Journal of Classical Analysis
Volume 11, Number 1 (2017), 83–89 doi:10.7153/jca-11-06

INITIAL BOUNDS FOR ANALYTIC AND BI–UNIVALENT FUNCTIONS BY MEANS OF CHEBYSHEV POLYNOMIALS

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Abstract. In this paper, we obtain initial coefficient bounds for functions belong to a subclass of bi-univalent functions by using the Chebyshev polynomials and also we find Fekete-Szegö inequalities for this class.

1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}$$

be the set of positive integers.

Let $\mathscr{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk

$$\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We also denote by $\mathscr{S}$ the class of all functions in the normalized analytic function class $\mathscr{A}$ which are univalent in Δ.

For two functions f and g, analytic in Δ, we say that the function f is subordinate to g in Δ, and write

$$f(z) \prec g(z) \qquad (z \in \Delta),$$

if there exists a Schwarz function ω, which is analytic in Δ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \Delta)$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \Delta).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \Delta) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Mathematics subject classification (2010): 30C45.
Keywords and phrases: Analytic functions, bi-univalent functions, coefficient bounds, Chebyshev polynomial, Fekete-Szegö problem, subordination.

Furthermore, if the function g is univalent in Δ, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in \Delta) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

It is well known (e.g. see Duren [9]) that every function $f \in \mathscr{S}$ has an inverse map f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \Delta)$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\, r_0(f) \geqslant \frac{1}{4}\right).$$

In fact, the inverse function $g = f^{-1}$ is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots. \tag{2}$$

A function $f \in \mathscr{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. We let Σ denote the class of bi-univalent functions in Δ given by (1). For a history and examples of functions which are (or which are not) in the class Σ, together with various other properties of subclasses of bi-univalent functions one can refer [1, 2, 3, 4, 11, 13, 15, 16, 18].

Ding et al. [7] introduced the class $Q_\lambda(\beta)$ of analytic functions defined as follows:

$$Q_\lambda(\beta) = \left\{ f \in \mathscr{A} : \Re\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \right) > \beta,\ \beta < 1,\ \lambda \geqslant 0 \right\}.$$

It is easy to see that

$$Q_{\lambda_1}(\beta) \subset Q_{\lambda_2}(\beta) \qquad \text{for} \quad \lambda_1 > \lambda_2 \geqslant 0.$$

Thus, for $\lambda \geqslant 1$ and $0 \leqslant \beta < 1$,

$$Q_\lambda(\beta) \subset Q_1(\beta) = \left\{ f \in \mathscr{A} : \Re\left(f'(z)\right) > \beta,\ 0 \leqslant \beta < 1 \right\}$$

and hence $Q_\lambda(\beta)$ is univalent class (see [5, 6, 12]). Recently, Srivastava et al. [16] and Frasin and Aouf [11] studied the bi-univalent results for above defined classes.

The significance of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. Out of four kinds of Chebyshev polynomials, many researchers dealing with orthogonal polynomials of Chebyshev. For a brief history of Chebyshev polynomials of first kind $T_n(t)$, the second kind $U_n(t)$ and their applications one can refer [8, 10, 14, 2]. The Chebyshev polynomials of the first and second kinds are well known and they are defined by

$$T_n(t) = \cos n\theta \quad \text{and} \quad U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta} \qquad (-1 < t < 1)$$

where n denotes the polynomial degree and $t = \cos\theta$.

DEFINITION 1. For $\lambda \geqslant 1$ and $t \in (1/2,1)$, a function $f \in \Sigma$ given by (1) is said to be in the class $\mathscr{B}_\Sigma(\lambda,t)$ if the following subordinations hold for all $z,w \in \Delta$:

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec H(z,t) := \frac{1}{1-2tz+z^2} \tag{3}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) \prec H(w,t) := \frac{1}{1-2tw+w^2}, \tag{4}$$

where the function $g = f^{-1}$ is defined by (2).

In particular, we set $\mathscr{B}_\Sigma(1,t) = \mathscr{B}_\Sigma(t)$ for the class of functions $f \in \Sigma$ given by (1) and satisfying the following subordination conditions for all $z,w \in \Delta$:

$$f'(z) \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$g'(w) \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is defined by (2).

We note that if $t = \cos\alpha$, where $\alpha \in (-\pi/3, \pi/3)$, then

$$H(z,t) = \frac{1}{1-2\cos\alpha z+z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\alpha}{\sin\alpha} z^n \quad (z \in \Delta).$$

Thus

$$H(z,t) = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + \dots \quad (z \in \Delta).$$

From [17], we can write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \dots \quad (z \in \Delta, \quad t \in (-1,1))$$

where

$$U_{n-1} = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}} \quad (n \in \mathbb{N})$$

are the Chebyshev polynomials of the second kind and we have

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$U_1(t) = 2t, \quad U_2(t) = 4t^2-1, \quad U_3(t) = 8t^3-4t, \quad U_4(t) = 16t^4-12t^2+1, \dots \tag{5}$$

The generating function of the first kind of Chebyshev polnomial $T_n(t)$, $t \in [-1,1]$, is given by

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1-tz}{1-2tz+z^2} \qquad (z \in \Delta).$$

The first kind of Chebyshev polnomial $T_n(t)$ and second kind of Chebyshev polnomial $U_n(t)$ are connected by:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t); \quad T_n(t) = U_n(t) - tU_{n-1}(t); \quad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

In this present paper, we use the Chebyshev polynomials expansions to provide the initial coefficients of bi-univalent functions in $\mathscr{B}_\Sigma(\lambda,t)$. We also solve Fekete-Szegö problem for functions in this class.

2. Coefficient bounds for the function class $\mathscr{B}_\Sigma(\lambda,t)$

THEOREM 1. *For $\lambda \geqslant 1$ and $t \in (1/2,1)$, let the function $f \in \Sigma$ given by (1) be in the class $\mathscr{B}_\Sigma(\lambda,t)$. Then*

$$|a_2| \leqslant \frac{2t\sqrt{2t}}{\sqrt{\left|(1+\lambda)^2 - 4\lambda^2 t^2\right|}}, \tag{6}$$

$$|a_3| \leqslant \frac{4t^2}{(1+\lambda)^2} + \frac{2t}{1+2\lambda}, \tag{7}$$

and for some $\eta \in \mathbb{R}$,

$$\left|a_3 - \eta a_2^2\right| \leqslant \begin{cases} \frac{4t}{1+2\lambda}, & |\eta - 1| \leqslant \frac{\left|(1+\lambda)^2 - 4\lambda^2 t^2\right|}{4(1+2\lambda)t^2} \\ \frac{8|\eta-1|t^3}{\left|(1+\lambda)^2 - 4\lambda^2 t^2\right|}, & |\eta - 1| \geqslant \frac{\left|(1+\lambda)^2 - 4\lambda^2 t^2\right|}{4(1+2\lambda)t^2} \end{cases}. \tag{8}$$

Proof. Let the function $f \in \Sigma$ given by (1) be in the class $\mathscr{B}_\Sigma(\lambda,t)$. From (3) and (4), we have

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = 1 + U_1(t)p(z) + U_2(t)p^2(z) + \cdots \tag{9}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) = 1 + U_1(t)q(w) + U_2(t)q^2(w) + \cdots \tag{10}$$

for some analytic functions

$$p(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \qquad (z \in \Delta), \tag{11}$$

and

$$q(w) = d_1 w + d_2 w^2 + d_3 w^3 + \cdots \qquad (w \in \Delta), \tag{12}$$

such that $p(0) = q(0) = 0$, $|p(z)| < 1$ $(z \in \Delta)$ and $|q(w)| < 1$ $(w \in \Delta)$. It is well-known that if $|p(z)| < 1$ and $|q(w)| < 1$, then

$$\left|c_j\right| \leqslant 1 \quad \text{and} \quad \left|d_j\right| \leqslant 1 \quad \text{for all} \quad j \in \mathbb{N}. \tag{13}$$

From (9), (10), (11) and (12), we have

$$(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)=1+U_1(t)c_1z+\left[U_1(t)c_2+U_2(t)c_1^2\right]z^2+\cdots \tag{14}$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)=1+U_1(t)d_1w+\left[U_1(t)d_2+U_2(t)d_1^2\right]w^2+\cdots. \tag{15}$$

Equating the coefficients in (14) and (15), we get

$$(1+\lambda)a_2=U_1(t)c_1 \tag{16}$$

$$(1+2\lambda)a_3=U_1(t)c_2+U_2(t)c_1^2 \tag{17}$$

$$-(1+\lambda)a_2=U_1(t)d_1 \tag{18}$$

and

$$(1+2\lambda)\left(2a_2^2-a_3\right)=U_1(t)d_2+U_2(t)d_1^2. \tag{19}$$

From (16) and (18) we obtain

$$c_1=-d_1 \tag{20}$$

and

$$2(1+\lambda)^2a_2^2=U_1^2(t)\left(c_1^2+d_1^2\right). \tag{21}$$

Also, by using (17) and (19), we obtain

$$2(1+2\lambda)a_2^2=U_1(t)(c_2+d_2)+U_2(t)\left(c_1^2+d_1^2\right). \tag{22}$$

By using (21) in (22), we get

$$\left[2(1+2\lambda)-\frac{2U_2(t)}{U_1^2(t)}(1+\lambda)^2\right]a_2^2=U_1(t)(c_2+d_2). \tag{23}$$

From (5), (13) and (23), we have the desired inequality (6). Next, by subtracting (19) from (17), we have

$$2(1+2\lambda)a_3-2(1+2\lambda)a_2^2=U_1(t)(c_2-d_2)+U_2(t)\left(c_1^2-d_1^2\right).$$

Further, in view of (20), we obtain

$$a_3=a_2^2+\frac{U_1(t)}{2(1+2\lambda)}(c_2-d_2). \tag{24}$$

Hence using (21) and applying (5), we get desired inequality (7).

Now, by using (23) and (24) for some $\eta\in\mathbb{R}$, we get

$$\begin{aligned}a_3-\eta a_2^2&=(1-\eta)\left[\frac{U_1^3(t)(c_2+d_2)}{2(1+2\lambda)U_1^2(t)-2(1+\lambda)^2U_2(t)}\right]+\frac{U_1(t)(c_2-d_2)}{2(1+2\lambda)}\\&=U_1(t)\left[\left(h(\eta)+\frac{1}{2(1+2\lambda)}\right)c_2+\left(h(\eta)-\frac{1}{2(1+2\lambda)}\right)d_2\right],\end{aligned}$$

where

$$h(\eta)=\frac{U_1^2(t)(1-\eta)}{2\left[(1+2\lambda)U_1^2(t)-(1+\lambda)^2U_2(t)\right]}$$

So, we conclude that

$$\left|a_3-\eta a_2^2\right|\leqslant\begin{cases}\frac{4t}{1+2\lambda}, & 0\leqslant|h(\eta)|\leqslant\frac{1}{2(1+2\lambda)}\\ 4|h(\eta)|t, & |h(\eta)|\geqslant\frac{1}{2(1+2\lambda)}\end{cases}.$$

This completes the proof of Theorem 1. □

Taking $\eta=1$ in Theorem 1, we get the following consequence.

COROLLARY 1. *For $\lambda\geqslant 1$ and $t\in(1/2,1)$, let the function $f\in\Sigma$ given by (1) be in the class $\mathscr{B}_\Sigma(\lambda,t)$. Then*

$$\left|a_3-a_2^2\right|\leqslant\frac{4t}{1+2\lambda}.$$

Taking $\lambda=1$ in Theorem 1, we get the following consequence.

COROLLARY 2. *For $t\in(1/2,1)$, let the function $f\in\Sigma$ given by (1) be in the class $\mathscr{B}_\Sigma(\lambda,t)$. Then*

$$|a_2|\leqslant\frac{t\sqrt{2t}}{\sqrt{1-t^2}},$$

$$|a_3|\leqslant t^2+\frac{2t}{3},$$

and for some $\eta\in\mathbb{R}$,

$$\left|a_3-\eta a_2^2\right|\leqslant\begin{cases}\frac{4t}{3}, & |\eta-1|\leqslant\frac{1-t^2}{3t^2}\\ \frac{2|\eta-1|t^3}{1-t^2}, & |\eta-1|\geqslant\frac{1-t^2}{3t^2}.\end{cases}.$$

Taking $\eta=1$ in Corollary 2, we get the following consequence.

COROLLARY 3. *For $t\in(1/2,1)$, let the function $f\in\Sigma$ given by (1) be in the class $\mathscr{B}_\Sigma(\lambda,t)$. Then*

$$\left|a_3-a_2^2\right|\leqslant\frac{4t}{3}.$$

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(Received July 14, 2016)

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