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SOME RESULTS ON POROUS SET RELATING TO RATIO SETS

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Abstract. An attempt has been made in this paper is to show that every Lebesgue measurable linear set with positive measure has a porous subset whose ratio set contains an interval. The category analogue of this result is also established.

1. Introduction

First we recall the definition of porous set [3] as bellow:

DEFINITION 1. ([3]) Let A be a non-empty subset of real line $\mathbb{R}$ and $x \in A$. A is said to be *porous* at x, if there exists a constant c, $0 < c \leqslant 1$ and a sequence of intervals $\{I_n\}$, each containing x, whose length tends to zero as n tends to infinity, such that each interval I_n contains an interval J_n that is disjoint from A and $\frac{\lambda(J_n)}{\lambda(I_n)} \geqslant c$, where $\lambda(A)$ denotes the Lebesgue measure of A. The set A is called *porous set* if it is porous at each of its points.

DEFINITION 2. ([2]) A set $A \subset \mathbb{R}$ is called p-*porous* for a $p \in (0,1)$ if for every $x \in \mathbb{R}$, $\limsup_{y \to 0} \dfrac{1}{y}$ (the length of the longest interval in $(x-y, x+y)$ which is contiguous to A) $\geqslant p$.

Porous set possesses the following properties:

- Every porous set is of Lebesgue measure zero.
- Every porous set is of first category.

It is to be noted that the converse may not be true. For example, the set of rational number $\mathbb{Q}$.

DEFINITION 3. ([5]) A set A is said to have the property of Baire if it can be expressed as symmetric difference of an open set and a set of first category.

H. I. Miller [3] established that every second category set A having the property of Baire contains a porous subset P such that difference set of P written as $D(P) = \{x - y : x, y \in P\}$ contains an interval. The measure theoretic analogue of this result was shown by Z.Buczolich [2].

In 1962, N. C. Bose Majumder [1] introduced the notion of ratio set in the following way:

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DEFINITION 4. The *ratio set* of a linear set A of non zero abscissa denoted by $R(A)$, is defined by $R(A)=\{\frac{a}{b}: a,b\in A\}$. Also ratio of two linear sets A and B is defined as $R(A,B)=\{\frac{a}{b}: a\in A, b\in B\setminus\{0\}\}$.

Bose Majumder [1] established that ratio set $R(A)$ of a linear set A with non zero abscissa having positive Lebesgue measure contains an interval with left hand end point 1.

In this paper we will show that every Lebesgue measurable set $A(\subset\mathbb{R})$ with positive measure contains a porous subset B whose ratio set $R(B)$ contains an interval. Also the category analogue of this result is proved.

2. Main results

Before going to establish main results we go through some lemmas.

LEMMA 1. ([2]) *For every closed set A of positive Lebesgue measure, $s\in\mathbb{N}$ and $t\in(0,1)$ there exists a closed set $A_r\subset A$ with the following properties: The set A_r has positive Lebesgue measure and there exists a sequence of natural numbers $n_1,n_2,\ldots,n_j,\ldots$ such that $s\mid n_j$ for every $j\in\mathbb{N}$ and, letting $d_j=\frac{1}{n_1\ldots n_j}$, we have either $[kd_j,(k+1)d_j]\cap A_r=\emptyset$ or $\lambda((kd_j,(k+1)d_j)\cap A_r))>t.d_j$ for every $j\in\mathbb{N}$ and $k\in\mathbb{Z}$. Here $\mathbb{N}$ and $\mathbb{Z}$ are the sets of natural numbers and integers respectively.*

LEMMA 2. *For every $p\in(0,\frac{1}{3})$ there exists a $t(p)\in(0,1)$ such that if $H_1\subset(0,1]$, $H_2\subset(0,1]$ and $\lambda(H_1)>t(p)$, $\lambda(H_2)>t(p)$ then $R(H_1,H_2)=R(G_1,G_2)$, where $G_1=H_1\setminus(\frac{1-p}{2},\frac{1+p}{2})$ and $G_2=H_2\setminus(\frac{1-p}{2},\frac{1+p}{2})$.*

Proof. Consider $t(p)=3p$ for $p\in(0,\frac{1}{3})$. Suppose H_1 and H_2 are two subsets of $(0,1]$ with $\lambda(H_1)>t(p)$ and $\lambda(H_2)>t(p)$. Clearly $R(H_1,H_2)$ is non-empty subset of $(0,\infty)$. If m is an element of $R(H_1,H_2)$ then there exist $y\in H_1$ and $x\in H_2$ such that $y=mx$. From the unit square $S=[0,1]\times[0,1]$, we obtained four squares each of length $\frac{1-p}{2}$ by deleting a horizontal strip and a vertical strip of breath $p\in(0,\frac{1}{3})$. These four squares are of the form

$$S_1=[0,\tfrac{1-p}{2}]\times[\tfrac{1+p}{2},1] \text{ which is upper left square,}$$

$$S_2=[0,\tfrac{1-p}{2}]\times[0,\tfrac{1-p}{2}] \text{ which is lower left square,}$$

$$S_3=[\tfrac{1+p}{2},1]\times[\tfrac{1+p}{2},1] \text{ which is upper right square,}$$

$$S_4=[\tfrac{1+p}{2},1]\times[0,\tfrac{1-p}{2}] \text{ which is lower right square.}$$

If we denote by l_m the graph of the line $y=mx$ then $l_m\cap(H_1\times H_2)\neq\emptyset$. Let P_x (resp. P_y) be the projection of the line $y=mx$ on x (resp. y) axis. For $m\in(0,\infty)$, clearly

$$\lambda(P_x(l_m\cap(S_1\cup S_2)))\geqslant\tfrac{1-3p}{2} \text{ and } \lambda(P_y(l_m\cap(S_1\cup S_2)))\geqslant\tfrac{1-3p}{2}.$$

Since $\lambda(H_1) > t(p)$, $\lambda(H_2) > t(p)$ and also $t(p) = 3p$ we have

$$\lambda(P_x(l_m \cap (S_1 \cup S_2)) \setminus H_2) < 1-3p \text{ and } \lambda(P_y(l_m \cap (S_1 \cup S_2)) \setminus H_1) < 1-3p.$$

Therefore $l_m \cap (S_1 \cup S_2) \cap (H_2 \times H_1) = l_m \cap (S_1 \cup S_2) \cap (G_2 \times G_1) \neq \emptyset$. Hence for $m \in R(H_1, H_2)$ implies $m \in R(G_1, G_2)$. Thus $R(H_1, H_2) = R(G_1, G_2)$. $\square$

THEOREM 1. *For every set $A(\subset \mathbb{R})$ having nonzero abscissa with positive Lebesgue measure and $p \in (0, \frac{1}{3})$ there exists a p-porous set $B \subset A$ such that $R(B) = \{\frac{a}{b} : a, b \in B\}$ contains an interval.*

Proof. With out loss of generality we consider A to be a closed set. It is enough to prove the theorem for rational $p \in (0, \frac{1}{3})$. Let $p = \frac{u}{v} \in (0, \frac{1}{3})$, where $u, v \in \mathbb{N}$. By Lemma 2 we choose a suitable $t(p)$ for $p \in (0, \frac{1}{3})$ and then applying the Lemma 1 with $s = 2v$ and $t = t(p)$ we obtain a closed set $A_r \subset A$ and the sequences $n_1, n_2, \ldots$ (of natural numbers) and $d_1, d_2, \ldots$, $(d_j = \frac{1}{n_1 n_2 \ldots n_j}$, $j \in \mathbb{N})$ such that either $[kd_j, (k+1)d_j] \cap A_r = \emptyset$ or $\lambda((kd_j, (k+1)d_j) \cap A_r)) > td_j$ for every $j \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Consider $B_0 = A_r$. We put

$$B_j = B_{j-1} \setminus \bigcup_{k \in \mathbb{Z}} \left(\left(k + \frac{1-p}{2}\right).d_j, \ \left(k + \frac{1+p}{2}\right).d_j \right).$$

Obviously $B = \bigcap_{j=1}^{\infty} B_j$ is p-porous. Since B_0 is of positive Lebesgue measure, according to Bose Majumder's result [1] the set $R(B_0)$ contains an interval with 1 as left hand end point. Again since all the sets B_j, $(j = 0, 1, 2, 3, \ldots)$ are compact, $R(B_j)$ are compact and hence closed. By Cantor Baire Stationary theorem [4], we have $R(B) = R(B_0)$. It is enough to prove $R(B_0) = R(B_1) = R(B_2) = \ldots$. We have to show that $R(B_{j-1}) = R(B_j)$ for all $j \in \mathbb{N}$.

We say that the set B_{j-1} possesses the property P_{j-1} if for every integer k we have either $(k.d_j, (k+1).d_j) \cap B_{j-1} = \emptyset$ or $\lambda((k.d_j, (k+1).d_j) \cap B_{j-1}) > t.d_j$.

If the set B_{j-1} possesses the property P_{j-1}, then by $p = \frac{u}{v}$ and $s = 2v$, we have $\frac{pd_j}{2d_{j+1}} \in \mathbb{N}$. So, in the definition of B_j which is obtained from B_{j-1} by deleting a subset of B_{j-1} which is a union of the intervals of the form $(k.d_{j+1}, (k+1).d_{j+1})$. According to Lemma 1, the set $B_0 = A_r$ has the property P_0. Thus by induction B_j has P_j property for every $j \in \mathbb{N}$. Let us take

$$H_{k,j-1} = [k.d_j, (k+1).d_j] \cap B_{j-1} \text{ and } G_{k,j-1} = H_{k,j-1} \setminus ((k + \tfrac{1-p}{2}).d_j, (k + \tfrac{1+p}{2}).d_j).$$

Then we have

$$R(B_{j-1}) = \bigcup_{k,m \in \mathbb{Z}} R(H_{k,j-1}, H_{m,j-1}),$$

(since the set B_{j-1} has the property P_{j-1}, the equality holds for those indices $m, k \in \mathbb{Z}$ for which $(k.d_j, (k+1).d_j) \cap B_{j-1} \neq \emptyset$ and $(m.d_j, (m+1).d_j) \cap B_{j-1} \neq \emptyset$).

Also we get

$$R(B_j) = \bigcup_{k,m \in \mathbb{Z}} R(G_{k,j-1}, G_{m,j-1}).$$

By Lemma 2, $R(H_{k,j-1}, H_{m,j-1}) = R(G_{k,j-1}, G_{m,j-1})$ for the indices $k,m \in \mathbb{Z}$ and $j \in \mathbb{N}$. Therefore $R(B_{j-1}) = R(B_j)$ for all $j \in \mathbb{N}$. Hence the result. □

To establish the category analogue of the Theorem 1 we need following lemmas.

LEMMA 3. *Let $0 < a < b$ and $F = \bigcup_{n=1}^{\infty} F_n$, where F_n's are nowhere dense closed subsets of $\mathbb{R}$. Then the ratio set $R((a,b) \setminus F) = (\frac{a}{b}, \frac{b}{a}) = R((a,b))$, where (a,b) is an open interval.*

Proof. Let $0 < a < b$. Let $x, y \in (a,b)$ and $x < y$. So, $0 < a < x < y < b \Rightarrow \frac{a}{y} < \frac{x}{y} < \frac{b}{y} \Rightarrow 0 < \frac{a}{b} < \frac{a}{y} < \frac{x}{y} < \frac{b}{y} < \frac{b}{a}$. Thus $\frac{x}{y}$ is an interior point of $(\frac{a}{b}, \frac{b}{a})$. Similarly $\frac{y}{x}$ is an interior point of $(\frac{a}{b}, \frac{b}{a})$. So, $R((a,b)) = (\frac{a}{b}, \frac{b}{a})$, where $0 < a < b$.

Since F_n's are nowhere dense closed subset of $\mathbb{R}$, the complement of each F_n is everywhere dense open subset of $\mathbb{R}$. By Baire category theorem we have, $(a,b) \setminus \bigcup_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} [(a,b) \setminus F_n]$ is dense in closed interval $[a,b]$.

Therefore $R((a,b) \setminus F) = R((a,b))$. □

LEMMA 4. *For any two open intervals $I, J (\subset \mathbb{R}^+)$, the ratio set $R(I,J) = \{\frac{x}{y} : x \in I, y \in J\}$ is an open set, where $\mathbb{R}^+$ denotes the set of positive real numbers.*

Proof. Let $I = (x', y')$ and $J = (x'', y'')$ be two open intervals in $\mathbb{R}^+$. So, $0 < x' < y'$ and $0 < x'' < y''$. Let $v \in R(I,J)$. So, $v = \frac{p}{q}$, i.e. $p = vq$, where $p \in I, q \in J$. Since p is an interior point of I, there exists $\delta' > 0$ such that $x' < p - \delta' < p < p + \delta' < y'$ $\Rightarrow x' < p - \delta' < vq < p + \delta' < y' \Rightarrow \frac{x'}{q} < v - \frac{\delta'}{q} < v < v + \frac{\delta'}{q} < \frac{y'}{q} \Rightarrow \frac{x'}{y''} < \frac{x'}{q} < v - \frac{\delta'}{q} < v < v + \frac{\delta'}{q} < \frac{y'}{q} < \frac{y'}{x''}$, since $q \in J$. This shows that v is an interior point of $(\frac{x'}{y''}, \frac{y'}{x''}) = R(I,J)$, where $I = (x', y')$, $0 < x' < y'$ and $J = (x'', y'')$, $0 < x'' < y''$. So, $R(I,J)$ is an open interval and hence an open set. □

THEOREM 2. *If $B \subseteq \mathbb{R}^+$ has the property of Baire and is of second category then there exists a porous set $P \subseteq B$ such that $R(P) = \{\frac{p}{q} : p, q \in P\}$ contains an interval, where $\mathbb{R}^+$ denotes the set of positive real numbers.*

Proof. Suppose B be a second category subset of positive reals having the property of Baire. So, there exist an open interval $I = (b,c)$, $0 < b < c$ and a sequence $\{F_n\}_{n=1}^{\infty}$ of closed nowhere dense subsets of $\mathbb{R}^+$ such that $B = (I \setminus F_1) \cup F_2$. i.e, $B \supseteq I \setminus F_1 \supseteq I \setminus \bigcup_{n=1}^{\infty} F_n$. Let $A = I \setminus \bigcup_{n=1}^{\infty} F_n$.

By Lemma 3 we have $R(A) = (\frac{b}{c}, \frac{c}{b})$. Clearly $I \setminus F_1$ is an open subset of $\mathbb{R}^+$. So, $I \setminus F_1 = \bigcup_{i=1}^{\infty} I_{i1}$, where $\{I_{i1} : i \in \mathbb{N}\}$ are pairwise disjoint open subintervals of I. Therefore $R(I \setminus F_1) = (\frac{b}{c}, \frac{c}{b})$. Again since

$$I \setminus F_1 = \bigcup_{i=1}^{\infty} I_{i1},\ R(I \setminus F_1) = R(\bigcup_{i=1}^{\infty} I_{i1}) = \bigcup_{i,j=1}^{\infty} R(I_{i1}, I_{j1}) \ = (\tfrac{b}{c}, \tfrac{c}{b}),$$

where

$$R(I_{i1}, I_{j1}) = \{\tfrac{x}{y} : x \in I_{i1}, y \in I_{j1}\}.$$

By Lemma 4, $R(I_{i1}, I_{j1})$ are open sets for each $i, j \in \mathbb{N}$. So, $\{R(I_{i1}, I_{j1}) : i, j \in \mathbb{N}\}$ forms an open cover for each closed subinterval of $(\frac{b}{c}, \frac{c}{b})$. Let $r \in (\frac{b}{c}, \frac{c}{b})$. Then $\{R(I_{i1}, I_{j1}) : i, j \in \mathbb{N}\}$ is an open cover of $[\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}]$ and therefore by Heine Borel Covering Theorem, the interval $[\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}]$ is covered by finitely many of $\{R(I_{i1}, I_{j1}) : i, j \in \mathbb{N}\}$. So, there exists $n_1 \in \mathbb{N}$ such that

$$\bigcup_{i,j=1}^{n_1} R(I_{i1}, I_{j1}) \supseteq \ [\tfrac{b}{c} + \tfrac{r}{2}, \tfrac{c}{b} - \tfrac{r}{2}].$$

For each open interval I_{i1}, $i = 1, 2, 3, \ldots, n_1$, there exists a natural number k_1 and a closed interval J_{i1} contained in I_{i1}, $i = 1, 2, 3, \ldots, n_1$, with end points are of the form $\frac{k}{3^{k_1}}$ (where $k \in \mathbb{N}$) such that

$$R(int(\bigcup_{i=1}^{n_1} J_{i1})) \supseteq \ [\tfrac{b}{c} + \tfrac{r}{2}, \tfrac{c}{b} - \tfrac{r}{2}].$$

Now each of the intervals J_{i1} is subdivided into closed sub-intervals of length $\frac{1}{3^{k_1}}$. Remove open middle ninth of each of these intervals of length $\frac{1}{3^{k_1}}$, obtaining intervals of length $\frac{4}{3^{k_1+2}}$. The union of the interiors of these closed intervals (of length $\frac{4}{3^{k_1+2}}$) is an open set, call it G_2. By argument of Utz [6], we can verify that $R(G_2) \supseteq [\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}]$. Since G_2 is a non-empty open subset of $\mathbb{R}^+$ and F_2 is nowhere dense and closed, it follows that $R(G_2 \setminus F_2) \supseteq [\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}]$. Again $G_2 \setminus F_2$ is an open set as F_2 is closed. So, $G_2 \setminus F_2 = \bigcup_{i=1}^{\infty} I_{i2}$, where $\{I_{i2}\}_{i=1}^{\infty}$ are pairwise disjoint countable open interval. Therefore

$$R(G_2 \setminus F_2) = R(\bigcup_{i=1}^{\infty} I_{i2}) = \bigcup_{i,j=1}^{\infty} R(I_{i2}, I_{j2}).$$

Again by Heine Borel Covering Theorem, $[\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}]$ is covered by finitely many of $\{R(I_{i2}, I_{j2}) : i, j \in \mathbb{N}\}$. So, there exists a natural number n_2, such that

$$\bigcup_{i,j=1}^{n_2} R(I_{i2},I_{j2}) \supseteq \left[\tfrac{b}{c}+\tfrac{r}{2},\tfrac{c}{b}-\tfrac{r}{2}\right].$$

Thus, there exist a sequence of closed intervals $\{J_{i2}\}_{i=1}^{n_2}$ and a natural number k_2 with $k_2 > k_1 + 2$ having the following properties.

Each J_{i2} is contained in I_{i2}, with end points are of the form $\frac{k}{3^{k_2}}$, $k \in \mathbb{N}$ and such that

$$R(int(\bigcup_{i=1}^{n_2} J_{i2})) \supseteq [\frac{b}{c}+\frac{r}{2},\frac{c}{b}-\frac{r}{2}].$$

Proceeding as before, we subdivide each of the intervals I_{i2} into closed subintervals of length $\frac{1}{3^{k_2}}$. Remove open middle ninth of each of these intervals of length $\frac{1}{3^{k_2}}$, obtaining intervals of length $\frac{4}{3^{k_2+2}}$. The union of the interiors of these closed intervals (of length $\frac{4}{3^{k_2+2}}$) is an open set, call it G_3. Again by same argument of Utz [6], we can verify that $R(G_3) \supseteq [\frac{b}{c}+\frac{r}{2},\frac{c}{b}-\frac{r}{2}]$. Continuing this process by finite induction, obtain

$$\{n_i\}_{i=1}^{\infty},\ \{k_i\}_{i=1}^{\infty},\ \{\{I_{ij}\}_{i=1}^{n_j}\}_{j=1}^{\infty},\ \{\{J_{ij}\}_{i=1}^{n_j}\}_{j=1}^{\infty} \text{ and } \{G_i\}_{i=2}^{\infty}$$

that satisfy the following conditions:

$$k_{i+1} > k_i + 2 \text{ for each } i.$$

For each j, $\{I_{ij}\}_{i=1}^{n_j}$ is sequence of pair wise disjoint open intervals such that

$$\bigcup[R(I_{ij},I_{kj}) : i,k \in \{1,2,3,\ldots,n_j\}] \supseteq [\tfrac{b}{c}+\tfrac{r}{2},\tfrac{c}{b}-\tfrac{r}{2}].$$

For each j, $\{J_{ij}\}_{i=1}^{n_j}$ is a sequence of closed intervals satisfying the following properties:

- The end points of J_{ij} are of the form $\frac{k}{3^{k_j}}$, $J_{ij} \subseteq I_{ij}$ and

$$R(int(\bigcup_{i=1}^{n_j} J_{ij})) \supseteq \left[\tfrac{b}{c}+\tfrac{r}{2},\tfrac{c}{b}-\tfrac{r}{2}\right].$$

Additionally, the formation of each G_j is as follows:

- Each of the closed intervals $J_{i,j-1}$ is divided into closed sub-intervals of length $\frac{1}{3^{k_{j-1}}}$. Remove the open middle ninth of each of these intervals of length $\frac{1}{3^{k_{j-1}}}$, obtaining closed intervals of length $\frac{4}{3^{k_{j-1}+2}}$. The union of the interiors of these intervals of length $\frac{4}{3^{k_{j-1}+2}}$ is defined as G_j. Clearly G_j is open for each j and $R(G_j) \supset [\frac{b}{c}+\frac{r}{2},\frac{c}{b}-\frac{r}{2}]$.

Finally we have

$$B \supset I \setminus F_1 \supset \bigcup_{i=1}^{n_1} I_{i1} \supset \bigcup_{i=1}^{n_1} J_{i1} \supset G_2 \supset G_2 \setminus F_2 \supset$$
$$\bigcup_{i=1}^{n_2} I_{i2} \supset \bigcup_{i=1}^{n_2} J_{i2} \supset G_3 \supset G_3 \setminus F_3 \supset \ldots$$

Let $P = \bigcap_{j=1}^{\infty}(\bigcup_{i=1}^{n_j} J_{ij})$. Clearly P is a compact subset of B. Also, since $P \subset \bigcup_{i=1}^{n_j} J_{ij}$ for each $j \geqslant 2$, P is porous. Finally, if $s \in [\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}]$, for each j, there exist $x_j, y_j \in \bigcup_{i=1}^{n_j} J_{ij}$ such that $x_j = sy_j$. By Bolzano-Weierstrass theorem, there exists a sequence $\{j_k\}$ of natural numbers such that $\lim_{k\to\infty} x_{j_k} = x$ and $\lim_{k\to\infty} y_{j_k} = y$. Clearly $x = sy$. Furthermore by definition of P, $x, y \in P$ and therefore $R(P) \supseteq [\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}]$. This completes the proof. □

QUESTION. *It is unknown whether the Theorem 2 is valid with out the property of Baire.*

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