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ON THE LOGARITHMIC PROXIMATE ORDER OF ANALYTIC FUNCTIONS OF SLOW GROWTH REPRESENTED BY LAPLACE–STIELTJES' TRANSFORMATIONS

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Abstract. In the present paper the characterizations of generalized type and generalized lower type of analytic functions represented by Laplace-Stieltjes transformation have been obtained. For this we introduce the concept of logarithmic proximate order.

1. Introduction

Consider Laplace-Stieltjes transformation

$$G(s)=\int_0^{\infty}\exp(-sx)d\alpha(x), \quad (s=\sigma+it), \tag{1.1}$$

where $\alpha(x)$ is a function of bounded variation on any finite interval $[0,X]$, $(0<X<+\infty)$, σ and t, are real variables. Let $\{\lambda_n\}$ be a sequence of real numbers satisfying the following conditions:

$$0=\lambda_1<\lambda_2<\lambda_3<\ldots<\lambda_n\uparrow+\infty, \tag{1.2}$$

$$\limsup_{n\to\infty}(\lambda_{n+1}-\lambda_n)<+\infty, \quad \limsup_{n\to\infty}\frac{n}{\lambda_n}=D<+\infty. \tag{1.3}$$

We put

$$K_n^*=\sup_{\lambda_n<x\leqslant\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^{x}e^{-ity}d\alpha(y)\right|.$$

Yu Jiarong [13] proved the following formula for the abscissa of uniform convergence of the integral in (1.1):

THEOREM A. *Suppose that the sequence* $\{\lambda_n\}$ *satisfies the conditions*

$$\limsup_{n\to\infty}(\lambda_{n+1}-\lambda_n)<+\infty \quad \text{and} \quad \limsup_{n\to\infty}\frac{\ln n}{\lambda_n}<+\infty,$$

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and σ_μ^F *denotes the abscissa of uniform convergence of Laplace-Stieltjes' transformation* $G(s)$*, given by* (1.1)*. Then*

$$\limsup_{n\to\infty}\frac{\ln K_n^*}{\lambda_n}\leqslant\sigma_\mu^F\leqslant\limsup_{n\to\infty}\frac{\ln K_n^*}{\lambda_n}+\lim_{n\to\infty}\sup\frac{\ln n}{\lambda_n}. \tag{1.4}$$

The result in (1.4) is known as Valiron-Knopp-Bohr formula. Suppose that

$$\limsup_{n\to\infty}\frac{\ln K_n^*}{\lambda_n}=0. \tag{1.5}$$

Then by (1.3), (1.4) and (1.5), it follows that $\sigma_\mu^F=0$ and $G(s)$ is analytic in the right half half plane $\mathrm{Re}s>0$. Various workers have studied the growth properties of these analytic L-S transformations (see e.g. [4], [5], [7], [9], [10], [11], [12]). Kong and Yang [6] considered the growth of entire functions represented by L-S transforms. In 2012, Luo and Kong [7] defined the Laplace-Stieltjes' transformations by taking positive exponent instead of negative ones in the integral term of (1.1). Thus they defined Laplace-Stieltjes' transformations as

$$F(s)=\int_0^{+\infty}\exp(sy)d\alpha(y),\quad (s=\sigma+it), \tag{1.6}$$

where $\alpha(y)$ is again a function of bounded variation and the sequence $\{\lambda_n\}$ satisfies the conditions given in (1.2) and (1.3). We put

$$A_n^*=\sup_{\lambda_n<x\leqslant\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^{x}e^{ity}d\alpha(y)\right|.$$

The result of Theorem A can be proved easily for the abscissa of convergence of $F(s)$ also. If the integral (1.6) converge absolutely in the half plane $\mathrm{Re}\,s<\alpha$ $(-\infty<\alpha<\infty)$, then it represents an analytic function in the half plane $\mathrm{Re}\,s<\alpha$ and since (1.3) holds, we have

$$\limsup_{n\to\infty}\frac{\ln A_n^*}{\lambda_n}=-\alpha. \tag{1.7}$$

Following Luo and Kong [7], we give some definitions pertaining to $F(s)$.

DEFINITION 1. The maximum modulus $M(\sigma,F)$, the maximum term $\mu(\sigma,F)$ and the index $N(\sigma,F)$ of maximum term of $F(s)$ given by (1.6) are defined as:

$$M(\sigma,F)=\sup_{-\infty<t<+\infty}|F(\sigma+it)|,$$

$$M_\mu(\sigma,F)=\sup_{0<x\leqslant+\infty,-\infty<t<+\infty}\left|\int_0^{x}e^{sy}d\alpha(y)\right|,\quad s=\sigma+it,\quad\sigma>0.$$

$$\mu(\sigma,F)=\max_{1\leqslant n<N}\left\{A_n^* e^{\lambda_n\sigma}\right\},\ \ \sigma>0,\quad N(\sigma,F)=\max\left\{\lambda_n;\ \mu(\sigma,F)=A_n^* e^{\lambda_n\sigma}\right\}.$$

The growth parameters such as order and type of an analytic function represented by Laplace-Stieltjes' transformations are defined in a manner similar to those used for the analytic functions represented by classical Dirichlet series. Let R_α^* denote the class of all the functions $F(s)$ of the form (1.6) which are analytic in the half plane $\mathrm{Re}\, s<\alpha$ $(-\infty<\alpha<\infty)$, and are of zero order. We also call them as functions of slow growth. Following Awasthi and Dixit [1], we define the logarithmic order ρ^* of $F(s)$ as

$$\limsup_{\sigma\to\alpha}\frac{\ln\ln M_\mu(\sigma,F)}{\ln[\ln\{1-\exp(\sigma-\alpha)\}^{-1}]}=\rho^* \tag{1.8}$$

Luo and Kong [7] have considered the order and type of entire functions of slow growth represented by (1.6).Wanchun and Caifeng [11] defined the logarithmic order and logarithmic proximate order for analytic function represented by (1.1). In this paper, we consider the analytic functions of slow growth represented by (1.6) and define their proximate order. Our results are more explicit.

2. Preliminary results

Let $F(s)\in R_\alpha^*$. We say that $F(s)\in Q_\alpha^*\subset R_\alpha^*$ if and only if there exists an $\varepsilon>0$ such that

$$\frac{\mu(\sigma,F)}{(1-e^{\sigma-\alpha})^{-\varepsilon}}\to\infty\quad\text{as}\quad\sigma\to\alpha. \tag{2.1}$$

Following Awasthi and Dixit [2], we define the concept of logarithmic H-proximate order and generalized logarithmic type of Laplace-Stieltjes' transformations $F(s)$ given by (1.6).

DEFINITION 2. A real valued function $\rho^*(\sigma)$ defined on $(-\infty,\alpha)$ is called a logarithmic H-proximate order if it satisfies the following properties:

i) $\rho^*(\sigma)$ is a positive, continuous and piecewise differentiable function for all σ

$$\text{such that } -\infty<\sigma_o<\sigma<\alpha, \tag{2.2}$$

ii) $$\lim_{\sigma\to\alpha}\rho^*(\sigma)=\rho^*,\quad(1<\rho^*<\infty), \tag{2.3}$$

iii) $$\lim_{\sigma\to\alpha}\rho^{*\prime}(\sigma)\{1-\exp(\sigma-\alpha)\}\times\ln\{1-\exp(\sigma-\alpha)\}^{-1}\times\ln\ln\{1-\exp(\sigma-\alpha)\}^{-1}=0, \tag{2.4}$$

where $\rho^{*\prime}(\sigma)$ denotes the derivative of $\rho^*(\sigma)$ and is either the left or right derivative of $\rho^*(\sigma)$ where these are different.

We then have [2, Lemma 1]:

LEMMA 1. *Let $\rho^*(\sigma)$ be a logarithmic H-proximate order. Then*

$$\left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)}$$

is monotonically increasing function of σ for $\sigma_o < \sigma < \alpha$.

In view of above result, a single valued real function $\phi(t)$ of t can be defined for $t > t_o$ such that

$$t = \{1-\exp(\sigma-\alpha)\}^{-1}, \tag{2.5}$$

if and only if,

$$\phi(t) = \left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)}. \tag{2.6}$$

From [2, Lemma 2], we have

LEMMA 2. *The function $\phi(t)$ defined as above satisfies the following properties:*

$$\lim_{t\to\infty} \frac{d\ln\phi(t)}{d\ln t} = \rho^*, \tag{2.7}$$

$$\lim_{t\to\infty} \frac{\phi(\eta t)}{\phi(t)} = 1, \quad 0 < \eta < \infty. \tag{2.8}$$

DEFINITION 3. The Laplace-Stieltjes' transformation $F(s) \in Q_\alpha^*$ is said to be of generalized logarithmic type $\bar{T}^*$ and generalized lower logarithmic type $\bar{t}^*$ with respect to a given logarithmic H-proximate order $\rho^*(\sigma)$ if

$$\begin{matrix} \bar{T}^* \\ \bar{t}^* \end{matrix} = \lim_{\sigma\to\alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\ln M_\mu(\sigma,F)}{\left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)}}, \quad \left(0 \leqslant \bar{t}^* \leqslant \bar{T}^* \leqslant \infty\right). \tag{2.9}$$

DEFINITION 4. The function $\rho^*(\sigma)$ is called a logarithmic H-proximate order of $F(s) \in Q_\alpha^*$ if $0 < \bar{T}^* < \infty$.

The function $F(s) \in Q_\alpha^*$ is said to be of perfectly regular growth with respect to its logarithmic H-proximate order $\rho^*(\sigma)$ if $0 < \bar{t}^* = \bar{T}^* < \infty$.

3. Main results

Now, first we show the existence of a logarithmic H-proximate order for every function belonging to the class Q_α^* and of finite logarithmic order.

THEOREM 1. *Let $F(s) = \int_0^\infty \exp(sy)d\alpha(y)$ belong to the class Q_α^* and be of logarithmic order ρ^* $(1 < \rho^* < \infty)$. Then for every $\bar{T}^*$, $0 < \bar{T}^* < \infty$, there exist a logarithmic H-proximate order $\rho^*(\sigma)$ satisfying (2.2) to (2.4) such that $F(s)$ is of generalized logarithmic type $\bar{T}^*$ with respect to $\rho^*(\sigma)$.*

Proof. The proof is similar to that of Theorem 1 of [2], hence we omit the proof. □

THEOREM 2. *Let* $F(s)=\int_0^{\infty}\exp(sy)d\alpha(y)$ *be a Laplace-Stieltjes' transformation belonging to the class* Q_α^* *and let* $\rho^*(\sigma)$ *be its logarithmic proximate order. Let* $\bar{T}^*$ *and* $\bar{t}^*$ *be the generalized logarithmic type and generalized lower logarithmic type of* $F(s)$ *with respect to* $\rho^*(\sigma)$. *Then*

$$\begin{matrix}\bar{T}^* \\ \bar{t}^*\end{matrix} = \lim_{\sigma\to\alpha} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\ln\mu(\sigma,F)}{\left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)}}. \tag{3.1}$$

Proof. Luo and Kong [7, p. 604] have shown that for $\varepsilon>0$,

$$\frac{1}{2}\mu(\sigma,F)\leqslant M_\mu(\sigma,F)\leqslant C\mu((1+2\varepsilon)\sigma,F) \tag{3.2}$$

where C is a finite constant. Consequently, for analytic function $F(s)$ of finite order,

$$\ln\mu(\sigma,F)\simeq\ln M_\mu(\sigma,F) \quad \text{as} \quad \sigma\to\alpha.$$

From the above and (2.9), we get the desired result of Theorem 2. □

We now obtain a coefficient characterization for the generalized type $\bar{T}^*$. For $x>0$ we denote $\ln^+x=\max(\ln x,0)$. We now prove

THEOREM 3. *Let* $F(s)$ *belonging to the class* Q_α^* *be of logarithmic order* ρ^*, $1<\rho^*<\infty$, *and let* $\bar{T}^*$ *be the generalized logarithmic type* $F(s)$ *with respect to the logarithmic* H*-proximate order* $\rho^*(\sigma)$. *Then*

$$\bar{T}^* = \limsup_{n\to\infty}\frac{\ln^+\{A_n^*\exp(\alpha\lambda_n)\}}{\phi(\lambda_n)} \tag{3.3}$$

where $\phi(\lambda_n)$ *is given by* (2.6).

Proof. Let

$$\limsup_{n\to\infty}\frac{\ln^+\{A_n^*\exp(\alpha\lambda_n)\}}{\phi(\lambda_n)}=W.$$

Then for a given $\varepsilon>0$ we have for all $n>n_o(\varepsilon)$,

$$A_n^*\exp(\alpha\lambda_n)<\exp[W_1\phi(\lambda_n)],$$

where $W_1=W+\varepsilon$. Hence for $\sigma<\alpha$,

$$A_n^*\exp(\sigma\lambda_n)<\exp[W_1\phi(\lambda_n)+(\sigma-\alpha)\lambda_n].$$

The above inequality holds for all σ and $n>n_0$. Hence we choose

$$\alpha-\sigma=\frac{W_1\rho^*}{\lambda_n}.$$

Then $\sigma \to \alpha$ as $n \to \infty$. Further, for $\sigma \to \alpha$, $1-\exp(\sigma-\alpha) \approx \alpha-\sigma$. Hence for σ sufficiently close to α, we have

$$\begin{aligned}\ln \mu(\sigma,F) &\leqslant \max_{n\geqslant 1}[W_1\phi(\lambda_n)-(\alpha-\sigma)\lambda_n] \\ &= W_1\left\{\ln\left(\frac{W_1\rho*}{1-\exp(\sigma-\alpha)}\right)^{\rho*(\sigma)}-\rho*\right\}.\end{aligned}$$

Thus for σ sufficiently close to α, we get

$$\frac{\ln \mu(\sigma,F)}{\left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)}} < W_1(1+o(1)).$$

Passing to limits and using (3.1), this gives $\bar{T}^* \leqslant W_1$. Since $\varepsilon > 0$ is arbitrary, we have,

$$\bar{T}^* \leqslant W = \limsup_{n\to\infty} \frac{\ln^+\{A_n^* \exp(\alpha\lambda_n)\}}{\phi(\lambda_n)}. \tag{3.4}$$

Now we obtain the reverse inequality. From (2.9), for a given $\varepsilon > 0$ and for all $\sigma > \sigma_o(\varepsilon)$, we have,

$$\ln M_\mu(\sigma,F) < \left(\bar{T}^*+\varepsilon\right)\left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)}.$$

From (3.2) we have

$$\ln^+\{A_n^*\exp(\alpha\lambda_n)\} \leqslant \ln\mu(\sigma,F) \leqslant \ln M_u(\sigma,F)+o(1).$$

Hence we have for all σ such that $\sigma_o < \sigma < \alpha$,

$$\ln^+\{A_n^*\exp(\alpha\lambda_n)\} < \left(\bar{T}^*+\varepsilon\right)\left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)} - (\sigma-\alpha)\lambda_n.$$

Now choose σ such that

$$(\sigma-\alpha) = \ln\left[1-\frac{\left(\bar{T}^*+\varepsilon\right)\rho^*}{\lambda_n}\right].$$

Then for all $n > n_0$, we have

$$\begin{aligned}\ln^+\{A_n^*\exp(\alpha\lambda_n)\} &< \left(\bar{T}^*+\varepsilon\right)\left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)} - \lambda_n \ln\left[1-\frac{\left(\bar{T}^*+\varepsilon\right)\rho^*}{\lambda_n}\right] \\ &= \left(\bar{T}^*+\varepsilon\right)\left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)} + O(1).\end{aligned}$$

Since

$$\lambda_n = \left(\bar{T}^* + \varepsilon\right)\rho^*\{1-\exp(\sigma-\alpha)\}^{-1},$$

$$\left(\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right)^{\rho*(\sigma)} = \phi\left(\frac{\lambda_n}{\left(\bar{T}^* + \varepsilon\right)\rho^*}\right).$$

Hence

$$\ln^+\{A_n^*\exp(\alpha\lambda_n)\} < \left(\bar{T}^* + \varepsilon\right)\phi\left(\frac{\lambda_n}{\left(\bar{T}^* + \varepsilon\right)\rho^*}\right) + O(1).$$

Now using (3.2), since $\varepsilon > 0$ is arbitrary, we obtain

$$\limsup_{n\to\infty}\frac{\ln^+\{A_n^*\exp(\alpha\lambda_n)\}}{\phi(\lambda_n)} \leqslant (\bar{T}^* + \varepsilon)\limsup_{n\to\infty}\frac{\phi\left(\lambda_n/\left(\bar{T}^* + \varepsilon\right)\rho^*\right)}{\phi(\lambda_n)}$$

Using (2.8), since $\varepsilon > 0$ is arbitrary, we get

$$\limsup_{n\to\infty}\frac{\ln^+\{A_n^*\exp(\alpha\lambda_n)\}}{\phi(\lambda_n)} \leqslant \bar{T}^*. \tag{3.5}$$

On combining (3.4) and (3.5) we get (3.3) and Theorem 3 is proved. □

Next we obtain the characterization of generalized lower type. We have

THEOREM 4. *Let* $F(s) = \int_0^\infty \exp(sy)d\alpha(y)$ *belong to the class* Q_α^* *with logarithmic order* ρ^* $(1 < \rho^* < \infty)$, *and logarithmic* H*-proximate order* $\rho^*(\sigma)$. *If* $\psi(n) = \frac{\ln\left(A_n^*/A_{n+1}^*\right)}{\lambda_{n+1}-\lambda_n}$ *is a non-decreasing function of* n *for* $n > n_o$ *then the generalized lower logarithmic type* $\bar{t}^*$ *of* $F(s)$ *satisfies the inequality*

$$\bar{t}^* \leqslant \liminf_{n\to\infty}\frac{\ln^+\{A_n^*\exp(\alpha\lambda_n)\}}{\phi(\lambda_n)} \tag{3.6}$$

where $\phi(t)$ *is defined in* (2.6). *Further if* $\phi(\lambda_{n-1}) \simeq \phi(\lambda_n)$ *as* $n\to\infty$ *then equality holds in* (3.6).

Proof. Since $1 < \rho^* < \infty$, the maximum term $\mu(\sigma, F)$ is unbounded. It can be seen easily that if $\mu(\sigma,F) = A_n^* e^{\lambda_n\sigma}$ is the maximum term for a given σ then

$$\psi(n-1) \leqslant \sigma < \psi(n)$$

Hence for infinitely many values of n, $\psi(n) > \psi(n-1)$ and $\psi(n)\to\alpha$ as $n\to\infty$. Now, first let $0 < \bar{t}^* < \infty$. In view of (3.1), for a given $\varepsilon > 0$ we have

$$\ln\mu(\sigma,F) > \left(\bar{t}^* - \varepsilon\right)\left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)}.$$

for all σ such that $-\infty < \sigma_o < \sigma < \alpha$, Let $A^*_{n_1}e^{\lambda_{n_1}\sigma}$ and $A^*_{n_2}e^{\lambda_{n_2}\sigma}$ be two consecutive maximum terms of $F(s)$ so that $n_1 \leqslant n_2 - 1$ and $\psi(n_1) = \psi(n_1+1) = \ldots = \psi(n_2-1) \leqslant \sigma < \psi(n_2)$. Hence

$$\ln A^*_{n_2} + \sigma\lambda_{n_2} > \left(\bar{t}^* - \varepsilon\right)\left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)} \text{ for } \psi(n_2-1) \leqslant \sigma < \psi(n_2).$$

Let $n_1 \leqslant n \leqslant n_2 - 1$. Then for $\sigma = \psi(n)$, $A^*_n \exp(\sigma\lambda_n) = A^*_{n_2}\exp(\sigma\lambda_{n_2})$. Hence

$$\ln^+\{A^*_n\exp(\alpha\lambda_n)\} > \left(\bar{t}^* - \varepsilon\right)\left[\ln\{1-\exp(\phi(n)-\alpha)\}^{-1}\right]^{\rho^*(\phi(n))} + \{\alpha - \psi(n)\}\lambda_n,$$

or, for $\psi(n)$ close to α,

$$\frac{\ln^+\{A^*_n\exp(\alpha\lambda_n)\}}{\phi(\lambda_n)} > \frac{\left(\bar{t}^* - \varepsilon\right)}{\phi(\lambda_n)}\left[\left[\ln\{1-\exp(\psi(n)-\alpha)\}^{-1}\right]^{\rho^*}(\psi(n)) + \frac{\lambda_n}{\left(\bar{t}^* - \varepsilon\right)}\{1-\exp(\psi(n)-\alpha)\}\right].$$

Let us put

$$\chi(\sigma) = \left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)} + \frac{\lambda_n}{\left(\bar{t}^* - \varepsilon\right)}\{1-\exp(\sigma-\alpha)\}.$$

Then

$$\chi'(\sigma) = \rho^*(\sigma)\left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)-1}\left(\frac{\exp(\sigma-\alpha)}{1-\exp(\sigma-\alpha)}\right) + (\rho^*(\sigma))'\left[\ln\{1-\exp(\sigma-\alpha)\}^{-1}\right]^{\rho^*(\sigma)}\ln\ln\{1-\exp(\sigma-\alpha)\}^{-1} - \frac{\lambda_n\exp(\sigma-\alpha)}{\left(\bar{t}^* - \varepsilon\right)}.$$

On simplifying the right hand side of above expression and using the condition (2.4) it follows that the minimum value of $\chi(\sigma)$ is attained at a point σ which is the root of the equation

$$\{1-\exp(\sigma-\alpha)\} = \frac{\left(\bar{t}^* - \varepsilon\right)\rho^*}{\lambda_n}$$

Now using (2.6), we thus have

$$\frac{\ln^+\{A^*_n\exp(\alpha\lambda_n)\}}{\phi(\lambda_n)} > \frac{\left(\bar{t}^* - \varepsilon\right)}{\phi(\lambda_n)}\left[\phi\left(\frac{\lambda_n}{\left(\bar{t}^* - \varepsilon\right)\rho^*}\right) + \rho*\right].$$

Now using (2.8) and proceeding to limits as $n \to \infty$, we obtain

$$\liminf_{n\to\infty}\frac{\ln^+\{A^*_n\exp(\alpha\lambda_n)\}}{\phi(\lambda_n)} \geqslant \bar{t}^*. \tag{3.7}$$

The inequality obviously holds if $\overline{t}^* = 0$.

Now we prove the reverse inequality. Let

$$\liminf_{n\to\infty} \frac{\ln^+ \{A_n^* \exp(\alpha\lambda_n)\}}{\phi(\lambda_{n+1})} = \beta.$$

Let $\beta > 0$, then for every β_1 such that $\beta > \beta_1$ we have for $n > n_o(\beta_1)$,

$$\ln^+ \{A_n^* \exp(\alpha\lambda_n)\} > \beta_1 \phi(\lambda_{n+1})$$

From (3.2) we have

$$\ln M_\mu(\mu, F) \geqslant \ln \mu(\sigma, F) + O(1) \geqslant \ln A_n^* + \sigma\lambda_n.$$

Now we choose a sequence $\{\sigma_n\}$ such that $\alpha - \sigma_n = \beta_1 \rho^*/\lambda_n$. Then $\sigma_n \to \alpha$ as $n \to \infty$. Hence

$$\ln M_\mu(\mu, F) \geqslant \beta_1 \phi(\lambda_{n+1}) - \beta_1 \rho * .$$

Let $\sigma_n \leqslant \sigma < \sigma_{n+1}$. Then

$$\begin{aligned} \frac{\ln M_\mu(\mu, F)}{\left\{\ln(1-\exp(\sigma-\alpha))^{-1}\right\}^{\rho*(\sigma)}} &\geqslant \frac{\beta_1 \phi(\lambda_{n+1})}{\left\{\ln(1-\exp(\sigma-\alpha))^{-1}\right\}^{\rho*(\sigma)}} - o(1) \\ &\geqslant \frac{\beta_1 \phi(\lambda_{n+1})}{\left\{\ln(1-\exp(\sigma_n-\alpha))^{-1}\right\}^{\rho*(\sigma)}} - o(1) \\ &\geqslant \frac{\beta_1 \phi(\lambda_{n+1})}{\phi(\lambda_n)} - o(1), \quad \text{on using (2.6).} \end{aligned}$$

Now proceeding to limits as $\sigma \to \alpha$, since $\varepsilon > 0$ is arbitrary, we get

$$\overline{t}^* \geqslant \beta \liminf_{n\to\infty} \frac{\phi(\lambda_{n+1})}{\phi(\lambda_n)} \tag{3.8}$$

Under the assumption $\phi(\lambda_{n-1}) \simeq \phi(\lambda_n)$ as $n \to \infty$ we get $\overline{t}^* \geqslant \beta$. Combining this with (3.7), we get the desired result of Theorem 4. □

REMARK 1. On taking $\rho^*(\sigma) = \rho^*$ we have the following corollary which gives a formula for the logarithmic type T^* and lower logarithmic type t^* of $F(s)$ belonging to Q_α^*.

COROLLARY 1. *Let $F(s) = \int_0^\infty \exp(sy)\, d\alpha(y)$ belong to the class Q_α^*. with logarithmic order ρ^* $(1 < \rho^* < \infty)$, logarithmic type T^* $(0 < T^* < \infty)$ and lower logarithmic type t^*. Then*

$$T^* = \limsup_{n\to\infty} \frac{\ln^+ \{A_n^* \exp(\alpha\lambda_n)\}}{(\ln \lambda_n)^{\rho^*}}.$$

Further, if $\psi(n) = \frac{\ln(A_n^*/A_{n+1}^*)}{\lambda_{n+1}-\lambda_n}$ *is a non-decreasing function of n for* $n > n_o$ *and* $\phi(\lambda_n) \simeq \phi(\lambda_{n+1})$ *as* $n \to \infty$ *then*

$$t^* = \liminf_{n\to\infty} \frac{\ln^+ \{A_n^* \exp(\alpha\lambda_n)\}}{(\ln\lambda_n)^{\rho^*}}.$$

REMARK 2. In the integral (1.6), let us take

$$\alpha(x) = \begin{cases} 0 & \text{when} \quad 0 \leqslant x < \lambda_1 \\ \sum_{k=1}^{n} a_k & \text{when} \quad \lambda_n \leqslant x < \lambda_{n+1} \end{cases}$$

Then it is easily seen that $F(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ and $A_n^* = a_n$, $n = 1,2,3,\ldots$ Using the Corollary above, we get Theorem 2 of Awasthi and Dixit [3]. Further using the formula for t^* above, we get the characterization of the lower logarithmic type of analytic functions represented by the Dirichlet series.

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