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VECTOR–STABILITY OF MULTIPLE VECTOR REFINABLE VECTORS

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Abstract. The stability is an expected property for refinable vectors, which is widely considered in the study of refinement equations. There are two types of stability for refinable vectors. One is *the ordinary-stability*, the other is *the vector-stability*. *The ordinary-stability* considers the stability of entries of refinable vectors, but *the vector-stability* considers the stability of refinable vectors themselves where they are considered as elements of super Hilbert spaces. In this paper, we give a necessary and sufficient condition for refinable vectors to be vector-stable. Our results improve some known ones.

1. Introduction and the main result

In this paper, we study vector refinement equations of the following form

$$\Phi(x) = \sum_{k\in\mathbb{Z}} \mathbf{P}_k \Phi(2x-k), \quad x \in \mathbb{R}. \tag{1}$$

Here, $\Phi = (\Phi_1, \cdots, \Phi_n)$, where $\Phi_j \in L^2(\mathbb{R})^{(r)}$, $1 \leqslant j \leqslant n$ and $\{\mathbf{P}_k : k \in \mathbb{Z}\}$ is the refinement mask such that each $\mathbf{P}_k$ is an $r \times r$ (complex) matrix. A nonzero solution of (1) is called a *refinable vector*.

Vector refinement equations are widely studied in the literature. Daubechies and Cohen [4], Heil and Colella [12], and Long, Chen and Yan [26, 27] studied the existence of solutions of (1). And Daubechies, Jia, Jiang, Lau, Micchelli, Shen, Zhou etc. discussed the regularity of refinable functions [1–3, 5–11, 16–19, 25, 28, 30–31]. In particular, the stability of solutions of vector refinement equations was characterized by Shen [29] and Jiang [22]. Hogan [14, 15] and Shen, Jiang and Lawton [23, 24] gave some characterizations for the stability of solutions of multiple vector refinement equations.

Recall that a vector $\Phi = (\phi_1, \cdots, \phi_r)^T \in L^2(\mathbb{R})^{(r)}$ is said to be stable [14] if there exist constants $0 < \beta_1 \leqslant \beta_2 < \infty$ such that for any $a = \{a_{p,k} : 1 \leqslant p \leqslant r, k \in \mathbb{Z}\} \in \ell^2(\mathbb{Z})$,

$$\beta_1 \|a\|^2_{\ell^2(\mathbb{Z})} \leqslant \left\| \sum_{k\in\mathbb{Z}} \sum_{p=1}^{r} a_{p,k} \phi_p(\cdot - k) \right\|^2_{L^2(\mathbb{R})} \leqslant \beta_2 \|a\|^2_{\ell^2(\mathbb{Z})}.$$

For convenience, we call the vector Φ *ordinarily stable* whenever the above conditions are satisfied.

In this paper, we study the stability of refinable vectors in the following sense.

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DEFINITION 1. Let $\Phi = (\Phi_1, \cdots, \Phi_n)$, where $\Phi_j \in L^2(\mathbb{R})^{(r)}$, $1 \leqslant j \leqslant n$. Φ is said to be *vector-stable* if there exist constants $0 < \beta_1 \leqslant \beta_2 < \infty$ such that for any $a = \left\{a_{j,k} : 1 \leqslant j \leqslant n, k \in \mathbb{Z}\right\} \in \ell^2(\mathbb{Z})$,

$$\beta_1 \|a\|^2_{\ell^2(\mathbb{Z})} \leqslant \left\| \sum_{j=1}^{n} \sum_{k\in\mathbb{Z}} a_{j,k} \Phi_j(\cdot - k) \right\|^2_{L^2(\mathbb{R})^{(r)}} \leqslant \beta_2 \|a\|^2_{\ell^2(\mathbb{Z})}.$$

Before going further, we introduce some notations used in this paper. The Fourier transform of a function in $L^1(\mathbb{R})$ is defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-ix\omega} dx.$$

For $F = (f_1, \cdots, f_r)^T$, $G = (g_1, \cdots, g_r)^T \in L^2(\mathbb{R})^{(r)}$, the inner product of F and G is given by

$$\langle F, G \rangle_{L^2(\mathbb{R})^{(r)}} = \int_{\mathbb{R}} \sum_{p=1}^{r} f_p(x) \overline{g_p(x)} dx,$$

the norm of F is defined by $\|F\|_{L^2(\mathbb{R})^{(r)}} = \langle F, F \rangle^{1/2}$ and $[F,G](\xi) = \sum_{j=1}^{r} \sum_{k\in\mathbb{Z}} \widehat{f_j}(\xi + 2\pi k) \overline{\widehat{g}_j(\xi + 2\pi k)}$.

In the Fourier domain, the refinement equation (1) can be written as

$$\widehat{\Phi}(\omega) = \mathbf{P}\left(\frac{\omega}{2}\right) \widehat{\Phi}\left(\frac{\omega}{2}\right),$$

where

$$\mathbf{P}(\omega) := \frac{1}{2} \sum_{k\in\mathbb{Z}} \mathbf{P}_k e^{-ik\omega},$$

and the Fourier transform of the vector-valued function Φ is defined componentwise.

The symbols $\mathbb{N}$ and $\mathbb{Z}_+$ denote the set of natural numbers and non-negative integers, respectively. We denote by $\mathbb{T}$ the quotient group $\mathbb{R}/2\pi\mathbb{Z}$.

For a given integer $m \geqslant 2$, we say that a point $\omega \in \mathbb{R}$ is m-cyclic in $\mathbb{T}$ if $2^m \omega = \omega \neq 0$ in $\mathbb{T}$. It was shown in [13] that if ω is m-cyclic in $\mathbb{T}$ for some integer $m \geqslant 2$, then for any $k \in \mathbb{Z}$,

$$\omega + 2k\pi = 2^{mn}\omega + 2^{mn-q}\nu\pi \tag{2}$$

for some $n \in \mathbb{N}$, $q \in \{0, \cdots, m-1\}$ and $\nu \in \mathbb{Z}\backslash 2\mathbb{Z}$. Also, if ω is cyclic, then $\omega + \pi$ is acyclic, i.e., is not m-cyclic in $\mathbb{T}$ for any integer m.

Let

$$\mathscr{P}_{n,k}(\cdot) := \prod_{n>\ell>k} \mathbf{P}\left(2^{\ell}\cdot\right) = \mathbf{P}\left(2^{n-1}\cdot\right) \mathbf{P}\left(2^{n-2}\cdot\right) \cdots \mathbf{P}\left(2^{k+1}\cdot\right), \quad \forall\, k, n \in \mathbb{Z}.$$

For the vector-stability of refinable vectors, Zhang and Sun gave some necessary and sufficient conditions [32, 33]. However, they considered only the case of single vector refinable vectors. In this paper, we extend their results to multiple vector refinable vectors.

Let $\Phi = (\Phi_1, \cdots, \Phi_n)$, where $\Phi_j \in L^2(\mathbb{R})^{(r)}$, $1 \leqslant j \leqslant n$. We denote by $S_0(\Phi)$ the linear span of $\{\Phi_j(\cdot - k) : 1 \leqslant j \leqslant n, k \in \mathbb{Z}\}$ and by $S(\Phi)$ the closure of $S_0(\Phi)$ in $L^2(\mathbb{R})^{(r)}$. Define

$$\operatorname{len} S(\Phi) := \min\{\sharp\Psi : S(\Psi) = S(\Phi)\}.$$

Our main result is the following.

THEOREM 1. *Assume that $\Phi = (\Phi_1, \cdots, \Phi_n)$ is a compactly supported solution of the refinement equation (1) and $\operatorname{len} S(\Phi) = n$. Then Φ is vector-stable if and only if for every $\lambda \in \mathbb{C}^n \backslash \{0\}$,*

(i) *if $\lambda \widehat{\Phi}(0) = 0$, then there exists $n \in \mathbb{Z}_+$ so that $\lambda \mathbf{P}^n(0)\mathbf{P}(\pi) \neq 0$;*

(ii) *if $\lambda \mathbf{P}(\omega) = 0$ for some $\omega \in \mathbb{R}$, then $\lambda \mathbf{P}(\omega + \pi) \neq 0$;*

(iii) *for any integer $m \geqslant 2$ and any $\omega \in \mathbb{R}$ which is m-cyclic in $\mathbb{T}$, there exist $n \in \mathbb{N}$ and $q \in \{0, \cdots, m-1\}$ so that*

$$\lambda \mathscr{P}_{mn,q}(\omega)\mathbf{P}(2^q \omega + \pi) \neq 0.$$

REMARK 1. Though Theorem 1 looks much like Theorem 1 in [13], they solve different problems. One of them gives a necessary and sufficient condition for refinable vectors to be vector-stable, another one gives a necessary and sufficient condition for refinable vectors to be ordinarily-stable. For the difference between two types of stability see [32, Example 4.1].

2. Proof of the main result

In this section, we give the proof of the main result. We begin with some preliminary results.

Given a vector function $F = (f_1, \cdots, f_r)^T$ on $\mathbb{R}$, we set

$$F^0 := \sum_{k \in \mathbb{Z}} \sum_{p=1}^{r} |f_p(\cdot - k)|.$$

Then F^0 is a 1-periodic function. Define

$$\mathscr{L}^2(\mathbb{R})^{(r)} := \left\{ F = (f_1, \cdots, f_r)^T : \|F\|_{\mathscr{L}^2(\mathbb{R})^{(r)}} := \|F^0\|_{L^2([0,1))} < \infty \right\}.$$

If $r = 1$, $\mathscr{L}^2(\mathbb{R})^{(1)}$ is written as $\mathscr{L}^2(\mathbb{R})$ for an abbreviation.

Given a function ϕ and a sequence a, the semi-convolution $a *_{sd} \phi$ is the sum

$$\sum_{k \in \mathbb{Z}} a(k)\phi(\cdot - k).$$

Next we give a necessary and sufficient condition on the vector-stability of single vector in $\mathscr{L}^2(\mathbb{R})^{(r)}$.

PROPOSITION 1. [32, Theorem 3.3] Let $\Psi=(\psi_1,\cdots,\psi_r)^T\in\mathscr{L}^2(\mathbb{R})^{(r)}$. Then Ψ is vector-stable if and only if

$$\sum_{k\in\mathbb{Z}}\sum_{p=1}^{r}|\hat{\psi}_p(\omega+2k\pi)|^2>0,\qquad \text{for all}\quad \omega\in\mathbb{R}.$$

The following proposition shows that $\psi*_{sd}$ maps $\ell^1(\mathbb{Z})$ to $\mathscr{L}^2(\mathbb{R})$ and maps $\ell^2(\mathbb{Z})$ to $L^2(\mathbb{R})$.

PROPOSITION 2. [16, Theorem 2.1] If $\psi\in\mathscr{L}^2(\mathbb{R})$, then

$$\|\psi*_{sd}c\|_{\mathscr{L}^2(\mathbb{R})}\leqslant\|\psi\|_{\mathscr{L}^2(\mathbb{R})}\|c\|_{\ell^1(\mathbb{Z})}$$

and

$$\|\psi*_{sd}c\|_{L^2(\mathbb{R})}\leqslant\|\psi\|_{\mathscr{L}^2(\mathbb{R})}\|c\|_{\ell^2(\mathbb{Z})}.$$

The following lemma is the generalized form of Proposition 2.

LEMMA 1. *If* $\Psi=(\psi_1,\cdots,\psi_r)^T\in\mathscr{L}^2(\mathbb{R})^{(r)}$, *then*

$$\|\Psi*_{sd}c\|_{\mathscr{L}^2(\mathbb{R})^{(r)}}\leqslant\sqrt{r}\|\Psi\|_{\mathscr{L}^2(\mathbb{R})^{(r)}}\|c\|_{\ell^1(\mathbb{Z})}\tag{3}$$

and

$$\|\Psi*_{sd}c\|_{L^2(\mathbb{R})^{(r)}}\leqslant\|\Psi\|_{\mathscr{L}^2(\mathbb{R})^{(r)}}\|c\|_{\ell^2(\mathbb{Z})}.\tag{4}$$

Proof. Since

$$\begin{aligned}\|\Psi*_{sd}c\|_{\mathscr{L}^2(\mathbb{R})^{(r)}}&=\|(\Psi*_{sd}c)^0\|_{L^2([0,1))}=\left\|\sum_{k\in\mathbb{Z}}\sum_{p=1}^{r}(\psi_p*_{sd}c)(\cdot-k)\right\|_{L^2([0,1))}\\&\leqslant\sum_{p=1}^{r}\left\|\sum_{k\in\mathbb{Z}}(\psi_p*_{sd}c)(\cdot-k)\right\|_{L^2([0,1))}=\sum_{p=1}^{r}\|\psi_p*_{sd}c\|_{\mathscr{L}^2(\mathbb{R})},\end{aligned}$$

by Proposition 2, we have

$$\begin{aligned}\|\Psi*_{sd}c\|_{\mathscr{L}^2(\mathbb{R})^{(r)}}&\leqslant\sum_{p=1}^{r}\|\psi_p\|_{\mathscr{L}^2(\mathbb{R})}\|c\|_{\ell^1(\mathbb{Z})}=\|c\|_{\ell^1(\mathbb{Z})}\sum_{p=1}^{r}\|(\psi_p)^0\|_{L^2([0,1))}\\&\leqslant\|c\|_{\ell^1(\mathbb{Z})}\sqrt{r}\left\|\sum_{p=1}^{r}(\psi_p)^0\right\|_{L^2([0,1))}=\sqrt{r}\|\Psi\|_{\mathscr{L}^2(\mathbb{R})^{(r)}}\|c\|_{\ell^1(\mathbb{Z})}.\end{aligned}$$

This proves (3).

(4) follows from [32, Lemma 3.1]. □

Let $F\in L^2(\mathbb{R})^{(r)}$, $G\in\mathscr{L}^2(\mathbb{R})^{(r)}$ and $c(F,G)(k)=\langle F,G(\cdot-k)\rangle_{L^2(\mathbb{R})^{(r)}}$. We have following lemma.

LEMMA 2. *The following inequality holds:*

$$\|c(F,G)\|_{\ell^2(\mathbb{Z})} \leqslant \|F\|_{L^2(\mathbb{R})^{(r)}} \|G\|_{\mathscr{L}^2(\mathbb{R})^{(r)}}. \tag{5}$$

Proof. For two sequences a and b, let

$$\langle a,b\rangle_{\ell^2(\mathbb{Z})} = \sum_{k\in\mathbb{Z}} a(k)\overline{b(k)}.$$

For any finitely supported sequence b, we obtain

$$\langle c(F,G),b\rangle_{\ell^2(\mathbb{Z})} = \langle F, G *_{sd} b\rangle_{L^2(\mathbb{R})^{(r)}}.$$

By Lemma 1, we have

$$|\langle c(F,G),b\rangle_{\ell^2(\mathbb{Z})}| \leqslant \|F\|_{L^2(\mathbb{R})^{(r)}} \|G *_{sd} b\|_{L^2(\mathbb{R})^{(r)}} \leqslant \|F\|_{L^2(\mathbb{R})^{(r)}} \|G\|_{\mathscr{L}^2(\mathbb{R})^{(r)}} \|b\|_{\ell^2(\mathbb{Z})}.$$

This proves (5). □

Now, we give a necessary and sufficient condition on the vector-stability of multiple vectors in $\mathscr{L}^2(\mathbb{R})^{(r)}$.

THEOREM 2. *Let $\Phi = (\Phi_1,\cdots,\Phi_n)$, where $\Phi_j \in \mathscr{L}^2(\mathbb{R})^{(r)}$, $1 \leqslant j \leqslant n$. Then Φ is vector-stable if and only if for any $\xi \in \mathbb{R}$, the sequences $\left\{\widehat{\Phi}_j(\xi+2\pi k) : k\in\mathbb{Z}\right\}$ $(j=1,\cdots,n)$ are linearly independent.*

Proof. $(\Rightarrow)$. We prove this by contradiction. If, for some $\xi \in \mathbb{R}$, the sequences $\left\{\widehat{\Phi}_j(\xi+2\pi k) : k\in\mathbb{Z}\right\}$ $(j=1,\cdots,n)$ are linearly dependent, then there exist constants r_j $(j=1,\cdots,n)$, not all zero, such that

$$\sum_{j=1}^{n} r_j \widehat{\Phi}_j(\xi+2\pi k) = 0 \quad \text{for all } k\in\mathbb{Z}.$$

Let $\widetilde{\Phi} := \sum_{j=1}^{n} r_j\Phi_j$. Then by Proposition 1, $\widetilde{\Phi}$ is not vector-stable, namely, $\Phi_1,\cdots,\Phi_n \in \mathscr{L}^2(\mathbb{R})^{(r)}$ are not vector-stable. This proves "$(\Rightarrow)$".

$(\Leftarrow)$. Given $a = \left\{a_{j,k} : 1\leqslant j\leqslant n, k\in\mathbb{Z}\right\} \in \ell^2(\mathbb{Z})$, then by Lemma 1

$$\begin{aligned}\left\|\sum_{j=1}^{n}\sum_{k\in\mathbb{Z}} a_{j,k}\Phi_j(\cdot-k)\right\|^2_{L^2(\mathbb{R})^{(r)}} &= \left\|\sum_{j=1}^{n} a_j *_{sd} \Phi_j\right\|^2_{L^2(\mathbb{R})^{(r)}} \leqslant n\sum_{j=1}^{n} \left\|a_j *_{sd} \Phi_j\right\|^2_{L^2(\mathbb{R})^{(r)}} \\ &\leqslant n\sum_{j=1}^{n} \|\Phi_j\|^2_{\mathscr{L}^2(\mathbb{R})^{(r)}} \|a_j\|^2_{\ell^2(\mathbb{Z})} \leqslant nC_2\sum_{j=1}^{n}\|a_j\|^2_{\ell^2(\mathbb{Z})} \\ &= nC_2\|a\|^2_{\ell^2(\mathbb{Z})},\end{aligned}$$

where $C_2 = \max\left\{\|\Phi_1\|^2_{\mathscr{L}^2(\mathbb{R})^{(r)}}, \cdots, \|\Phi_n\|^2_{\mathscr{L}^2(\mathbb{R})^{(r)}}\right\}$ and $a_j = \{a_j(k) = a_{j,k} : k \in \mathbb{Z}\}$ $(1 \leqslant j \leqslant n)$.

Note that $\Phi_1, \cdots, \Phi_n \in \mathscr{L}^2(\mathbb{R})^{(r)}$, we have $\left\{\widehat{\Phi}_j(\xi + 2\pi k) : k \in \mathbb{Z}\right\} \in \ell^2(\mathbb{Z})$ $(1 \leqslant j \leqslant n)$. Since the sequences $\left\{\widehat{\Phi}_j(\xi + 2\pi k) : k \in \mathbb{Z}\right\}$ $(1 \leqslant j \leqslant n)$ are linearly independent, their Gram matrix $([\Phi_j, \Phi_k](\xi))_{1 \leqslant j,k \leqslant n}$ is nonsingular for $\xi \in \mathbb{T}$ and has its all entries in $\mathscr{B}$. Here

$$\mathscr{B} = \left\{\sum_{k \in \mathbb{Z}} a_k e^{-2\pi k i \xi} : a = \{a_k : k \in \mathbb{Z}\} \in \ell^1(\mathbb{Z}) \text{ is a sequence}\right\}.$$

By Wiener's lemma, the inverse matrix of $([\Phi_j, \Phi_k](\xi))_{1 \leqslant j,k \leqslant n}$ also has its all entries in $\mathscr{B}$. Take $b_{j,k} \in \ell^1(\mathbb{Z})$ $(j,k = 1, \cdots, n)$ such that the matrix $\left(\widehat{b}_{j,k}(\xi)\right)_{1 \leqslant j,k \leqslant n}$ is the inverse of $([\Phi_j, \Phi_k](\xi))_{1 \leqslant j,k \leqslant n}$. For $1 \leqslant j \leqslant n$, let

$$\Psi_j := \sum_{k=1}^{n} \Phi_k *_{sd} b_{j,k}.$$

Then by Lemma 1, $\Psi_j \in \mathscr{L}^2(\mathbb{R})^{(r)}$ and for $1 \leqslant j, m \leqslant n$

$$[\Psi_j, \Phi_m](\xi) = \sum_{k=1}^{n} \widehat{b}_{j,k}(\xi)[\Phi_k, \Phi_m](\xi) = \delta_{j,m} \quad \text{for all } \xi \in \mathbb{T}.$$

Hence

$$\left\langle \Psi_j, \Phi_k(\cdot - \alpha) \right\rangle = \delta_{j,k}\delta_{0,\alpha}$$

and

$$a_{j,k} = \left\langle \sum_{m=1}^{n} \sum_{\alpha \in \mathbb{Z}} a_{m,\alpha} \Phi_m(\cdot - \alpha), \Psi_j(\cdot - k) \right\rangle \quad \text{for all } 1 \leqslant j \leqslant n, k \in \mathbb{Z}.$$

Therewith, by Lemma 2

$$\begin{aligned} \|a\|^2_{\ell^2(\mathbb{Z})} &= \sum_{j=1}^{n} \|a_j\|^2_{\ell^2(\mathbb{Z})} \\ &\leqslant \sum_{j=1}^{n} \left\| \sum_{m=1}^{n} \sum_{\alpha \in \mathbb{Z}} a_{m,\alpha} \Phi_m(\cdot - \alpha) \right\|^2_{L^2(\mathbb{R})^{(r)}} \|\Psi_j\|^2_{\mathscr{L}^2(\mathbb{R})^{(r)}} \\ &= \left\| \sum_{m=1}^{n} \sum_{\alpha \in \mathbb{Z}} a_{m,\alpha} \Phi_m(\cdot - \alpha) \right\|^2_{L^2(\mathbb{R})^{(r)}} \sum_{j=1}^{n} \|\Psi_j\|^2_{\mathscr{L}^2(\mathbb{R})^{(r)}}, \end{aligned}$$

where $a_j = \{a_j(k) = a_{j,k} : k \in \mathbb{Z}\}$ $(1 \leqslant j \leqslant n)$. Let $C_1 = 1/(\sum_{j=1}^{n} \|\Psi_j\|^2_{\mathscr{L}^2(\mathbb{R})^{(r)}})$. Then $C_1 \|a\|^2_{\ell^2(\mathbb{Z})} \leqslant \|\sum_{m=1}^{n} \sum_{\alpha \in \mathbb{Z}} a_{m,\alpha} \Phi_m(\cdot - \alpha)\|^2_{L^2(\mathbb{R})^{(r)}}$. From the above argument, Φ is vector-stable. $\square$

Denote by $\Pi(\mathbb{C})$ the ring of polynomials over $\mathbb{C}$. Let S be the linear space of all sequences $h:\mathbb{Z}\to\mathbb{C}$. For $\theta\in\mathbb{C}\setminus\{0\}$, the sequence given by

$$h_\theta : k \to \theta^k, \quad k\in\mathbb{Z},$$

is an element of S, which we shall denote by h_θ. We define $\tau h = h(\cdot+1)$. If $p\in\Pi(\mathbb{C})$, $p(x)=\sum_{k\geqslant 0} a_k x^k$, then p induces the linear partial difference operator $p(\tau):=\sum_{k\geqslant 0} a_k\tau^k$. Let $P=(p_{ij})_{1\leqslant i\leqslant m,1\leqslant j\leqslant n}$ be a matrix with all its entries $p_{ij}\in\Pi(\mathbb{C})$. For $(h_1,\cdots,h_n)\in S^n$, consider the system of linear homogeneous partial difference equations

$$\sum_{j=1}^{n} p_{ij}(\tau)h_j = 0, \quad i=1,\cdots,m.$$

All the solutions to this system of equations form a subspace of S^n which we shall denote by $\tau(P)$. We have the following proposition.

PROPOSITION 3. [20, Theorem 2.1] Let P be an $m\times n$ matrix whose entries are elements of $\Pi(\mathbb{C})$. Then the following conditions are equivalent.

(i) $\tau(P)\neq 0$.

(ii) There exist some $\theta\in\mathbb{C}\backslash\{0\}$ and $(a_1,\cdots,a_n)\in\mathbb{C}^n\backslash\{0\}$ such that

$$(a_1h_\theta,\cdots,a_nh_\theta)\in\tau(P).$$

Let $\Upsilon_1,\cdots,\Upsilon_n\in L^2(\mathbb{R})$ or $\Upsilon_1,\cdots,\Upsilon_n\in L^2(\mathbb{R})^{(r)}$. Assumption that $\Upsilon_1,\cdots,\Upsilon_n$ are compactly supported. We define

$$K(\Upsilon_1,\cdots,\Upsilon_n):=\left\{(h_1,\cdots,h_n)\in S^n:\ \sum_{j=1}^{n}[\Upsilon_j,h_j]=0\right\}$$

and

$$H(\Upsilon_1,\cdots,\Upsilon_n):=\left\{\sum_{j=1}^{n}[\Upsilon_j,h_j]:\ (h_1,\cdots,h_n)\in S^n\right\}.$$

Here, $[\Upsilon_j,h_j]:=\sum_{k\in\mathbb{Z}}\Upsilon_j(\cdot-k)h_j(k)$.

Let $\Phi_1,\cdots,\Phi_n\in L^2(\mathbb{R})^{(r)}$ be compactly supported. The following lemma gives a characterization for $K(\Phi_1,\cdots,\Phi_n)$.

LEMMA 3. *There exists a matrix P with n columns whose entries are elements of $\Pi(\mathbb{C})$ such that $K(\Phi_1,\cdots,\Phi_n)=\tau(P)$.*

Proof. Let $G=(-1,1)$. Since $\mathbb{R}=\cup_{k\in\mathbb{Z}}(G+k)$, we have that $(h_1,\cdots,h_n)\in K(\Phi_1,\cdots,\Phi_n)$ if and only if

$$\tau^\alpha\left(\sum_{j=1}^{n}[\Phi_j,h_j]\right)\Bigg|_G=0, \quad \forall\,\alpha\in\mathbb{Z}.$$

Observe that $\tau^\alpha[F,h] = [F,\tau^\alpha h]$. Hence, $(h_1,\cdots,h_n) \in K(\Phi_1,\cdots,\Phi_n)$ if and only if

$$\sum_{j=1}^{n}\sum_{k\in\mathbb{Z}}(\tau^\alpha h_j)(k)\Phi_j(\cdot - k)\Big|_G = 0, \quad \forall\, \alpha \in \mathbb{Z}. \tag{6}$$

Since $\Phi_1,\cdots,\Phi_n$ are compactly supported, there exists a positive integer N such that $|k| > N$ implies

$$\Phi_j(\cdot - k)|_G = 0, \quad 1 \leqslant j \leqslant n.$$

This shows that the restriction of the linear space $H(\Phi_1,\cdots,\Phi_n)$ to G is finite dimensional. Choose a basis $\Psi_1,\cdots,\Psi_m$ for it. For $k\in\mathbb{Z}$, $1\leqslant j\leqslant n$, $\Phi_j(\cdot-k)|_G$ can be uniquely represented as follows:

$$\Phi_j(\cdot - k)|_G = \sum_{i=1}^{m} a_{ij}(k)\Psi_i, \tag{7}$$

where the coefficients $a_{ij}(k)\in\mathbb{C}$ and are zero for $|k| > N$. In terms of (7), (6) is equivalent to

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\sum_{|k|\leqslant N} a_{ij}(k)\tau^\alpha h_j(k)\right)\Psi_i = 0, \quad \forall\, \alpha\in\mathbb{Z}. \tag{8}$$

Note that $\tau^\alpha h(k) = h(k+\alpha) = \tau^k h(\alpha)$. Since $\Psi_1,\cdots,\Psi_m$ are linearly independent, (8) is equivalent to

$$\sum_{j=1}^{n}\left(\sum_{|k|\leqslant N} a_{ij}(k)\tau^k\right) h_j = 0, \quad i = 1,\cdots,m.$$

Let $P = (p_{ij})_{1\leqslant i\leqslant m,\, 1\leqslant j\leqslant n}$, where

$$p_{ij}(x) = \sum_{|k|\leqslant N} a_{ij}(k)x^{k+N}, \quad i=1,\cdots,m, \quad j=1,\cdots,n.$$

Then we have $K(\Phi_1,\cdots,\Phi_n) = \tau(P)$. □

The following proposition is a consequence of the Poisson summation formula.

PROPOSITION 4. [20, Lemma 3.2] Let ϕ be a compactly supported distribution on $\mathbb{R}$. Then for a given $\omega\in\mathbb{C}$, the sequence $\{e^{ik\omega} : k\in\mathbb{Z}\}$ lies in $K(\phi)$ if and only if

$$\hat{\phi}(\omega + 2k\pi) = 0, \quad \forall\, k\in\mathbb{Z}.$$

Based on Proposition 4, we give the following lemma.

LEMMA 4. *Let $\Psi = (\psi_1,\cdots,\psi_r)^T \in L^2(\mathbb{R})^{(r)}$ be compactly supported. Then for a given $\omega\in\mathbb{C}$, the sequence $\{e^{ik\omega} : k\in\mathbb{Z}\}$ lies in $K(\Psi)$ if and only if*

$$\widehat{\Psi}(\omega + 2k\pi) = 0, \quad \forall\, k\in\mathbb{Z}.$$

Proof. ($\Rightarrow$) If $\{e^{ik\omega}:k\in\mathbb{Z}\}\in K(\Psi)$, then $\{e^{ik\omega}:k\in\mathbb{Z}\}\in K(\psi_j)$, $1\leqslant j\leqslant r$. By Proposition 4, we have $\hat{\psi}_j(\omega+2k\pi)=0$, $1\leqslant j\leqslant r$, $k\in\mathbb{Z}$, that is $\widehat{\Psi}(\omega+2k\pi)=0$, $k\in\mathbb{Z}$.

($\Leftarrow$) If $\widehat{\Psi}(\omega+2k\pi)=0$, $k\in\mathbb{Z}$, then $\hat{\psi}_j(\omega+2k\pi)=0$, $1\leqslant j\leqslant r$, $k\in\mathbb{Z}$. By Proposition 4, we have $\{e^{ik\omega}:k\in\mathbb{Z}\}\in K(\psi_j)$, $1\leqslant j\leqslant r$, that is $\{e^{ik\omega}:k\in\mathbb{Z}\}\in K(\Psi)$. $\square$

Now, we give two equivalent conditions of $K(\Phi_1,\cdots,\Phi_n)\neq 0$.

LEMMA 5. *Let $\Phi_1,\cdots,\Phi_n\in L^2(\mathbb{R})^{(r)}$ be compactly supported. Then the following conditions are equivalent.*

(i) $K(\Phi_1,\cdots,\Phi_n)\neq 0$.

(ii) *There exist some $\theta\in\mathbb{C}\backslash\{0\}$ and $(a_1,\cdots,a_n)\in\mathbb{C}^n\backslash\{0\}$ such that*

$$(a_1h_\theta,\cdots,a_nh_\theta)\in K(\Phi_1,\cdots,\Phi_n). \tag{9}$$

(iii) *There exists some $\omega\in\mathbb{C}$ such that the sequences $\left\{\widehat{\Phi}_j(\omega+2k\pi):k\in\mathbb{Z}\right\}$ $(j=1,\cdots,n)$ are linearly dependent.*

Proof. By Lemma 3, $K(\Phi_1,\cdots,\Phi_n)=\tau(P)$ for some matrix P of polynomials. Hence, the equivalence between (i) and (ii) follows from Proposition 3.

Suppose (9) is true. Choose $\omega\in\mathbb{C}$ so that $e^{i\omega}=\theta$, and set

$$\tilde{\Phi}:=\sum_{j=1}^{n}a_j\Phi_j. \tag{10}$$

Then (9) and (10) imply

$$\sum_{k\in\mathbb{Z}}\theta^k\tilde{\Phi}(\cdot-k)=\sum_{k\in\mathbb{Z}}\theta^k\sum_{j=1}^{n}a_j\Phi_j(\cdot-k)=\sum_{j=1}^{n}[\Phi_j,a_jh_\theta]=0.$$

In other words, $h_\theta\in K(\tilde{\Phi})$. Hence, by Lemma 4,

$$\widehat{\tilde{\Phi}}(\omega+2k\pi)=0,\quad\forall\,k\in\mathbb{Z}.$$

It follows from (10) that

$$\sum_{j=1}^{n}a_j\widehat{\Phi}_j(\omega+2k\pi)=0,\quad\forall\,k\in\mathbb{Z}. \tag{11}$$

Since $(a_1,\cdots,a_n)\neq 0$, this proves that (ii) implies (iii).

Finally, suppose (iii) holds. Then there exists some $(a_1,\cdots,a_n)\in\mathbb{C}^n\backslash\{0\}$ such that (11) is true. With $\theta=e^{i\omega}$ and $\tilde{\Phi}$ given by (10), we obtain

$$\widehat{\tilde{\Phi}}(\omega+2k\pi)=0,\quad\forall\,k\in\mathbb{Z}.$$

Hence, $h_\theta \in K(\tilde{\Phi})$. Thus (9) follows. □

A compactly supported $\Phi = (\Phi_1, \cdots, \Phi_n)$ $(\Phi_j \in L^2(\mathbb{R})^{(r)}, 1 \leqslant j \leqslant n)$ is said to have linearly independent shifts if the map

$$(a_1, \cdots, a_n) \to \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}} a_j(k) \Phi_j(\cdot - k)$$

is one-to-one on $\mathbb{C}^{\mathbb{Z}}$. $S_0(\Phi)$ denotes the linear span of $\{\Phi_j(\cdot - k) : k \in \mathbb{Z}, 1 \leqslant j \leqslant n\}$. Let $[r_\Phi, s_\Phi]$ be the smallest integer-bounded interval containing $\operatorname{supp}\Phi$. The length of the interval $[r_\Phi, s_\Phi]$ is

$$\ell(\Phi) = s_\Phi - r_\Phi.$$

We call $\ell(\Phi)$ the length of Φ. Let $\Phi = (\Phi_1, \cdots, \Phi_n)$ be a finite collection of compactly supported vector functions on $\mathbb{R}$. The length of Φ, denoted $\ell(\Phi)$, is defined by

$$\ell(\Phi) = \sum_{j=1}^{n} \ell(\Phi_j).$$

The following lemma shows that a compactly supported vector function is linear combination of a collection of linearly independent compactly supported vector functions.

LEMMA 6. *Let $\Phi = (\Phi_1, \cdots, \Phi_n)$ $(\Phi_j \in L^2(\mathbb{R})^{(r)},\ 1 \leqslant j \leqslant n)$ be compactly supported. Then there exists a compactly supported Ψ with the following properties:*

(i) *The shifts of the Ψ are linearly independent;*

(ii) $\Phi \subset S_0(\Psi)$.

Proof. If $K(\Phi_1, \cdots, \Phi_n) = 0$, then we may take $\Psi = \Phi$.

Suppose $K(\Phi_1, \cdots, \Phi_n) \neq 0$. By Lemma 5, there exist some $\theta \in \mathbb{C} \backslash \{0\}$ and $(a_1, \cdots, a_n) \in \mathbb{C}^n \backslash \{0\}$ such that

$$(a_1 h_\theta, \cdots, a_n h_\theta) \in K(\Phi_1, \cdots, \Phi_n),$$

that is

$$\sum_{j=1}^{n} \sum_{k \in \mathbb{Z}} a_j \theta^k \Phi_j(\cdot - k) = 0. \tag{12}$$

After shifting the Φ_j appropriately, we may assume that all $r_{\Phi_j} = 0$. Then $s_{\Phi_j} = \ell(\Phi_j)$. Let

$$\ell = \max\{\ell(\Phi_j) : a_j \neq 0\}.$$

For simplicity, we assume that $a_1 \neq 0$ and $\ell(\Phi_1) = \ell$. Let $\rho = \sum_{j=1}^{n} a_j \Phi_j$ and $\Psi = \sum_{k=0}^{\infty} \theta^k \rho(\cdot - k)$. By our choice of ρ, we deduce from (12) that

$$\sum_{k \in \mathbb{Z}} \theta^k \rho(\cdot - k) = 0.$$

Let $\Psi = (\Psi, \Phi_2, \cdots, \Phi_n)$. We have

$$\Psi - \theta\Psi(\cdot - 1) = \sum_{k=0}^{\infty} \theta^k \rho(\cdot - k) - \sum_{k=0}^{\infty} \theta^{k+1} \rho(\cdot - k - 1) = \rho = \sum_{j=1}^{n} a_j \Phi_j.$$

Since $a_1 \neq 0$, we have $\Phi_1 \in S_0(\Psi)$, that is $\Phi \subset S_0(\Psi)$. Clearly, $\operatorname{supp}\Psi \subseteq [0, \infty)$. Note that

$$\begin{aligned}\Psi(x) &= \sum_{k=0}^{\infty} \theta^k \rho(x-k) = \sum_{j=1}^{n} \sum_{k=0}^{\infty} a_j \theta^k \Phi_j(\cdot - k) = \sum_{j=1}^{n} \sum_{k\in\mathbb{Z}} a_j \theta^k \Phi_j(\cdot - k) \\ &= \sum_{k\in\mathbb{Z}} \theta^k \rho(x-k) = 0, \quad x > \ell(\Phi) - 1.\end{aligned}$$

Consequently, $\operatorname{supp}\Psi \subseteq [0, \ell - 1]$, that is $\ell(\Psi) < \ell(\Phi)$. Repeat the preceding process until $\ell(\Psi)$ achieves its minimum. The resulting vector function Ψ has the property that the shifts of Ψ are linearly independent. □

Proof of Theorem 1. Suppose that Φ is vector-stable. Note that every element of $2\pi\mathbb{Z} \setminus \{0\}$ has the form $2^{n+1}(2k+1)\pi$ for some $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. Then we have

$$\widehat{\Phi}(2^{n+1}(2k+1)\pi) = \mathbf{P}^n(0)\mathbf{P}(\pi)\widehat{\Phi}((2k+1)\pi).$$

If the condition (i) is false, then $\left\{\lambda\widehat{\Phi}(2k\pi) : k \in \mathbb{Z}\right\} = 0$. Also, if the condition (ii) is false, then

$$\lambda\widehat{\Phi}(2\omega + 4k\pi) = \lambda\mathbf{P}(\omega)\widehat{\Phi}(\omega + 2k\pi) = 0, \quad \forall\, k \in \mathbb{Z}$$

and

$$\lambda\widehat{\Phi}(2\omega + 2\pi + 4k\pi) = \lambda\mathbf{P}(\omega + \pi)\widehat{\Phi}(\omega + 2k\pi + \pi) = 0, \quad \forall\, k \in \mathbb{Z}.$$

Therefore, we have $\{\lambda\widehat{\Phi}(2\omega + 2k\pi) : k \in \mathbb{Z}\} = 0$.

Now, if $\mathbf{P}$ does not satisfy condition (iii), then we show that

$$\lambda\widehat{\Phi}(\omega + 2k\pi) = 0, \quad \forall\, k \in \mathbb{Z}.$$

Suppose ω is m-cyclic in $\mathbb{T}$ for some integer $m \geqslant 2$, and let $k \in \mathbb{Z}$ be given. Then by (2), there exist $n \in \mathbb{N}$, $q \in \{0, \cdots, m-1\}$ and $\nu \in \mathbb{Z}\backslash 2\mathbb{Z}$ such that

$$\begin{aligned}\lambda\widehat{\Phi}(\omega + 2k\pi) &= \lambda\widehat{\Phi}(2^{mn}\omega + 2^{mn-q}\nu\pi) = \lambda\mathscr{P}_{mn,q}(\omega)\widehat{\Phi}(2^{q+1}\omega + 2\nu\pi) \\ &= \lambda\mathscr{P}_{mn,q}(\omega)\mathbf{P}(2^q\omega + \nu\pi)\widehat{\Phi}(2^q\omega + \nu\pi).\end{aligned}$$

So, $\lambda\widehat{\Phi}(\omega + 2k\pi)$ is zero if condition (iii) is not satisfied. By arbitrariness of k, we have $\{\lambda\widehat{\Phi}(2\omega + 2k\pi) : k \in \mathbb{Z}\} = 0$. This completes the proof of necessity.

To prove sufficiency, suppose that the shifts of Φ is not vector-stable. Moreover, assume that $\mathbf{P}$ satisfies conditions (i) and (ii). Then we show that (iii) is violated.

Since the shifts of Φ is not vector-stable, by Theorem 2, there exists $\omega_0 \in \mathbb{R}$ such that

$$\lambda\widehat{\Phi}(\omega_0 + 2k\pi) = 0, \quad \forall\, k \in \mathbb{Z}.$$

Lemma 6 implies that Φ is a finite linear combination of the shifts of the Ψ. In the Fourier domain, this implies the existence of a trigonometric polynomial matrix U satisfying

$$\widehat{\Phi}(\omega) = U(\omega)\widehat{\Psi}(\omega). \tag{13}$$

Note that the shifts of the Ψ are linearly independent. Then by (13), we get $\lambda U(\omega_0) = 0$. Since that $\operatorname{len} S(\Phi) = n$, we have that $\det U$ is a non-trivial trigonometric polynomial, which in turn implies that the rational trigonometric polynomials matrix B is well-defined by the relation

$$U(2\omega)B(\omega) = \mathbf{P}(\omega)U(\omega). \tag{14}$$

Therewith, equation (14) implies that

$$\lambda \mathbf{P}\left(\frac{\omega_0}{2}\right)U\left(\frac{\omega_0}{2}\right) = \lambda U(\omega_0)B\left(\frac{\omega_0}{2}\right) = 0$$

and

$$\lambda \mathbf{P}\left(\frac{\omega_0}{2}+\pi\right)U\left(\frac{\omega_0}{2}+\pi\right) = \lambda U(\omega_0)B\left(\frac{\omega_0}{2}+\pi\right) = 0.$$

By condition (ii), it follows that U is singular at either $\frac{\omega_0}{2}$ or $\frac{\omega_0}{2}+\pi$. Since $\det U$ is a trigonometric polynomial, it has only finitely many zeros in $\mathbb{T}+i\mathbb{R}$ and the arguments used in the proof of [21, Lemma 1] imply that $2^m\omega_0 - \omega_0 \in 2\pi\mathbb{Z}$ for some $m \geqslant 2$. If $\omega_0 = 0$, then $\lambda\widehat{\Phi}(0) = 0$ and (i) implies that $\lambda \mathbf{P}^n(0)\mathbf{P}(\pi)$ is not zero for some $n \in \mathbb{Z}_+$. Evidently,

$$\lambda \mathbf{P}^n(0)\mathbf{P}(\pi)U(\pi) = U(0)B^n(0)B(\pi) = 0.$$

But this is a contradiction, since π is acyctic, so $\det U(\pi) \neq 0$.

Accepting the existence of the integer $m \geqslant 2$ and the ω_0 is m-cyclic, we proceed to prove that this violates condition (iii).

Observe that

$$2^{mn}\omega_0 - \omega_0 = \frac{2^{mn}-1}{2^m-1}(2^m\omega_0 - \omega_0) \in 2\pi\mathbb{Z}.$$

Then $\lambda U(\omega_0) = 0$ implies that $\lambda U(2^{mn}\omega_0) = 0$. Therefore, by (14) and the 2π periodicity of U

$$\begin{aligned}
&\lambda \mathscr{P}_{mn,q}(\omega_0)\mathbf{P}(2^q\omega_0+\pi)U(2^q\omega_0+\pi)\\
&= \lambda \mathscr{P}_{mn,q}(\omega_0)U(2^{q+1}\omega_0+2\pi)B(2^q\omega_0+\pi)\\
&= \lambda \mathscr{P}_{mn,q}(\omega_0)U(2^{q+1}\omega_0)B(2^q\omega_0+\pi)\\
&= \lambda U(2^{mn}\omega_0)\mathscr{B}_{mn,q}(\omega_0)B(2^q\omega_0+\pi)\\
&= 0,
\end{aligned}$$

where

$$\mathscr{B}_{n,k} := \prod_{n>\ell>k} B\left(2^\ell\cdot\right) = B\left(2^{n-1}\cdot\right)B\left(2^{n-2}\cdot\right)\cdots B\left(2^{k+1}\cdot\right), \quad \forall\, k,n \in \mathbb{Z}.$$

Since $2^q\omega_0+\pi$ is acyclic, $\det U(2^q\omega_0+\pi)\neq 0$, so

$$\lambda\mathscr{P}_{mn,q}(\omega_0)\mathbf{P}(2^q\omega_0+\pi)=0$$

for every $n\in\mathbb{N}$ and $q\in\{0,\cdots,m-1\}$. $\square$

3. Results and discussion

A wavelet system is generally derived from a refinable function via a multiresolution analysis. Stability is an important property of refinable function. In this paper, we discuss the vector-stability of refinable vectors and we give a necessary and sufficient condition for refinable vectors to be vector-stable. Our results improve some known ones. Studying vector-stability of refinable vectors in $L^p(\mathbb{R})^{(r)}$ is the goal of future work.

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