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HOMOGENEOUS BETA–TYPE FUNCTIONS

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Abstract. All beta-type functions, i.e. the functions $B_f : (0,\infty)^2 \to (0,\infty)$ of the form

$$B_f(x,y) = \frac{f(x)f(y)}{f(x+y)}$$

for some $f : (0,\infty) \to (0,\infty)$, which are p-homogeneous, are determined. Applying this result, we show that a beta-type function is a homogeneous mean iff it is the harmonic one. A reformulation of a result due to Heuvers in terms of a Cauchy difference and the harmonic mean is given.

1. Introduction

For a given $f : (0,\infty) \to (0,\infty)$, the function $B_f : (0,\infty)^2 \to (0,\infty)$ defined by

$$B_f(x,y) = \frac{f(x)f(y)}{f(x+y)}, \qquad x,y > 0,$$

is called the *beta-type function,* and f is called its *generator* ([7]). The notion the beta-type function arises from the well-known relation between the Euler Beta function $B : (0,\infty)^2 \to (0,\infty)$ and the Euler Gamma function $\Gamma :\ (0,\infty) \to (0,\infty)$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad x,y > 0.$$

Given $p \in \mathbb{R}$, we examine when the beta-type function B_f is p-homogeneous, i.e. when

$$B_f(tx,ty) = t^p B_f(x,y), \qquad x,y > 0.$$

Theorem 1, the main result, says that, under some regularity assumptions of the generator f, the beta-type function is p-homogeneous if, and only if, there exist $a,b > 0$ such that $f(x) = bxa^x$ for all $x > 0$. As a corollary we obtain that a beta-type function is a homogeneous pre-mean if, and only if, there exists $a > 0$ such that $f(x) = 2xa^x$ for all $x > 0$, or, equivalently, that B_f is the harmonic mean, that is $B_f = H$, where

$$H(x,y) = \frac{2xy}{x+y}, \qquad x,y > 0.$$

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A related companion of the beta-type function is the Cauchy difference $C_g : (0,\infty)^2 \to \mathbb{R}$ defined by

$$C_g(x,y) = g(x+y) - g(x) - g(y)$$

for a function $g : (0,\infty) \to \mathbb{R}$. The relationship

$$B_f = \exp \circ \left(-C_{\log \circ f}\right)$$

allows to reformulate Theorem 1 in terms of logarithmical homogeneity of the Cauchy difference (Corollary 3).

At the end we remark that Heuvers result [4] on a characterization of logarithmic functions can be reformulated in terms of the Cauchy difference and the harmonic mean.

2. Main result

THEOREM 1. *Let a function $f : (0,\infty) \to (0,\infty)$ be continuous or Lebesgue measurable. Then the following conditions are equivalent:*

(i) the beta-type function B_f is p-homogeneous, i.e.

$$B_f(tx,ty) = t^p B_f(x,y), \qquad x,y,t > 0;$$

(ii) there exist $a,b \in (0,\infty)$ such that

$$f(x) = bxa^x, \quad x > 0$$

and

$$B_f(x,y) = b\left(\frac{xy}{x+y}\right)^p, \qquad x,y > 0.$$

Proof. Assume (i) holds. Hence, by the definition of B_f, we have

$$\frac{f(tx)f(ty)}{f(t(x+y))} = t^p \frac{f(x)f(y)}{f(x+y)}, \qquad x,y,t > 0, \tag{2.1}$$

which can be written in the form

$$\frac{f(t(x+y))}{t^p f(x+y)} = \frac{f(tx)}{t^p f(x)} \frac{f(ty)}{t^p f(y)}, \qquad x,y,t > 0. \tag{2.2}$$

For every fixed $t > 0$ define $\varphi_t : (0,\infty) \to (0,\infty)$ by

$$\varphi_t(x) := \frac{f(tx)}{t^p f(x)}, \qquad x > 0.$$

Thus, from (2.2), for arbitrary fixed $t > 0$, it holds

$$\varphi_t(x+y) = \varphi_t(x)\varphi_t(y), \qquad x,y > 0,$$

stating that φ_t is an exponential function. Hence (see, for instance, [1] p. 39), for every $t>0$, there exists a unique additive function $\alpha_t : \mathbb{R} \to \mathbb{R}$ such that

$$\varphi_t(x) = e^{\alpha_t(x)}, \qquad x>0.$$

From the definition of φ_t, we have

$$e^{\alpha_t(x)} t^p f(x) = f(tx), \qquad x>0.$$

Since the right hand side is symmetric in x and t, so is the left hand side; thus

$$e^{\alpha_x(t)} x^p f(t) = f(xt) = f(tx) = e^{\alpha_t(x)} t^p f(x), \qquad x,t>0.$$

Setting here $t=1$ gives

$$e^{\alpha_1(x)} f(x) = f(x) = e^{\alpha_x(1)} x^p f(1), \qquad x>0,$$

and as, by assumption, f is positive, it follows that

$$\alpha_1(x) = 0, \qquad x>0.$$

and, consequently,

$$f(x) = f(1) x^p e^{\alpha_x(1)}, \qquad x>0.$$

Putting, for convenience, $\lambda : (0,\infty) \to \mathbb{R}$,

$$\lambda(x) := \alpha_x(1), \qquad x>0,$$

we have

$$f(x) = f(1) x^p e^{\lambda(x)}, \qquad x>0. \tag{2.3}$$

Inserting this into (2.1), we obtain,

$$\frac{f(1)(tx)^p e^{\lambda(tx)} f(1)(ty)^p e^{\lambda(ty)}}{f(1)[t(x+y)]^p e^{\lambda(t(x+y))}} = t^p \frac{f(1) x^p e^{\lambda(x)} f(1) y^p e^{\lambda(y)}}{f(1)(x+y)^p e^{\lambda(x+y)}}, \qquad x,y,t>0,$$

that reduces to

$$e^{\lambda(tx)+\lambda(ty)-\lambda(t(x+y))} = e^{\lambda(x)+\lambda(y)-\lambda(x+y)}, \qquad x,y,t>0,$$

whence

$$\lambda(tx) + \lambda(ty) - \lambda(t(x+y)) = \lambda(x) + \lambda(y) - \lambda(x+y), \qquad x,y,t>0.$$

Writing this in the form

$$\lambda(t(x+y)) - \lambda(x+y) = [\lambda(tx) - \lambda(x)] + [\lambda(ty) - \lambda(y)], \qquad x,y,t>0,$$

we conclude that, for any $t>0$, the function $\omega = \omega_t : (0,\infty) \to \mathbb{R}$, defined by

$$\omega(x) := \lambda(tx) - \lambda(x), \qquad x>0, \tag{2.4}$$

is additive. From (2.3) and the assumed regularity of f we get that ω is continuous or Lebesgue measurable. Thus, ω, being additive and continuous or measurable, is of the form ([6], p. 129, see also [1])

$$\omega(x) = \omega(1)x, \qquad x > 0,$$

and hence, by (2.4),

$$\lambda(tx) - \lambda(x) = (\lambda(t) - \lambda(1))x, \qquad x,t > 0,$$

whence

$$\lambda(tx) = \lambda(x) + (\lambda(t) - \lambda(1))x, \qquad x,t > 0.$$

The symmetry in t and x of the left hand side implies that

$$\lambda(x) + (\lambda(t) - \lambda(1))x = \lambda(t) + (\lambda(x) - \lambda(1))t, \qquad x,t > 0,$$

whence

$$\lambda(x)(1-t) + \lambda(1)t = \lambda(t)(1-x) + \lambda(1)x, \qquad x,t > 0.$$

Subtracting $\lambda(1)$ from both sides yields

$$\lambda(x)(1-t) + \lambda(1)t - \lambda(1) = \lambda(t)(1-x) + \lambda(1)x - \lambda(1), \qquad x,t > 0,$$

whence

$$\lambda(x)(1-t) - \lambda(1)(1-t) = \lambda(t)(1-x) - \lambda(1)(1-x), \qquad x,t > 0,$$

and, consequently,

$$\frac{\lambda(x) - \lambda(1)}{1-x} = \frac{\lambda(t) - \lambda(1)}{1-t}, \qquad x,t > 0,\ x \neq 1 \neq y.$$

It follows that there exists $c \in \mathbb{R}$ such that

$$\frac{\lambda(x) - \lambda(1)}{1-x} = -c, \qquad x > 0, x \neq 1,$$

whence,

$$\lambda(x) = c(x-1) + \lambda(1), \qquad x > 0,$$

and we obtain

$$\lambda(x) = cx + d, \qquad x > 0,$$

where $d := \lambda(1) - c$. Inserting this function λ into (2.3), we obtain

$$f(x) = f(1)e^d x^p (e^c)^x, \qquad x > 0,$$

whence, setting

$$a := e^c, \qquad b := f(1)e^d,$$

we get

$$f(x) = bx^p a^x, \qquad x > 0,$$

and

$$B_f(x,y) = b\left(\frac{xy}{x+y}\right)^p, \qquad x,y > 0,$$

which proves (ii). The implication $(ii) \Longrightarrow (i)$ is obvious.

3. Applications to pre-means

DEFINITION 1. Let $I \subseteq \mathbb{R}$ be an interval and $M : I^2 \to \mathbb{R}$. The M is reflexive, if

$$M(x,x) = x, \qquad x \in I;$$

M is called a *pre-mean in* I ([8]), if it is reflexive and $M\left(I^2\right) \subseteq I$;
M is called a *mean in* I, *if*

$$\min(x,y) \leqslant M(x,y) \leqslant \max(x,y), \qquad x,y \in I.$$

REMARK 1. If $M : I^2 \to \mathbb{R}$ is reflexive, then $I \subseteq M\left(I^2\right)$; so a reflexive function is a pre-mean if, and only if, $M\left(I^2\right) = I$.

REMARK 2. Obviously, every mean is a pre-mean, but, in general, not vice versa. Indeed, the function $M : (0,\infty)^2 \to (0,\infty)$ defined by

$$M(x,y) = \frac{2x^2 + y^2}{x + 2y}$$

is a pre-mean. Since $M(2,1) = 3 \notin [2,1]$ the function is not a mean. So M is not increasing in both variables because, otherwise, it would be a mean.

REMARK 3. If $M : (0,\infty)^2 \to \mathbb{R}$ is reflexive and, for some $p \in \mathbb{R}$, p-homogenous, then $p = 1$.

COROLLARY 1 *Let* $f : (0,\infty) \to (0,\infty)$ *be a continuous function. Then the following conditions are equivalent:*
(i) the beta-type function B_f *is a homogeneous pre-mean;*
(ii) there exists $a \in (0,\infty)$ *such that*

$$f(x) = 2xa^x, \quad x > 0; \tag{3.1}$$

(iii) the beta-type function coincides with the harmonic mean, i.e.

$$B_f(x,y) = \frac{2xy}{x+y}, \qquad x,y > 0.$$

Proof. Assume (i). By Theorem 1 and remark 3, its generator f is of the form

$$f(x) = bxa^x, \quad x > 0,$$

for some $a,b \in (0,\infty)$. Since B_f is reflexive, that is $B_f(x,x) = x$ for all $x \in (0,\infty)$. Substituting here $x = 2$ and using Theorem 1 (ii), yields

$$2 = B_f(2,2) = \frac{f(2)f(2)}{f(2+2)} = \frac{b \cdot 2 \cdot 2}{2+2} = b,$$

whence we get (3.1), which proves (ii).

Assume (ii). From (3.1) and the definition of B_f we get (iii).

The implication $(iii) \Longrightarrow (i)$ is obvious.

Because every homogeneous quasi-arithmetic mean is a power mean ([2], p. 249), our result implies the following

COROLLARY 2 *A homogeneous beta-type function is a quasi-arithmetic mean if, and only if, it is the harmonic mean.*

For another result connecting harmonic mean and the Euler Gamma function see [3].

4. Cauchy differences and a corollary

Applying our main result, we obtain the following

COROLLARY 3 *Let $g:(0,\infty)\to\mathbb{R}$ be an arbitrary continuous function and let $p\in\mathbb{R}$. The following conditions are equivalent:*

(i) the Cauchy difference is $p\log t$-homogeneous, that is

$$C_g(tx,ty) = C_g(x,y) + p\log t, \qquad x,y,t>0; \tag{4.1}$$

(ii) there exist $c,d\in\mathbb{R}$ such that

$$g(x) = cx + d - p\log t, \qquad x>0$$

and

$$C_g(x,y) = \log\left(\frac{xy}{x+y}\right)^p - d, \qquad x,y>0.$$

Proof. Setting $f := \exp\circ g$, we observe that condition (i) is equivalent to

$$B_f(tx,ty) = t^{-p}B_f(x,y), \qquad x,y,t>0,$$

since, using the definition of beta-type function, we have, for all $x,y>0$,

$$e^{g(tx)+g(ty)-g(t(x+y))} = t^{-p}e^{g(x)+g(y)-g(x+y)}.$$

Taking the logarithm of both sides, we indeed obtain

$$-C_g(tx,ty) = \log t^{-p} - C_g(x,y), \qquad x,y>0,$$

and thus g satisfies (4.1).

By Theorem 1, there exist $a,b>0$

$$f(x)=bx^{-p}a^{x}, \quad x>0.$$

Thus, by the definition of f, we get, for all $x>0$,

$$g(x)=\log b+p\log x+x\log a;$$

whence, putting $c:=\log a$ and $d:=\log b$, we obtain,

$$g(x)=cx+d+p\log x, \quad x>0,$$

and consequently, for all $x,y>0$,

$$\begin{aligned} C_g(x,y) &= g(x+y)-g(x)-g(y) \\ &= \log\left(\frac{xy}{x+y}\right)^p - d, \end{aligned}$$

which proves the implication $(i) \Longrightarrow (ii)$.

The second implication is easy to verify.

In connection with Cauchy differences and harmonic mean, let us note that Heuvers result [4] (see also Kannappan [5], p. 31) can be reformulated as

REMARK 4. The Cauchy difference of a function $f:(0,\infty)\to\mathbb{R}$ satisfies the functional equation

$$C_f(x,y)=f\left(\frac{2}{H(x,y)}\right), \quad x,y>0 \tag{4.2}$$

if, and only if, f is a logarithmic function, i.e.

$$f(xy)=f(x)+f(y), \quad x,y>0.$$

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