zeta(3)^2 + 8\zeta(6)$.

The second integral:

$$\begin{aligned}
I_2 &= \int_0^1 \frac{\log^3(1-x)}{x}\,dx = \int_0^1 \frac{\log^3 x}{1-x}\,dx \\
&= \sum_{n=1}^{\infty} \int_0^1 x^{n-1}\log^3 x\,dx = -6\sum_{n=1}^{\infty} \frac{1}{n^4} = -6\zeta(4).
\end{aligned}$$

The third integral: Using $\dfrac{\log(1-x)}{1-x} = -\sum_{n=1}^{\infty} H_n x^n$ we have,

$$\begin{aligned}
I_3 &= \int_0^1 \frac{\log x\log^4(1-x)}{x}\,dx = \int_0^1 \frac{\log(1-x)\log^4 x}{1-x}\,dx \\
&= -\sum_{n=1}^{\infty} H_n \int_0^1 x^n \log^4 x\,dx = -24\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^5} = -24\sum_{n=1}^{\infty} \frac{H_n}{n^5} + 24\zeta(6).
\end{aligned}$$

Substituting the value of $E(5,1)$ we have $I_3 = -18\zeta(6) + 12\zeta(3)^2$.

Thus,

$$I = 10\zeta(6) - 6\zeta(3)^2 - 6\zeta(2)\zeta(4) = -\frac{1}{2}\zeta(6) - 6\zeta(3)^2.$$

Substituting the value of *the second Euler sum* from Theorem 2.3. and $\sum_{n=1}^{\infty} \dfrac{H_n^{(3)}}{n^3} = \dfrac{1}{2}\zeta(6) + \dfrac{1}{2}\zeta(3)^2$ along with the value of the integral I in (3) we get the desired result.

The sum $\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^3} = \frac{1}{2}\zeta(6) + \frac{1}{2}\zeta(3)^2$ is a special case of the summation formula

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{n} a_k \right) = \frac{1}{2} \sum_{n=1}^{\infty} a_n^2 + \frac{1}{2} \left(\sum_{n=1}^{\infty} a_n \right)^2,$$

where we took $a_n = \frac{1}{n^3}$. □

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Ramya Dutta
Undergraduate student
Department of Mathematics
Chennai Mathematical Institute
H1, SIPCOT IT Park, Siruseri, Kelambakkam 603103, India
e-mail: dattaramya.ramya@gmail.com